# ON THE UNIQUENESS OF ENERGY MINIMIZERS IN FINITE ELASTICITY

JEYABAL SIVALOGANATHAN AND SCOTT J. SPECTOR

Abstract. The uniqueness of absolute minimizers of the energy of a compressible, hyperelastic body subject to a variety of dead-load boundary conditions in two and three dimensions is herein considered. Hypotheses under which a given solution of the corresponding equilibrium equations is the unique absolute minimizer of the energy are obtained. The hypotheses involve uniform polyconvexity and pointwise bounds on derivatives of the stored-energy density when evaluated on the given equilibrium solution. In particular, an elementary proof of the uniqueness result of Fritz John [*Comm. Pure Appl. Math.* **25** (1972), 617–634] is obtained for uniformly polyconvex stored-energy densities.

## 1. Introduction

In this manuscript we consider the uniqueness of absolute minimizers of the energy of a compressible, hyperelastic body under dead loads. Although one does not always expect such uniqueness, for example, when a thin rod is subjected to uniaxial compression there should be more than one buckled minimizer, a result of Zhang [48] for the displacement problem shows that there is exactly one absolute minimizer of the elastic energy for certain boundary displacements.

In addition to the displacement problem we also consider both the traction and the mixed problem for energy functions that are uniformly polyconvex, that is, when $n = 3$, stored-energy densities of the form

$$W(\mathbf{x}, \mathbf{F}) = \frac{\omega(\mathbf{x})}{p}|\mathbf{F}|^p + \Phi(\mathbf{x}, \mathbf{F}, \operatorname{cof} \mathbf{F}, \det \mathbf{F}),$$

where $\omega(\mathbf{x}) \geq \omega_{\rm o} > 0$, $p \geq 3$, $\mathbf{N} \mapsto \Phi(\mathbf{x}, \mathbf{N})$ is convex, $\det \mathbf{F}$ denotes the determinant of the 3 by 3 matrix $\mathbf{F}$, $\operatorname{cof} \mathbf{F}$ its cofactor matrix, and $|\mathbf{F}|$ the square-root of the sum of the squares of the elements of $\mathbf{F}$. Our main result, Theorem 4.2, shows (using elementary methods) that for such energies any (weak) solution of the equilibrium equations that satisfies a certain pointwise bound will be the unique absolute minimizer of the energy. Moreover, there can be no other solution of the equilibrium equations that satisfies this bound.

We note, in Remark 4.3, that our proof of Theorem 4.2 is also valid when $\Phi$ is not globally convex. If instead a weak solution of the equilibrium equations lies at a point of convexity of $\Phi$ (see (4.16)) and satisfies the required pointwise bound, then that deformation must be a, potentially nonunique, absolute minimizer of the energy. Theorem 4.2 therefore has implications for stored-energies that admit phase transitions (see Ball and James [5] or, e.g., Grabovsky and Truskinovsky [19] and the references therein).

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*Date*: 5 May 2016.

2010 *Mathematics Subject Classification.* 74B20, 35A02, 49J40, 74G30, 74G65, 35J57.

*Key words and phrases.* Finite Elasticity, Nonlinear Elasticity, Uniqueness, Equilibrium Solutions, Energy Minimizers, Nonuniqueness, Uniform Polyconvexity, Strict Polyconvexity, Strongly Polyconvex.

This material is based upon work supported by the National Science Foundation under Grant No. DMS–1107899.

