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ASYMPTOTIC SPEED OF SPREAD AND TRAVELING WAVES FOR A NONLOCAL EPIDEMIC MODEL

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Abstract. By applying the theory of asymptotic speeds of spread and traveling waves to a nonlocal epidemic model, we established the existence of minimal wave speed for monotone traveling waves, and show that it coincides with the spreading speed for solutions with initial functions having compact supports. The numerical simulations are also presented.

1. **Introduction.**

Consider the epidemic model proposed in [10, 8]

$$\begin{cases} \partial_t u_1(t,x) &= d\triangle u_1(t,x) - a_{11}u_1(t,x) + a_{12}u_2(t,x), \\ \partial_t u_2(t,x) &= -a_{22}u_2(t,x) + g(u_1(t,x)), \end{cases} \tag{1}$$

where d, a_{11}, a_{12} and a_{22} are positive constants, $u_1(t,x)$ and $u_2(t,x)$ denote the spatial densities of infectious agents and the infective human population at time $t \geq 0$, respectively. $1/a_{11}$ is the mean lifetime of the agents in the environment, $1/a_{22}$ is the mean infectious period of the infective human, a_{12} is the multiplicative factor of the infectious agents due to the human population, $g(u_1)$ is the force of infection on human population due to a concentration u_1 of the infectious agents. This model has some basic assumptions: (i) the total susceptible human population is large enough, with respect to the infective population, to be considered as constant; (ii) the infectious agents diffuse randomly in the habitat Ω due to a particular transmission mechanism; (iii) the infective population at $x \in \Omega$ only contributes to the infectious agents at the same spatial point.

Note that some infection agents u_1 (e.g., bacteria or viruses in the air), at a spatial point x, depend on the infective humans u_2 not only at the spatial point x, but also at spatial neighbor points of x, and even points in the whole region Ω. As mentioned in [1], to deal with indirect transmission diseases, typical of infectious diseases transmitted via the pollution of the environment due to the infective population (typhoid fever, schistosomiasis, malaria, etc.), a different approach should be used to model the mechanism of production of the pollutants. A possible model is the one proposed in [5]. Assume that the growth rate of bacteria or pollutants due to the infective population can be modelled by

$$\int_\Omega K(x,y)u_2(t,y)dy, \quad x \in \Omega, t \geq 0,$$

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where $K(x, y)$ describes the transfer kernel of infective agents produced by the infective humans located at y and made available at x. From the viewpoint of statistics, normal distribution is one of the most common probability distributions. Many phenomena generate random variables with probability distributions which are very well approximated by a normal distribution. Therefore, it is natural to assume that the transfer kernel $K(x, y)$, just like the standard normal density function, depends only on the distance between the two spatial points x and y, i.e., $K(x, y) = K(x - y)$, and $K(u) = K(v)$ if $|u| = |v|, \forall u, v \in \Omega$, where $|\cdot|$ denotes the usual norm on $\mathbb{R}^n, n = 1, 2, 3$. A kernel with this property is said to be isotropic. A typical isotropic function is the standard normal density function. The whole model system is then governed by

$$\begin{cases} \partial_t u_1(t,x) &= d\triangle u_1(t,x) - a_{11}u_1(t,x) + \int_\Omega K(x-y)u_2(t,y)dy, \\ \partial_t u_2(t,x) &= -a_{22}u_2(t,x) + g(u_1(t,x)). \end{cases} \tag{2}$$

Systems (1) and (2) have been widely analyzed. In the case where there is at most one nontrivial epidemic equilibrium, [10, 8, 7, 6] investigated the threshold dynamics for system (1): above some parameter threshold all epidemic outbreaks tend to the nontrivial equilibrium, below the parameter threshold all epidemics tend to extinction. In the case where there are two nontrivial equilibria, system (1) may admit a saddle point structure [9, 11, 15]. For the monotone increasing infection rate g and a general kernel $K(x, y)$, the stabilities of trivial solution and the unique nontrivial equilibrium solution of (2) were studied in [5], and [1] provides conditions for exponential decay of the epidemics for (2). The purpose of this paper is to study the asymptotic speed of spread, traveling waves and minimal wave speed for system (2) with $\Omega = \mathbb{R}^n$.

The existence of Fisher type monotone traveling waves and minimal wave speed of (1) were obtained in [22] via the method of upper and lower solutions, while the existence, uniqueness and exponential stability of bistable monotone traveling waves of (1) were established in [21] via phase plane techniques and spectrum analysis. Recently, the theory of asymptotic speeds of spread and monotone traveling waves, developed in [3, 2, 4, 12, 14, 13, 17, 18], has been generalized to a large class of scalar nonlinear integral equations in [19]. As an application example, a time-delayed version of (1) was also analyzed in [19]. We will use this theory to obtain the asymptotic speed of spread for solutions and minimal wave speed of monotone traveling waves for (2).

The paper is organized as follows. In section 2, we first reduce the system into an integral equation, and then obtain the asymptotic speed of spread under appropriate assumptions. Section 3 is devoted to the existence and nonexistence of traveling wave solutions. Our results show that the asymptotic speed of spread is exactly the minimal wave speed for monotone traveling waves. Finally, some numerical simulations for the solutions and the asymptotic speed of spread are presented.

