

To solve an exact eqn., the key is to find  $\psi$ !

$$\because \psi_x = M, \psi_y = N.$$

$$\therefore \psi_x = M \Rightarrow \int \psi_x dx = \int M dx \Rightarrow \psi = \int M dx + h(y), \text{ (holding } y \text{ constant)}$$

[Here,  $h(y)$  plays the role of the arbitrary const.]

$$\text{Thus } \psi_y = \frac{\partial}{\partial y} \int M dx + \frac{d}{dy} h(y) = N.$$

$$\text{i.e. } \frac{d}{dy} h(y) = N - \frac{\partial}{\partial y} \int M dx \Rightarrow h(y) = \int [N - \frac{\partial}{\partial y} \int M dx] dy.$$

$$\text{ex: } (y \cos x + 2x e^y) + (\sin x + x^2 e^y - 1) y' = 0.$$

$$\text{Here: } M = y \cos x + 2x e^y, N = \sin x + x^2 e^y - 1.$$

$$\text{check: } M_y = \cos x + 2x e^y, N_x = \cos x + 2x e^y.$$

$\therefore$  it is an exact eqn. ( $\because M_y = N_x$ ).

Thus  $\exists \psi$  s.t.  $\psi_x = M, \psi_y = N$ .

$$\begin{aligned} \text{Integrating } \psi_x = M: \quad \psi &= \int (y \cos x + 2x e^y) dx + h(y) \\ &= y \sin x + x^2 e^y + h(y) \end{aligned}$$

Differentiate it:  $\psi_y = \sin x + x^2 e^y + \frac{d}{dy} h(y)$ , which is

equal to  $N$ . That is:

$$\sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1.$$

$$\text{i.e. } h'(y) = -1, \Rightarrow h(y) = -y.$$

Thus  $\psi(x, y) = y \sin x + x^2 e^y - y = c$  gives the general solution IMPLICITLY.

6x: (82.6=1) Find  $b$  s.t.  
 $(ye^{2xy} + x)dx + bxe^{2xy}dy = 0$  is an exact eqn. Solve it.

$$M = ye^{2xy} + x, N = bxe^{2xy}.$$

To make the ODE be exact, we need  $M_y = N_x$ . (i.e.  $\frac{\partial}{\partial y}M = \frac{\partial}{\partial x}N$ ).

$$\therefore M_y = 2xye^{2xy} + e^{2xy}, N_x = be^{2xy} + 2bxye^{2xy}$$

$\therefore b=1$ , then  $M_y = N_x$ .

Solve the ODE when  $b=1$ .

$$\begin{aligned}\psi_x = M &\Rightarrow \psi = \int M dx + h(y) \\ &= \int (ye^{2xy} + x) dx + h(y) \\ &= \frac{1}{2}x^2 + \frac{1}{2}e^{2xy} + h(y).\end{aligned}$$

$$\text{Thus } \psi_y = xe^{2xy} + \frac{d}{dy}h(y) = N.$$

$$\text{i.e. } xe^{2xy} + h'(y) = xe^{2xy}$$

$$h'(y) = 0, \Rightarrow h = C.$$

Choosing  $h=0$ , then

$$\psi(x, y) = \frac{1}{2}x^2 + \frac{1}{2}e^{2xy} = C.$$

gives the general solution of the ODE implicitly.

Transfer an ODE into an exact eqn. when

either  $\frac{M_y - N_x}{N}$  depends on only  $x$ , or  $\frac{M_y - N_x}{M}$  depends on only  $y$ .

Ex (Ex 2.6: 28) Solve  $y dx + (2xy - e^{-2y}) dy = 0$ .

Here  $M = y$ ,  $N = 2xy - e^{-2y}$ .

$M_y = 1$ ,  $N_x = 2y$ . Thus, the ODE is not exact.

check (i)  $\frac{M_y - N_x}{N} = \frac{1 - 2y}{2xy - e^{-2y}}$ , depends on both  $x$  and  $y$ .

(ii)  $-\frac{M_y - N_x}{M} = -\frac{1 - 2y}{y}$  depends on only  $y$ .

Then solve  $M_y = -\frac{1 - 2y}{y} M$  for  $M$ .

$$\ln M = \int \frac{1 - 2y}{y} dy = -\ln y + 2y.$$

$$M = y^{-1} e^{2y}.$$

Multiply the original ODE:

$$e^{2y} dx + (2xe^{2y} - y^{-1}) dy = 0$$

Now:  $M = e^{2y}$ ,  $N = 2xe^{2y} - y^{-1}$ .

$M_y = 2e^{2y}$ ,  $N_x = 2e^{2y}$ . Thus, the new ODE is an exact eqn.

Set  $\psi_x = M$ ,  $\Rightarrow \psi = \int e^{2y} dx + h(y) \Rightarrow \psi = x e^{2y} + h(y)$ .

$\Rightarrow \psi_y = 2x e^{2y} + h'(y) = N$ , i.e.  $2x e^{2y} + h' = 2x e^{2y} - y^{-1}$ .

$\Rightarrow h' = -y^{-1}$ ,  $\Rightarrow h = -\ln|y|$ .

Thus  $\psi = x e^{2y} - \ln|y| = C$  gives the general solution implicitly.