In the special case when the stored-energy density $W$ of a homogeneous body $\mathscr{B} \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, is given by

$$W(\mathbf{F}) = \frac{\omega_{\mathrm{o}}}{n}|\mathbf{F}|^n + \Phi(\mathbf{F}, \det \mathbf{F}), \tag{1.1}$$

where $\omega_{\mathrm{o}} > 0$ and $\Phi$ is convex, then our results show that if $\mathbf{u}_{\mathrm{e}}$ is a weak solution of the equilibrium equations that satisfies (see (4.6)–(4.8) and Remark A.2)

$$||\Lambda(\nabla \mathbf{u}_{\mathrm{e}}, \det \nabla \mathbf{u}_{\mathrm{e}})||_{L^\infty(\mathscr{B})} < \omega_{\mathrm{o}}/(2n-3), \qquad \Lambda(\mathbf{F}, \det \mathbf{F}) := \frac{\partial}{\partial \lambda}\Phi(\mathbf{F}, \lambda)\Big|_{\lambda = \det \mathbf{F}}, \tag{1.2}$$

then $\mathbf{u}_{\mathrm{e}}$ is the unique absolute minimizer of the elastic energy. Moreover, no other solution of the equilibrium equations can satisfy $(1.2)_1$. Furthermore, for the pure-displacement problem, Theorem 4.2 also shows that the same results are valid if $(1.2)_1$ is replaced by

$$||\Lambda(\nabla \mathbf{u}_{\mathrm{e}}, \det \nabla \mathbf{u}_{\mathrm{e}}) - \nu||_{L^\infty(\mathscr{B})} < \omega_{\mathrm{o}}/(2n-3),$$

where $\nu \in \mathbb{R}$ denotes an arbitrary constant.

For the displacement problem we obtain additional results. We first show, in Theorem 5.1, that a result of Zhang [48] is a simple consequence of our main theorem; we prove that if an equilibrium solution, $\mathbf{u}_{\mathrm{e}}$, is sufficiently close to a homogeneous deformation (in the Sobolev space $W^{1,\infty}$), then $\mathbf{u}_{\mathrm{e}}$ is the unique absolute minimizer of the energy and there are no other equilibrium solutions nearby.

We also consider, in Theorem 5.3, a uniqueness result of John [26], who proved that there is at most one equilibrium solution with (sufficiently) small strain: $\mathbf{E} := \frac{1}{2}[(\nabla \mathbf{u})^{\mathrm{T}}\nabla \mathbf{u} - \mathbf{I}]$. We use a recent result of Šilhavý [37], which produces a polyconvex representative that is invariant under rotations, to show that John's result is a direct consequence of our proof of the above-mentioned result of Zhang. We thus provide an elementary proof of a version[1] of the result in [26] that does not require the use and properties of BMO [27].

In §6 we present some examples of classical equilibrium solutions that satisfy the hypotheses of our theorems. In particular we construct two explicit examples in 2-dimensions, one for a mixed problem and one for a pure-traction problem, of homogeneous solutions that are each the unique absolute minimizer of the energy when the stored-energy density is compressible neo-Hookean. In §7 we briefly mention a recent alternative approach to the uniqueness of minimizers due to Gao, Neff, Roventa, and Thiel [16].

Most of the prior literature on uniqueness in finite elasticity considers the uniqueness of equilibrium solutions rather than energy minimizers. For example, results of Gurtin and Spector [23] imply that there is at most one solution of the equilibrium equations that lies in any convex set where the second variation of the energy is strictly positive. Knops and Stuart [28] (also see Bevan [8] and Taheri [45]) have shown that, for a star-shaped body, a homogeneous deformation is the unique smooth equilibrium solution that satisfies a homogeneous pure-displacement boundary condition whenever the energy is strictly quasiconvex at that deformation and globally rank-one convex.

Alternatively, there are a number of results that establish the nonuniqueness of equilibrium solutions for compressible materials.[2] For example, Post and Sivaloganathan [32] (verifying a

[1] Our hypotheses on the stored-energy differs from that in [26]. See Remark 5.4.