2. **The asymptotic speed of spread.** Recall that a number $c^* > 0$ is called the asymptotic speed of spread for a function $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ if $\lim_{t\to\infty, |x|\geq ct} u(t, x) = 0$ for every $c > c^*$, and there exists some $\bar{u} > 0$ such that $\lim_{t\to\infty, |x|\leq ct} u(t, x) = \bar{u}$ for every $c \in (0, c^*)$, where $|\cdot|$ denotes the usual norm in $\mathbb{R}^n$. In this section, we will find such a c^* for solutions of system (2). By scaling the variables x, t and u_2, we

reduce system (2) to the following form

$$\begin{cases} \partial_t u_1(t,x) &= \triangle u_1(t,x) - u_1(t,x) + \int_{\mathbb{R}^n} K(x-y)u_2(t,y)dy, \\ \partial_t u_2(t,x) &= -\beta u_2(t,x) + g(u_1(t,x)), \end{cases} \tag{3}$$

where $\beta = a_{22}/a_{11}$, the new g is the $1/a_{11}^2$ times of g in (2), and the new $K(x)$ is $(\sqrt{d/a_{11}})^n K(\sqrt{d/a_{11}}\,x)$. System (3) is supplemented by initial conditions

$$u_1(0,x) = \phi_1(x) \geq 0,\ u_2(0,x) = \phi_2(x) \geq 0,\ x \in \mathbb{R}^n. \tag{4}$$

In what follows we reduce (3)-(4) to an integral equation for u_1. Let $\Gamma(t,x)$ and $\Gamma_1(t,x)$ be the Green functions associated with the parabolic equations $\partial_t u = \triangle u$ and $\partial_t u = \triangle u - u$, respectively. Then $\Gamma_1(t,x) = \Gamma(t,x)e^{-t}$. Integrating (3) together with (4), we have

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy + \\ &\quad \int_0^t ds \int_{\mathbb{R}^n} \Gamma_1(t-s,x-y) \int_{\mathbb{R}^n} K(y-z)u_2(s,z)dzdy, \end{aligned} \tag{5}$$

$$u_2(t,x) = e^{-\beta t}\phi_2(x) + \int_0^t e^{-\beta(t-r)} g(u_1(r,x))dr. \tag{6}$$

Changing the order of spatial integration in (5),

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy + \\ &\quad \int_0^t ds \int_{\mathbb{R}^n} u_2(s,z) \int_{\mathbb{R}^n} \Gamma_1(t-s,x-y)K(y-z)dydz. \end{aligned}$$

After a substitution,

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy + \\ &\quad \int_0^t ds \int_{\mathbb{R}^n} u_2(s,z) \int_{\mathbb{R}^n} \Gamma_1(t-s,x-z-y)K(y)dydz. \end{aligned}$$

Let

$$k_1(t,\xi) = \int_{\mathbb{R}^n} \Gamma_1(t,\xi-y)K(y)dy. \tag{7}$$

Then

$$u_1(t,x) = \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy + \int_0^t ds \int_{\mathbb{R}^n} u_2(s,z)k_1(t-s,x-z)dz.$$

Inserting (6) into the above equation,

$$u_1(t,x) = u_0(t,x) + \int_0^t ds \int_{\mathbb{R}^n} k_1(t-s,x-z) \int_0^s e^{-\beta(s-r)} g(u_1(r,z))drdz,$$

where

$$u_0(t,x) = \int_{\mathbb{R}^n} \Gamma_1(t,x-y)\phi_1(y)dy + \int_0^t ds \int_{\mathbb{R}^n} k_1(t-s,x-z)e^{-\beta s}\phi_2(z)dz. \tag{8}$$

Changing the order of the time integration,

$$\begin{aligned} u_1(t,x) &= u_0(t,x) + \int_0^t dr \int_{\mathbb{R}^n} \int_r^t k_1(t-s, x-z) e^{-\beta(s-r)} g(u_1(r,z)) ds dz \\ &= u_0(t,x) + \int_0^t dr \int_{\mathbb{R}^n} \int_0^{t-r} k_1(t-r-s, x-z) e^{-\beta s} g(u_1(r,z)) ds dz. \end{aligned}$$

After a substitution, we have

$$\begin{aligned} u_1(t,x) &= u_0(t,x) + \int_0^t dr \int_{\mathbb{R}^n} \int_0^{t-r} k_1(t-r-s, y) e^{-\beta s} g(u_1(r, x-y)) ds dy \\ &= u_0(t,x) + \int_0^t dr \int_{\mathbb{R}^n} \int_0^{r} k_1(r-s, y) e^{-\beta s} g(u_1(t-r, x-y)) ds dy. \end{aligned}$$

Letting $k_2(t,x) = \int_0^t k_1(t-s,x) e^{-\beta s} ds$, we have

$$u_1(t,x) = u_0(t,x) + \int_0^t ds \int_{\mathbb{R}^n} g(u_1(t-s, x-y)) k_2(s,y) dy, \tag{9}$$

where

$$k_2(t,x) = \int_0^t \int_{\mathbb{R}^n} \Gamma_1(t-s, x-y) K(y) e^{-\beta s} dy ds. \tag{10}$$

