Ex: Find the eigenvalues and eigenvector of

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = -\lambda^3 + 1 + 1 + \lambda + \lambda + \lambda$$

$$= -\lambda^3 + 3\lambda + 2 = -\lambda^3 + \lambda + 2\lambda + 2 = -\lambda(\lambda^2 - 1) + 2(\lambda + 1)$$

$$= -\lambda(\lambda+1)(\lambda-1) + 2(\lambda+1) = (\lambda+1)(-\lambda^2 + \lambda + 2)$$

$$= (\lambda+1)(-\lambda+2)(\lambda+1) = 0$$

$\therefore \lambda_1 = 2,\ \lambda_2 = \lambda_3 = -1$

Eigenvector corresponding to $\lambda_1 = 2$.

$$(A - 2I)X = 0 \Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\therefore \begin{cases} x_1 = \frac{1}{2}(x_2 + x_3) \\ x_2 = x_3 \end{cases} \qquad \therefore \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$

Set $x_3 = 1$, then $x_1 = x_2 = 1$. $\therefore (1, 1, 1)^T$ is an eigenvector of $\lambda_1 = 2$.

Eigenvectors corresponding to $\lambda_2 = \lambda_3 = -1$

$$(A + I)X = 0 \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow x_1 + x_2 + x_3 = 0.$$

Thus x_1, x_2, x_3 can be chosen arbitrarily as long as $x_1 + x_2 + x_3 = 0$.

Set $x_2 = 1$, $x_3 = 0$, then $x_1 = -1$, Thus $X^{(1)} = (-1, 1, 0)^T$

Set $x_2 = 0$, $x_3 = 1$, then $x_1 = -1$, Thus $X^{(2)} = (-1, 0, 1)^T$.

Obviously, $X^{(1)}, X^{(2)}$ are linearly independent. Thus $X^{(1)}, X^{(2)}$ are linearly independent eigenvectors of $\lambda_2 = \lambda_3 = -1$.