[2] For interesting examples of nonuniqueness for both compressible and incompressible materials see, e.g., [2, §9], [9, §5.8], [1, 7, 21, 22, 23, 32, 33], and the references therein.

conjecture of John [25, 26]) proved that there are (at least) a countably infinite number of equilibrium solutions for certain pure-displacement problems for an annulus. Antman [1] has shown that, for the pure-traction problem, a thick spherical shell without loads has a second equilibrium solution corresponding to an everted deformation. Simpson and Spector [38] have proven that, in addition to the homogeneous equilibrium solution, there are indeed two distinct buckled equilibrium solutions for certain 2-dimensional isotropic bars subject to uniaxial compression.

We mention that there is an interesting result of Spadaro [42] for the pure-displacement problem in 2-dimensions for constitutive relations of the form (1.1) with $n = 2$. Spadaro shows that there must be at least two absolute minimizers of the energy to a certain boundary-value problem when the body is a disk. However (as he notes), his construction is not compatible with finite elasticity, since it requires negative Jacobians.

Finally, we note that a possible interesting extension, which we have not addressed, is a uniqueness result for live loads.[3]

## 2. Preliminaries; The Nonlinear Problem

2.1. **Preliminaries.** We consider a body that for convenience we identify with the region $\overline{\mathscr{B}} \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, that it occupies in a fixed reference configuration. We assume that $\mathscr{B} \neq \varnothing$ is a connected, bounded, open set whose boundary, $\partial\mathscr{B}$, is Lipschitz[4] (see, e.g., [15]). A *deformation* of $\mathscr{B}$ is a mapping that lies in the space

$$\text{Def} := \{\mathbf{u} \in W^{1,1}(\mathscr{B};\mathbb{R}^n) : \det \nabla\mathbf{u} > 0 \ a.e.\},$$

where $\det \mathbf{F}$ denotes the determinant of $\mathbf{F} \in \text{Lin}_n$ (the space of linear maps from $\mathbb{R}^n$ into $\mathbb{R}^n$) and for $1 \le p \le \infty$, $W^{1,p}(\mathscr{B};\mathbb{R}^n)$ denotes the usual Sobolev space of (Lebesgue) measurable (vector-valued) functions $\mathbf{u} \in L^p(\mathscr{B};\mathbb{R}^n)$ whose distributional derivative, $\nabla\mathbf{u}$, is also contained in $L^p$. We write $\delta_{ij}$ for the Kronecker delta: thus,

$$\delta_{ij} := \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

We assume that the body is composed of a hyperelastic material whose *stored-energy density* $W : \overline{\mathscr{B}} \times \text{Lin}_n \to [0,\infty]$ with $\mathbf{x} \mapsto W(\mathbf{x},\mathbf{F})$ (Lebesgue) measurable[5] for every $\mathbf{F} \in \text{Lin}_n$. $W(\mathbf{x},\nabla\mathbf{u}(\mathbf{x}))$ gives the elastic energy stored at almost every point $\mathbf{x} \in \mathscr{B}$ of a deformation $\mathbf{u} \in \text{Def}$. We assume that the response of the material is *invariant under a change in observer* and hence that, for *a.e.* $\mathbf{x} \in \mathscr{B}$,

$$W(\mathbf{x},\mathbf{QF}) = W(\mathbf{x},\mathbf{F}) \quad \text{for every } \mathbf{F} \in \text{Lin}_n^{\succ} \text{ and } \mathbf{Q} \in \text{Orth}_n^{\succ}, \tag{2.1}$$

where $\text{Lin}_n^{\succ}$ denotes those $\mathbf{F} \in \text{Lin}_n$ with $\det \mathbf{F} > 0$ and $\text{Orth}_n^{\succ}$ denotes those $\mathbf{Q} \in \text{Lin}_n^{\succ}$ that satisfy $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$ ($\mathbf{I} \in \text{Lin}_n^{\succ}$ denotes the *identity*, i.e, $\mathbf{Ia} = \mathbf{a}$ for all $\mathbf{a} \in \mathbb{R}^n$).

[3] See, e.g., [31, 34, 43], [9, §2.7], or [36, §13.3].