Before making some assumptions on system (3), we need to compute some Laplace-like transforms of integral kernels. Define

$$k(t,x) = g'(0) k_2(t,x), \ \ f(u) = \frac{g(u)}{g'(0)}.$$

For any function $\psi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, let

$$\mathcal{K}_\phi(c,\lambda) := \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs+y_1)} \phi(s,y) dy ds, \quad c, \lambda \geq 0,$$

where y_1 is the first coordinate of y. By [19, Proposition 4.2], we have

$$\begin{aligned} \mathcal{K}_{\Gamma_1}(c,\lambda) &= \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs+y_1)} \Gamma_1(s,y) dy ds \\ &= \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs+y_1)-s} \Gamma(s,y) dy ds \\ &= \int_0^\infty e^{\lambda^2 s - \lambda c s - s} ds \\ &= \begin{cases} \frac{1}{1+\lambda c - \lambda^2}, & \text{when } 1 + \lambda c - \lambda^2 > 0, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

By [19, Proposition 4.1 (1)], we further obtain

$$\begin{aligned} \mathcal{K}_k(c,\lambda) &= g'(0) \mathcal{K}_{\Gamma_1}(c,\lambda) \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(cs+y_1)} K(y) e^{-\beta s} dy ds \\ &= \frac{g'(0)}{\lambda c + \beta} \mathcal{K}(\lambda) \mathcal{K}_{\Gamma_1}(c,\lambda), \end{aligned} \tag{11}$$

where $\mathcal{K}(\lambda) = \int_{\mathbb{R}^n} e^{-\lambda y_1} K(y) dy$. In particular, $\mathcal{K}_k(0,\lambda) = \frac{g'(0)}{(1-\lambda^2)\beta} \mathcal{K}(\lambda)$. Let $k^* = \mathcal{K}_k(c,0) = \frac{g'(0)}{\beta} \mathcal{K}(0)$. We now can make the following assumptions for system (3).

(H1) $K : \mathbb{R}^n \to \mathbb{R}_+$ is continuous, and K is isotropic, i.e., $K(x) = K(y)$ if $|x| = |y|$, where $|\cdot|$ is the usual norm on $\mathbb{R}^n$.

(H2) $\mathcal{K}(0) > 0$, and there exists some $\lambda_0 > 0$ such that $\mathcal{K}(\lambda_0) = \infty$, and $\mathcal{K}(\lambda) < \infty$ for all $\lambda \in [0, \lambda_0)$, where λ_0 may be infinity.

(H3) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous with $g(0) = 0$, differentiable at 0, and satisfies $g'(0)\mathcal{K}(0) > \beta$, $0 < g(u) \le g'(0)u, \forall u > 0$.

Note that the condition $g'(0)\mathcal{K}(0) > \beta$ in (H3) implies the instability of the equilibrium zero of system (3). Since $\mathcal{K}(0) > 0$ and $\Gamma_1(t, \cdot) > 0, \forall t > 0$, $k(t, \cdot) > 0$ for all $t > 0$. One can easily check that (H1)-(H3) imply the assumptions (A) and (B) in [19] with $F(u, s, y) = f(u)k(s, y)$. Our assumptions also imply that system (3) is quasi-monotone. By [16, Corollary 5] (see also [20, Corollary 8.1.3]) and [20, Corollary 2.2.5], for any bounded, uniformly continuous and nonnegative function $\phi(x) = (\phi_1(x), \phi_2(x))$, system (3) with (4) admits a unique and nonnegative mild solution $u(t, x) = (u_1(t, x), u_2(t, x))$, and it is a classic solution for $t > 0$. Note that $u_1(t, x)$ is also a solution of (9).

With assumption (H2), the expression (11) shows that if $\lambda^\sharp(c) = \min(\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1}, \lambda_0)$, then $\mathcal{K}_k(c, \lambda) < \infty$ for all $\lambda \in [0, \lambda^\sharp(c))$, and $\lim_{\lambda \nearrow \lambda^\sharp(c)} \mathcal{K}_k(c, \lambda) = \infty$ for every $c \ge 0$. Define

$$c^* := \inf\{c \ge 0 : \mathcal{K}_k(c, \lambda) < 1 \text{ for some } \lambda > 0\}.$$

According to [19, Proposition 2.3], c^* can be uniquely determined as the solution of the system

$$\mathcal{K}_k(c, \lambda) = 1, \quad \frac{d}{d\lambda}\mathcal{K}_k(c, \lambda) = 0.$$

That is, (c^*, λ^*) is the unique positive solution of the system

$$\begin{aligned} &(\beta + \lambda c)(1 + \lambda c - c^2) = g'(0)\mathcal{K}(\lambda), \\ &\frac{2\lambda - c}{1 + \lambda c - c^2} - \frac{c}{\beta + \lambda c} = \frac{1}{\mathcal{K}(\lambda)} \int_{\mathbb{R}^n} y_1 e^{-\lambda y_1} K(y) dy. \end{aligned}$$

The following results show that c^* is the asymptotic speed of spread for solutions of (3) with initial functions having compact supports. In order to obtain the convergence for $0 < c < c^*$, we need one of the following two additional conditions:

(H4) $\lim_{u \to \infty} \frac{g(u)}{u} = 0$, and there exists $u^* > 0$ such that g is increasing on $[0, u^*]$, $g(u)\mathcal{K}(0) > \beta u$ for $u \in (0, u^*)$, and $g(u)\mathcal{K}(0) < \beta u$ for $u > u^*$.

(H5) $\limsup_{u \to \infty} \frac{g(u)}{u} < \frac{\beta}{\mathcal{K}(0)}$, $\frac{g(u)}{u}$ is strictly decreasing, and $ug(u)$ is strictly increasing for $u > 0$.

It is easy to see that the first condition of (H4) or (H5) implies the boundedness of solutions of system (3) with bounded initial data.

Theorem 1. *Let (H1)-(H3) hold and c^* be defined as above. Denote by $u(t, x, \phi)$ the unique solution of system (3)-(4). Then the following statements are valid:*

(i) *For any continuous function $\phi = (\phi_1, \phi_2) : \mathbb{R}^n \to \mathbb{R}^2_+$ with the property that for every $\kappa_1 > 0$, there exists $\kappa_2 > 0$ such that $\phi_1(y) + \phi_2(y) \le \kappa_2 e^{-\kappa_1 |y|}, \forall y \in \mathbb{R}^n$, we have*

$$\lim_{t \to \infty, \, |x| \ge ct} u(t, x, \phi) = (0, 0), \quad \forall c > c^*.$$

(ii) *Assume in addition that either (H4) or (H5) holds. Then for any bounded and uniformly continuous function $\phi = (\phi_1, \phi_2) : \mathbb{R}^n \to \mathbb{R}^2_+$ with $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$, we have*

$$\lim_{t \to \infty, |x| \le ct} u(t, x, \phi) = (u^*, v^*), \quad \forall c \in (0, c^*),$$

where u^* is the unique solution of $g(u)\mathcal{K}(0) = \beta u$, and $v^* = \frac{g(u^*)}{\beta}$.

Proof. Let $\phi = (\phi_1, \phi_2) : \mathbb{R}^n \to \mathbb{R}^2_+$ be a bounded continuous function with $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$. For convenience, we set $u(t, x, \phi) := (u_1(t, x), u_2(t, x))$. Note that $\Gamma_1(t, \cdot) > 0, \forall t > 0$, and $\mathcal{K}(0) > 0$. Recall that $u_0(t, \cdot) > 0$ for $t > 0$, and $u_0(t, x) = u_{01}(t, x) + u_{02}(t, x)$, where

$$\begin{aligned} u_{01}(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\phi_1(y)dy, \\ u_{02}(t,x) &= \int_0^t ds \int_{\mathbb{R}^n} k_1(t-s, x-y)e^{-\beta s}\phi_2(y)dy. \end{aligned}$$

In what follows, we show that $\lim_{t\to\infty} u_0(t, x) = 0$ uniformly in $x \in \mathbb{R}^n$. In view of (7) and the fact that $\int_{\mathbb{R}^n} \Gamma(t, x-y)dy = 1, \forall t \geq 0, x \in \mathbb{R}^n$, we have

$$\begin{aligned} u_{02}(t,x) &= \int_0^t ds \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Gamma_1(t-s, x-y-z)K(z)e^{-\beta s}\phi_2(y)dzdy \\ &= \int_0^t ds \int_{\mathbb{R}^n} e^{-\beta s}K(z) \int_{\mathbb{R}^n} e^{-(t-s)}\Gamma(t-s, x-y-z)\phi_2(y)dydz \\ &\leq M_1 \int_0^t e^{-t-\beta s+s}ds \int_{\mathbb{R}^n} K(z)dz \\ &= M_1\mathcal{K}(0)\int_0^t e^{-t-\beta s+s}ds, \end{aligned}$$

where $M_1 = \sup_{y\in\mathbb{R}^n} \phi_2(y)$. Therefore, $\lim_{t\to\infty} u_{02}(t, x) = 0$ uniformly in $x \in \mathbb{R}^n$. Since $\int_{\mathbb{R}^n} \Gamma_1(t, y)dy = e^{-t}$, it follows that $\lim_{t\to\infty} u_{01}(t, x) = 0$, and hence $\lim_{t\to\infty} u_0(t, x) = 0$ uniformly in x. By [19, Proposition 2.1], $u_1(t, x)$ is the unique solution of (9).