[4] This assumption allows for a piecewise $C^1$ boundary, for example, a rectangle.

[5] In particular, the stored-energy density may therefore be piecewise continuous.

We further assume that, for *a.e.* $\mathbf{x} \in \mathscr{B}$,

$$\mathbf{F} \mapsto W(\mathbf{x}, \mathbf{F}) \in C(\mathrm{Lin}_n; [0, \infty]) \cap C^1(\mathrm{Lin}_n^{\succ}; \mathbb{R}^{\succeq}),$$
$$\lim_{|\mathbf{F}| \to \infty} W(\mathbf{x}, \mathbf{F}) = \lim_{\det \mathbf{F} \to 0^+} W(\mathbf{x}, \mathbf{F}) = +\infty,$$
$$W(\mathbf{x}, \mathbf{F}) = +\infty \quad \text{if and only if} \quad \det \mathbf{F} \leq 0,$$

where $\mathbb{R}^{\succeq} := [0, \infty)$. The (Piola-Kirchhoff) *stress* is then the derivative

$$\mathbf{S}(\mathbf{x}, \mathbf{F}) := \frac{\partial}{\partial \mathbf{F}} W(\mathbf{x}, \mathbf{F}) : \overline{\mathscr{B}} \times \mathrm{Lin}_n^{\succ} \to \mathrm{Lin}_n,$$

for *a.e.* $\mathbf{x} \in \mathscr{B}$. We call the body *homogeneous* if the stored-energy function $W$ is independent of $\mathbf{x}$. We call the reference configuration *stress free* if, for *a.e.* $\mathbf{x} \in \mathscr{B}$, $\mathbf{S}(\mathbf{x}, \mathbf{I}) = \mathbf{0}$.

2.2. **The Nonlinear Problem.** We assume the body is subject to dead loads. We take

$$\partial\mathscr{B} = \overline{\mathscr{D}} \cup \overline{\mathscr{S}} \quad \text{with } \mathscr{D} \text{ and } \mathscr{S} \text{ relatively open and } \mathscr{D} \cap \mathscr{S} = \varnothing.$$

If $\mathscr{D} \neq \varnothing$ we assume that a function $\mathbf{d} \in C(\overline{\mathscr{D}}; \mathbb{R}^n)$ is prescribed; $\mathbf{d}$ will give the deformation of $\mathscr{D}$. If $\mathscr{S} \neq \varnothing$ we assume that a function $\mathbf{s} \in L^1(\mathscr{S}; \mathbb{R}^n)$ is prescribed; for $\mathscr{H}^{n-1}$-*a.e.* $\mathbf{x} \in \mathscr{S}$, $\mathbf{s}(\mathbf{x})$ will give the surface force (per unit area, when $n = 3$, and per unit length, when $n = 2$) exerted on the body, at the point $\mathbf{x}$, by its environment. Finally, we suppose that a function $\mathbf{b} \in L^1(\mathscr{B}; \mathbb{R}^n)$ is prescribed; for *a.e.* $\mathbf{x} \in \mathscr{B}$, $\mathbf{b}(\mathbf{x})$ will give the body force (per unit volume, when $n = 3$, and per unit area, when $n = 2$) exerted on the body, at the point $\mathbf{x}$, by its environment. Here, and in the sequel, $\mathscr{H}^k$ denotes $k$-dimensional Hausdorff measure. The set of *admissible deformations* will be denoted by

$$\mathscr{A} := \{\mathbf{u} \in \mathrm{Def} \cap W^{1,n}(\mathscr{B}; \mathbb{R}^n) \cap C(\overline{\mathscr{B}}; \mathbb{R}^n) : \mathbf{u} = \mathbf{d} \text{ on } \mathscr{D}\}.$$

**Remark 2.1.** A result of Vodop′yanov and Gol′dšhteĭn [47] (see, also, [44, Theorem 4]) implies that each $\mathbf{u} \in W^{1,n}(\mathscr{B}; \mathbb{R}^n)$ with strictly positive Jacobian has a continuous representative. Thus, discontinuities such as cavitation (see, e.g., Ball [3]) are not allowed in this manuscript.

The *total energy* $E$ of a deformation $\mathbf{u} \in \mathscr{A}$ is defined by

$$E(\mathbf{u}) := \int_{\mathscr{B}} \Big[W(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\Big] \mathrm{d}\mathbf{x} - \int_{\mathscr{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, \mathrm{d}\mathscr{H}^{n-1}_{\mathbf{x}}. \tag{2.2}$$

Under suitable additional hypotheses on $W$ one might hope to show that any $\mathbf{u}$ that is a minimizer (local in an appropriate topology or global) of $E$ has first variation zero, i.e.,

$$0 = \int_{\mathscr{B}} \Big[\mathbf{S}(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})) : \nabla\mathbf{w}(\mathbf{x}) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x})\Big] \mathrm{d}\mathbf{x} - \int_{\mathscr{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x})\, \mathrm{d}\mathscr{H}^{n-1}_{\mathbf{x}} \tag{2.3}$$

for all *variations* $\mathbf{w} \in \mathrm{Var}$, where

$$\mathrm{Var} := \{\mathbf{w} \in W^{1,n}(\mathscr{B}; \mathbb{R}^n) \cap C(\overline{\mathscr{B}}; \mathbb{R}^n) : \mathbf{w} = \mathbf{0} \text{ on } \mathscr{D}\},$$

$\mathbf{F} : \mathbf{G} := \mathrm{tr}(\mathbf{F}\mathbf{G}^{\mathrm{T}})$, $\mathrm{tr}\, \mathbf{M}$ denotes the trace of $\mathbf{M} \in \mathrm{Lin}_n$, and $\mathbf{M}^{\mathrm{T}}$ denotes its transpose.

Moreover, one would then want to show that $\mathbf{u}$ is a *classical solution of the equations of equilibrium*, that is,[6] $\mathbf{u} \in C^2(\mathscr{B}; \mathbb{R}^n) \cap C^1(\overline{\mathscr{B}}; \mathbb{R}^n) \cap \mathscr{A}$ satisfies

$$\mathrm{Div}\, \mathbf{S}(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})) + \mathbf{b}(\mathbf{x}) = \mathbf{0} \quad \text{for } \mathbf{x} \in \mathscr{B} \tag{2.4}$$

[6]If $\mathscr{S} = \varnothing$, then $\mathbf{u} \in C^2(\mathscr{B}; \mathbb{R}^n) \cap \mathscr{A}$ suffices.

and the *traction boundary conditions*

$$\mathbf{S}(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x}))\mathbf{n}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathscr{S}. \tag{2.5}$$

Unfortunately, such results[7] have not been obtained for arbitrary minimizers. In general, *in this manuscript we will therefore assume that one or more solutions of* (2.3) *are given.*

**Remark 2.2.** There are a number of well-known classical equilibrium solutions that are of interest. Among these are:

(1) Homogeneous solutions (see Remarks 2.7 and 4.6, Proposition 4.5, and §6.2);
(2) Solutions obtained using the implicit function theorem (see Remark 5.2); and
(3) Radial solutions when the body in its reference configurations is an annulus or a thick spherical shell (see §6.1).