(i). For given $c, \lambda > 0$ with $\mathcal{K}_k(c, \lambda) < 1$, $\mathcal{K}_{\Gamma_1}(c, \lambda)$ and $\mathcal{K}(\lambda)$ are finite numbers. Therefore, $\lambda^2 - \lambda c - 1 < 0$. Note that for every $w \in \mathbb{R}^n$ with $|w| = 1$, $-|y| \leq w \cdot y \leq |y|, \forall y \in \mathbb{R}^n$, where $\cdot$ is the inner product on $\mathbb{R}^n$. By the assumption on ϕ_1 and ϕ_2, there exists $\gamma > 0$ such that $\phi_i(y) \leq \gamma e^{-\lambda|y|} \leq \gamma e^{\lambda w\cdot y}, \forall y \in \mathbb{R}^n, i = 1, 2$. In the following, we show that $u_0(t, x)$ is admissible in the sense that there exists a constant $\gamma' > 0$ such that $u_0(t, x) \leq \gamma' e^{\lambda(ct-|x|)}, \forall t \geq 0, x \in \mathbb{R}^n$. Note that

$$\int_{\mathbb{R}^n} \Gamma(t, y)e^{-\lambda w\cdot y}dy = \int_{\mathbb{R}^n} \Gamma(t, y)e^{-\lambda y_1}dy = e^{\lambda^2 t},\ \forall w \in \mathbb{R}^n \text{ with } |w| = 1.$$

We then have

$$\begin{aligned} u_{01}(t,x) &= \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\phi_1(y)dy \\ &\leq \int_{\mathbb{R}^n} \Gamma_1(t, x-y)\gamma e^{\lambda w\cdot y}dy \\ &= \gamma \int_{\mathbb{R}^n} \Gamma_1(t, y)e^{\lambda w\cdot(x-y)}dy \\ &= \gamma e^{\lambda w\cdot x - t}\int_{\mathbb{R}^n} \Gamma(t, y)e^{-\lambda w\cdot y}dy \\ &= \gamma e^{\lambda w\cdot x}e^{(\lambda^2-1)t}. \end{aligned}$$

Letting $w = -\frac{x}{|x|}$, and using the inequality $\lambda^2 - 1 < \lambda c$, we obtain

$$u_{01}(t,x) \leq \gamma e^{\lambda(ct-|x|)}, \quad \forall t \geq 0, x \in \mathbb{R}^n.$$

Applying similar arguments to $u_{02}(t,x)$, we have

$$\begin{aligned}
u_{02}(t,x) &= \int_0^t ds \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\beta s} \phi_2(y) \Gamma_1(t-s, x-y-z) K(z) dz dy \\
&= \int_0^t ds \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\beta s} \phi_2(y) \Gamma_1(t-s, x-y-z) K(z) dy dz \\
&\leq \gamma \int_0^t ds \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\beta s} e^{\lambda w \cdot y} \Gamma_1(t-s, x-y-z) K(z) dy dz \\
&\leq \gamma \int_0^t ds \int_{\mathbb{R}^n} e^{-\beta s} e^{\lambda w \cdot (x-z)} e^{(\lambda^2-1)(t-s)} K(z) dz \\
&\leq \gamma e^{\lambda(ct + w\cdot x)} \int_0^t e^{-(\lambda c+\beta)s} ds \int_{\mathbb{R}^n} e^{-\lambda z_1} K(z) dz \\
&\leq \frac{\mathcal{K}(\lambda)}{\lambda c + \beta} e^{\lambda(ct+w\cdot x)}.
\end{aligned}$$

Letting $w = -\frac{x}{|x|}$,

$$u_{02}(t,x) \leq \frac{\mathcal{K}(\lambda)}{\lambda c + \beta} e^{\lambda(ct-|x|)}, \quad \forall t \geq 0, x \in \mathbb{R}^n.$$

Therefore, $u_0(t,x)$ is admissible. By [19, Theorem 2.1], it follows that

$$\lim_{t\to\infty, |x|\geq ct} u_1(t,x) = 0, \quad \text{for each } c > c^*,$$

and hence (6) implies the result.

(ii). Assume in addition that either (H4) or (H5) holds. Then we can find some constants $c_1, c_2 \geq 0$ such that $c_1 k^* < 1$ and $g(u) \leq g'(0)(c_2 + c_1 u), \forall u \geq 0$. Therefore, [19, Proposition 2.1 (a)] implies that every solution of (9) is bounded. Note that the monotonicity of g on $[0, u^*]$ implies that there is no pair $w > u^* > v > 0$ such that $\beta w = \mathcal{K}(0) g(v)$ and $\beta v = \mathcal{K}(0) g(w)$. Thus, by [19, Theorem 2.3 and Theorem 2.5], we have

$$\lim_{t\to\infty, |x|\leq ct} u_1(t,x) = u^*, \quad \forall c \in (0, c^*).$$

Therefore,

$$\lim_{t\to\infty, |x|\leq ct} u_2(t,x) = g(u^*) \int_0^\infty e^{-\beta s} ds = \frac{g(u^*)}{\beta} = v^*, \quad \forall c \in (0, c^*).$$

This completes the proof. □

Remark 1. Theorem 1 implies that c^* is the asymptotic speed for solutions of system (3) with initial functions having compact supports. Let $u(t,x) = (u_1(t,x), u_2(t,x))$ be such a solution. For any given $\rho \in (0, u^*)$, denote by $x_+^\rho(t)$ and $x_-^\rho(t)$ the most right and left points with $u_1(t, x_\pm^\rho(t)) = \rho$, respectively. Clearly, $x_+^\rho(t)$ and $x_-^\rho(t)$ are well defined for all large t because of the two limit formulas in Theorem 1. We claim that $\lim_{t\to\infty} \frac{x_+^\rho(t)}{t} = c^*$. Indeed, by Theorem 1, it follows that for any $0 < \varepsilon < \min(\rho, u^* - \rho)$, there exists some $t_0 = t_0(\varepsilon) > 0$ such that $u_1(t,x) < \varepsilon$ for all $t \geq t_0, |x| \geq (c^* + \varepsilon)t$, and $|u_1(t,x) - u^*| < \varepsilon$ for all $t \geq t_0, |x| \leq (c^* - \varepsilon)t$. Therefore, $x_+^\rho(t) < (c^* + \varepsilon)t$ and $x_+^\rho(t) > (c^* - \varepsilon)t$, and hence, $|\frac{x_+^\rho(t)}{t} - c^*| < \varepsilon$, for all $t > t_0$. By a similar argument, we can prove that $\lim_{t\to\infty} \frac{x_-^\rho(t)}{t} = c^*$. We will use this observation to compute c^* numerically.

3. **Traveling wave solutions.** In this section, we consider the traveling wave solutions of system (3) with $n = 1$. Recall that a solution $u(t,x)$ of (3) is said to be a traveling wave solution if it is of the form $u(t,x) = U(x+ct)$. The parameter c is called the wave speed, and the function $U(\cdot)$ is called the wave profile. We will impose the following conditions on the wave profile:

$$U(\cdot) \text{ is positive and bounded on } \mathbb{R}, \text{ and } \lim_{\xi\to-\infty} U(\xi) = 0. \tag{12}$$

Suppose that $(u_1(t,x), u_2(t,x)) = (U_1(x+ct), U_2(x+ct))$ is a traveling wave solution of system (3). Then $u_i(t,x)$ are uniformly bounded for all $t \in \mathbb{R}$. Integrating the second equation of (3), we have

$$u_2(t,x) = e^{-\beta(t-r)}u_2(r,x) + \int_r^t e^{-\beta(t-s)}g(u_1(s,x))ds,\ \forall t \geq r,\ r \in \mathbb{R}.$$

Setting $r \to -\infty$, we obtain

$$u_2(t,x) = \int_0^\infty e^{-\beta s}g(u_1(t-s,x))ds. \tag{13}$$

Integrating the first equation of (3) from r to t, and then setting $r \to -\infty$, we see that

$$u_1(t,x) = \int_0^\infty ds \int_{\mathbb{R}} u_2(t-s,y)k_1(s,x-y)dy.$$

Substituting (13) into the above equation, we further have

$$u_1(t,x) = \int_0^\infty ds \int_{\mathbb{R}} f(u_1(t-s,x-y))k(s,y)dy. \tag{14}$$

Note that $f(u)k(t,x) \equiv g(u)k_2(t,x)$ according to our notations. It follows that any traveling wave solution of (3) is a traveling wave solution of (14)-(13). The following lemma shows that the converse is also true.

Lemma 1. *If system (14)-(13) admits a traveling wave $U(x+ct)$ subject to (12), then $U(x+ct)$ is also a traveling wave of (3) subject to (12).*

Proof. Let $(u_1(t,x), u_2(t,x)) = (U_1(x+ct), U_2(x+ct))$ be a traveling wave solution of (14)-(13). Then

$$u_2(t,x) = \int_0^\infty e^{-\beta s}g(u_1(t-s,x))ds = U_2(x+ct).$$

In view of (10), we have

$$
\begin{aligned}
u_1(t,x) &= \int_0^\infty ds \int_{\mathbb{R}} f(u_1(t-s,x-y))k(s,y)dy \\
&= \int_0^\infty ds \int_{\mathbb{R}} dy \int_0^s dr \int_{\mathbb{R}} g(u_1(t-s,x-y))\Gamma_1(s-r,y-z)K(z)e^{-\beta r}dz \\
&= \int_0^\infty dr \int_{\mathbb{R}} dy \int_r^\infty ds \int_{\mathbb{R}} g(u_1(t-s,x-y))\Gamma_1(s-r,y-z)K(z)e^{-\beta r}dz \\
&= \int_0^\infty dr \int_{\mathbb{R}} dy \int_0^\infty ds \int_{\mathbb{R}} g(u_1(t-r-s,x-y))\Gamma_1(s,y-z)K(z)e^{-\beta r}dz \\
&= \int_0^\infty dr \int_{\mathbb{R}} dy \int_0^\infty ds \int_{\mathbb{R}} g(u_1(t-r-s,x-y))\Gamma_1(s,z)K(y-z)e^{-\beta r}dz \\
&= \int_0^\infty ds \int_{\mathbb{R}} \Gamma_1(s,z) \int_0^\infty \int_{\mathbb{R}} g(u_1(t-r-s,x-y))K(y-z)e^{-\beta r}dydrdz \\
&= \int_0^\infty ds \int_{\mathbb{R}} \Gamma_1(s,z) \int_{\mathbb{R}} \int_0^\infty g(u_1(t-r-s,x-y))K(y-z)e^{-\beta r}drdydz \\
&= \int_0^\infty T_1(s) \int_{\mathbb{R}} K(y-z)u_2(t-s,x-y)dyds, \qquad (15)
\end{aligned}
$$