**Definition 2.3.** We say that $\mathbf{u}_\mathrm{e} \in \mathscr{A}$ is a *weak equilibrium solution* if $E(\mathbf{u}_\mathrm{e}) < +\infty$,

$$\mathbf{x} \mapsto \mathbf{S}(\mathbf{x}, \nabla\mathbf{u}_\mathrm{e}(\mathbf{x})) \in L^{n'}(\mathscr{B}; \mathrm{Lin}_n), \qquad n' := \frac{n}{n-1}, \tag{2.6}$$

$\mathbf{u}_\mathrm{e}$ satisfies (2.3) for all $\mathbf{w} \in \mathrm{Var}$, and if $\mathscr{D} = \varnothing$,

$$\int_{\mathscr{B}} [\mathbf{u}_\mathrm{e}(\mathbf{x}) - \mathbf{x}]\mathrm{d}\mathbf{x} = \mathbf{0}. \tag{2.7}$$

If $\mathscr{D} = \partial\mathscr{B}$ we will call $\mathbf{u}_\mathrm{e}$ a weak solution of the *(pure) displacement problem.* If $\mathscr{S} = \partial\mathscr{B}$ we will call $\mathbf{u}_\mathrm{e}$ a weak solution of the *(pure) traction problem.* Otherwise, we will refer to such a $\mathbf{u}_\mathrm{e}$ as a weak solution of the *(genuine) mixed problem.*

**Remark 2.4.** When $\mathscr{S} = \partial\mathscr{B}$ any translation of a weak equilibrium solution $\mathbf{u}_\mathrm{e}$ will satisfy both (2.3) and (2.6). Equation (2.7) eliminates this nonuniqueness.

**Remark 2.5.** Our assumption that $\mathbf{S} \in L^{n'}$ is, in general, more stringent than expected for an absolute minimizer $\mathbf{u} \in W^{1,n}$ of $E$. However, it is necessitated by (2.3) which requires $\mathbf{S} : \nabla\mathbf{w}$ to be integrable for $\mathbf{w} \in \mathrm{Var} \subset W^{1,n}$. As will become evident, our conditions for uniqueness, e.g., (4.6) and (4.7), may sometimes require a weak equilibrium solution $\mathbf{u}_\mathrm{e}$ to satisfy $\mathbf{u}_\mathrm{e} \in W^{1,\infty}(\mathscr{B}; \mathbb{R}^n)$. See Remark 4.4.

**Remark 2.6.** Any classical solution of (2.4) and (2.5) is also a weak equilibrium solution.

**Remark 2.7.** Let the body be homogeneous and $\mathscr{D} \neq \varnothing$. Fix $\mathbf{F}_\mathrm{e} \in \mathrm{Lin}_n^{\succ}$, $\mathbf{a} \in \mathbb{R}^n$, and define $\mathbf{u}_\mathrm{e}(\mathbf{x}) := \mathbf{F}_\mathrm{e}\mathbf{x} + \mathbf{a}$ for all $\mathbf{x} \in \overline{\mathscr{B}}$. Then $\mathbf{u}_\mathrm{e}(\mathbf{x}) = \mathbf{d}(\mathbf{x}) := \mathbf{F}_\mathrm{e}\mathbf{x} + \mathbf{a}$ for all $\mathbf{x} \in \mathscr{D}$. If $\mathscr{S} \neq \varnothing$ assume, in addition, that $\mathbf{s}(\mathbf{x}) := \mathbf{S}(\mathbf{F}_\mathrm{e})\mathbf{n}(\mathbf{x})$ for $\mathscr{H}^{n-1}$-*a.e.* $\mathbf{x} \in \mathscr{S}$, where, for such $\mathbf{x}$, $\mathbf{n}(\mathbf{x})$ denotes the outward unit normal to the boundary. Then $\mathbf{u}_\mathrm{e}$ is an admissible deformation that satisfies both the equilibrium equations (2.4) (with $\mathbf{b} \equiv \mathbf{0}$) and the traction boundary conditions (2.5); thus $\mathbf{u}_\mathrm{e}$ is a classical equilibrium solution.

Although it is not known if an arbitrary minimizer of the energy is a solution of the equilibrium equations, it will be if the mapping happens to satisfy certain additional conditions. In order to illustrate this we first formally define what we mean by a local minimizer.

[7] In general, one can only prove that a minimizer is a weak solution of alternative forms of the equilibrium equations. See [4, Theorem 2.4] and the references therein. However, Lemma 2.9 shows that additional hypotheses may imply that a minimizer is in fact a weak equilibrium solution.

**Definition 2.8.** Let $\mathbf{u}_\mathrm{m} \in \mathscr{A}$. We say that $\mathbf{u}_\mathrm{m}$ is a *weak relative minimizer*[8] of the energy $E$ provided that there exists a $\delta > 0$ such that

$$E(\mathbf{u}_\mathrm{m}) \leq E(\mathbf{u}_\mathrm{m} + \mathbf{w})$$

for all variations $\mathbf{w} \in \mathrm{Var} \cap W^{1,\infty}(\mathscr{B};\mathbb{R}^n)$ that satisfy $||\mathbf{w}||_{L^\infty(\mathscr{B})} + ||\nabla \mathbf{w}||_{L^\infty(\mathscr{B})} < \delta$.

The next lemma then illustrates a circumstance where such a $\mathbf{u}_\mathrm{m}$ does indeed satisfy the equilibrium equations. The proof follows from the mean-value theorem together with the bounded convergence theorem (see, e.g., Ball [4, §2.4] or [12, §3.4.2]).

**Lemma 2.9.** *Let* $\mathbf{u}_\mathrm{m} \in \mathscr{A} \cap W^{1,\infty}(\mathscr{B};\mathbb{R}^n)$ *be a weak relative minimizer of* $E$ *that satisfies*

$$\det \nabla \mathbf{u} > \epsilon \quad a.e. \tag{2.8}$$

*for some* $\epsilon > 0$. *Suppose, in addition, that either*

$$\mathbf{S} \in C(\overline{\mathscr{B}} \times \mathrm{Lin}^\succ) \quad \text{or} \quad \mathbf{S} \text{ is bounded on compact subsets of } \overline{\mathscr{B}} \times \mathrm{Lin}^\succ. \tag{2.9}$$

*Then* $\mathbf{u}_\mathrm{m}$ *is a weak equilibrium solution.*

## 3. Uniform Polyconvexity

Let $n = 2$ or $n = 3$. Define

$$\mathscr{E}_2 := \mathrm{Lin}_2, \qquad \mathscr{E}_3 := \mathrm{Lin}_3 \times \mathrm{Lin}_3,$$

$$\mathscr{E}_2^\succ := \mathrm{Lin}_2^\succ, \quad \mathscr{E}_3^\succ := \mathrm{Lin}_3^\succ \times \mathrm{Lin}_3^\succ.$$

We assume that the stored-energy density is *uniformly polyconvex,*[9] that is, there is a constant $p \geq n$ and functions $\omega : \mathscr{B} \to \mathbb{R}^\succ$ and $\Phi^{(n)} : \mathscr{B} \times \mathscr{E}_n \times \mathbb{R}^\succ \to \mathbb{R}$, $\mathbb{R}^\succ := (0, \infty)$, that satisfy, for all $\mathbf{F} \in \mathrm{Lin}^\succ$ and *a.e.* $\mathbf{x} \in \mathscr{B}$,

$$W(\mathbf{x}, \mathbf{F}) = \frac{\omega(\mathbf{x})}{p}|\mathbf{F}|^p + \begin{cases} \Phi^{(2)}(\mathbf{x}, \mathbf{F}, \det \mathbf{F}), & \text{if } n = 2, \\ \Phi^{(3)}(\mathbf{x}, \mathbf{F}, \mathrm{cof}\, \mathbf{F}, \det \mathbf{F}), & \text{if } n = 3, \end{cases} \tag{3.1}$$

where $|\mathbf{F}| := \sqrt{\mathbf{F} : \mathbf{F}}$,

1. $\omega \in L^\infty(\mathscr{B})$ satisfies $\omega \geq \omega_\mathrm{o}$ for some constant $\omega_\mathrm{o} > 0$;
2. $\mathbf{x} \mapsto \Phi^{(n)}(\mathbf{x}, \mathbf{M}, \lambda)$ is measurable for every $\mathbf{M} \in \mathscr{E}_n$ and $\lambda > 0$; and, for *a.e.* $\mathbf{x} \in \mathscr{B}$,
3. $(\mathbf{M}, \lambda) \mapsto \Phi^{(n)}(\mathbf{x}, \mathbf{M}, \lambda)$ is convex on its domain and differentiable on $\mathscr{E}_n^\succ \times \mathbb{R}^\succ$.