where $T_1(t)$ is the semigroup on $BUC(\mathbb{R},\mathbb{R})$ generated by the parabolic equation $\partial_t u = \triangle u - u$, and $BUC(\mathbb{R},\mathbb{R})$ is the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to itself. By [19, Proposition 4.3], it follows that $(u_1(t,\cdot),u_2(t,\cdot))$ satisfies the abstract integral equations

$$u_1(t) = T_1(t-r)u_1(r) + \int_r^t T_1(t-s)\int_{\mathbb{R}} K(y-z)u_2(s)dyds, \qquad (16)$$

$$u_2(t) = e^{-\beta(t-r)}u_2(r) + \int_r^t e^{-\beta(t-s)}g(u_1(s))ds, \quad \forall t \geq r,\, r\in\mathbb{R}. \qquad (17)$$

Clearly, $u_2(t,x)$ satisfies the second equation of system (3). By the form $u_1(t,x) = U_1(x+ct)$ and the smoothing property of parabolic operators (see, e.g., [20, Corollary 2.2.5] with $r=0$), it follows that $u_1(t,x)$ satisfies the first equation of (3). Thus, $(u_1(t,x),u_2(t,x)) = (U_1(x+ct),U_2(x+ct))$ is a traveling wave solution of system (3) with speed c. □

Theorem 2. *Let (H1)-(H3) hold, and let c^*, v^* be defined as in Theorem 1. Then the following statements are valid.*

(i) *There is no traveling wave solution for (3) and (12) with wave speed $c \in (0,c^*)$.*
(ii) *Assume in addition that (H4) holds, and that $|g(u)-g(v)| \leq g'(0)|u-v|, \forall u,v \in [0,u^*]$, and $g''(0)$ exists. Then system (3) with (12) admits a monotone traveling wave connecting $(0,0)$ and (u^*,v^*) with speed $c \geq c^*$. Moreover, the monotone traveling wave with speed $c > c^*$ is unique up to translation.*

Proof. (i). Note that (H1)-(H3) imply the assumption (A) and (B) in [19]. The result is a straight forward consequence of [19, Theorem 3.5].

(ii). Since $g''(0)$ exists, we can find two numbers $\delta > 0, b > 0$ such that $g(u) \geq g'(0)(u-bu^2), \forall u \in [0,\delta]$. By [19, Theorems 3.3-3.4], as applied to (14) with $F(u,s,x) = f(u)k(s,x)$, it follows that for each $c \geq c^*$, (14) admits a monotone traveling wave $u_1(t,x) = U_1(x+ct)$ connecting 0 and u^*. Define $u_2(t,x)$ as in (13), we then have

$$u_2(t,x) = \int_0^\infty e^{-\beta s} g(U_1(x + c(t-s)))ds := U_2(x+ct), \tag{18}$$

where $U_2(\xi) = \int_0^\infty e^{-\beta s} g(U_1(\xi - cs))ds$. Obviously, $U_2'(\xi) > 0$. By the dominant convergence theorem, $\lim_{\xi\to-\infty} U_2(\xi) = 0$, and $\lim_{\xi\to\infty} U_2(\xi) = v^*$. Therefore, $(u_1(t,x), u_2(t,x))$ is a traveling wave of (14)-(13), and hence Lemma 1 implies the result. The uniqueness of traveling waves with $c > c^*$ follows from [19, Theorem 3.2], as applied to (14) with $F(u,s,x) = f(u)k(s,x)$. □

Numerical simulation. We numerically simulate system (3) with $n = 1$. Let the transfer kernel K be the standard normal density function, i.e., $K(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and set $g(u) = \frac{2u}{1+u}$, $\beta = 1$. It is easy to see that system (3) satisfies assumptions (H1)-(H4) with $u^* = v^* = 1$. By Theorem (1), for any continuous initial functions ϕ_1, ϕ_2 with compact supports, we have

$$\lim_{t\to\infty, |x|\geq ct} u(t,x) = (0,0), \quad \forall c > c^*,$$

and

$$\lim_{t\to\infty, |x|\leq ct} u(t,x) = (1,1), \quad \forall c \in (0, c^*).$$

We discretise system (3) by the finite difference method coupled with composite integration formulas on a finite spatial interval $[-L, L]$ with the Neumann boundary condition, where $L > 0$ is sufficiently large in comparison with the domain in which the solutions rapidly change shapes. Let

$$\phi_1(x) = \phi_2(x) = \begin{cases} 0, & \text{if } x \leq -\pi/2, \\ \frac{1}{2}\cos x, & \text{if } x \in (-\pi/2, \pi/2), \\ 0, & \text{if } x \geq \pi/2. \end{cases}$$

Figure 1 illustrates the corresponding numerical solution $u(t,x) = (u_1(t,x), u_2(t,x))$. Obviously, the result is consistent with the above two limit formulas. In order to get the asymptotic speed c^*, we use Remark 1 to approximate c^*. Figure 2 shows the curves $x_+^{0.25}(t)/t$ and $x_-^{0.25}(t)/t$ versus t. Thus, $c^* \approx 1.0$. To get a traveling wave, we choose the initial condition as

$$\phi_1(x) = \phi_2(x) = \begin{cases} 0, & \text{if } x \leq -1, \\ \frac{1}{2}(1+x), & \text{if } x \in (-1, 1), \\ 1, & \text{if } x \geq 1. \end{cases}$$

The evolution of the solution is shown in Figure 3. The solution becomes smooth immediately. The shape of the solution promptly converges to a traveling wave. The wave moves in the negative x-direction as the time t increases (shown as in Figure 4), and the wave speed is about 1.0, which coincides with the spreading speed.

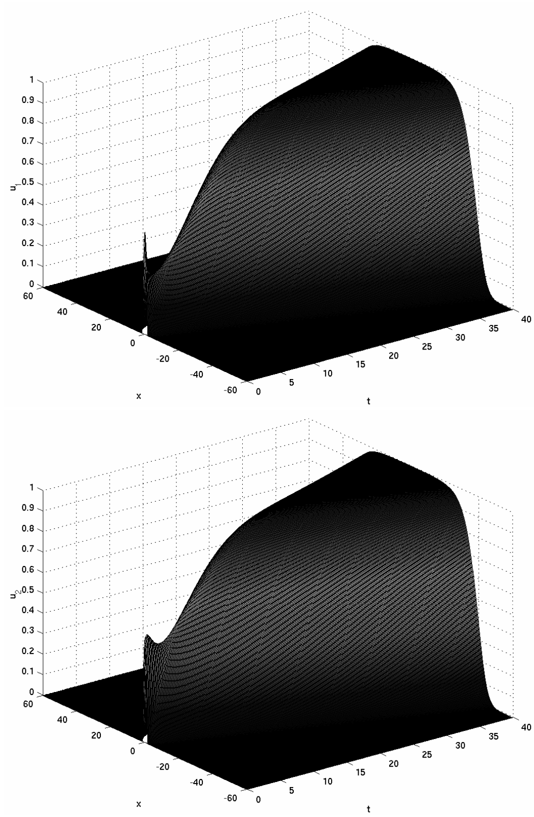

FIGURE 1. Two components of the numerical solution $u(t,x) = (u_1(t,x), u_2(t,x))$.

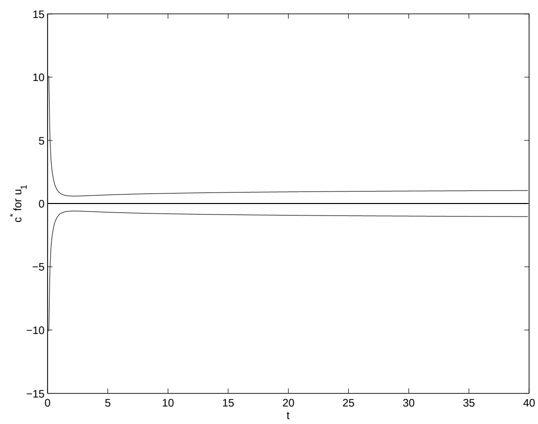

FIGURE 2. The curves $x_+^{0.25}(t)/t$ (upper) and $x_-^{0.25}(t)/t$ (lower) versus t.

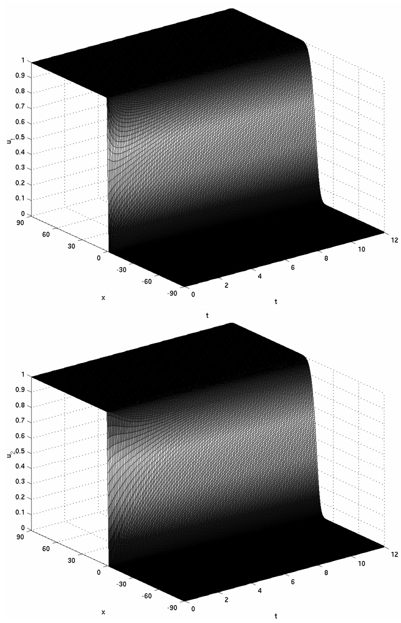

FIGURE 3. Two components of the numerical solution $u(t, x) = (u_1(t, x), u_2(t, x))$.

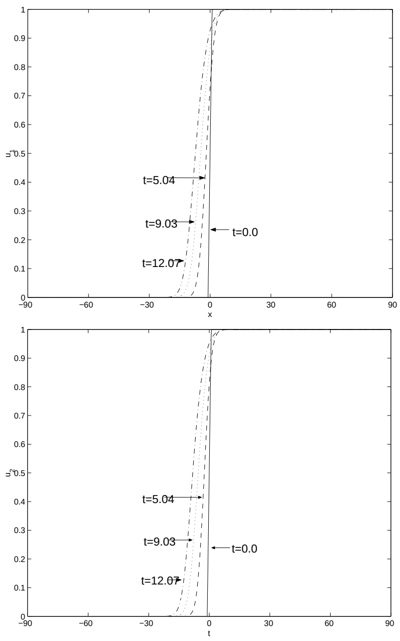

FIGURE 4. The solution $u(t, x)$ at some specific times.

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