Moreover, if the body is *homogeneous* we assume that *both* $\omega \equiv \omega_\mathrm{o}$ *and* $\Phi^{(n)}$ *are independent of* $\mathbf{x}$. Here, and in the sequel, $\mathrm{cof}\, \mathbf{F} \in \mathrm{Lin}_n^\succ$ denotes the tensor of cofactors of $\mathbf{F} \in \mathrm{Lin}_n^\succ$; thus,

$$\mathrm{cof}\, \mathbf{F} = (\det \mathbf{F})\mathbf{F}^{-\mathrm{T}} \quad \text{for all } \mathbf{F} \in \mathrm{Lin}_n^\succ.$$

In general, invariance under a change in observer, (2.1), *does **not** imply* that the function $\Phi^{(n)}$ must satisfy[10]

$$\Phi^{(n)}(\mathbf{x}, \mathbf{QM}, \lambda) = \Phi^{(n)}(\mathbf{x}, \mathbf{M}, \lambda) \quad \text{for every } \mathbf{M} \in \mathscr{E}_n^\succ,\ \mathbf{Q} \in \mathrm{Orth}_n^\succ,\ \lambda \in \mathbb{R}^\succ, \tag{3.2}$$

[8] See, e.g., Del Piero and Rizzoni [13] and the references therein for results concerning weak relative minimizers in Elasticity. See Grabovsky and Mengesha [17, 18] for results concerning the relationship between such minimizers and strong relative minimizers, although not for Elasticity.

[9] This terminology for (3.1) has previously been used in [40].

[10] For $\mathbf{K} \in \mathrm{Lin}$ and $\mathbf{M} = (\mathbf{F}, \mathbf{A}) \in \mathscr{E}_3$ we write $\mathbf{KM} := (\mathbf{KF}, \mathbf{KA})$.

and *a.e.* $\mathbf{x} \in \mathscr{B}$. In fact, $\Phi^{(n)}$ *is not uniquely determined* by (3.1) (see, e.g., [12, p. 158]). A particular choice of $\Phi^{(n)}$ may satisfy (3.2), while another does not. However, a recent result of Šilhavý [37] identifies a particular $\Phi^{(n)}$ that satisfies (3.2). In §5.2 we will have occasion to require that this $\Phi^{(n)}$ be used in (3.1).

Let $n = 2$. For such stored-energy functions the Piola-Kirchhoff stress is given by

$$\mathbf{S}(\mathbf{x}, \mathbf{F}) = \omega(\mathbf{x})|\mathbf{F}|^{p-2}\mathbf{F} + \mathbf{B}(\mathbf{x}, \mathbf{F}) + \Lambda(\mathbf{x}, \mathbf{F}) \operatorname{cof} \mathbf{F}$$

for *a.e.* $\mathbf{x} \in \mathscr{B}$ and every $\mathbf{F} \in \mathrm{Lin}_2^{\succ}$, where

$$\mathbf{B}(\mathbf{x}, \mathbf{F}) := \frac{\partial \Phi^{(2)}(\mathbf{x}, \mathbf{F}, \lambda)}{\partial \mathbf{F}}\bigg|_{\lambda = \det \mathbf{F}}, \qquad \Lambda(\mathbf{x}, \mathbf{F}) := \frac{\partial \Phi^{(2)}(\mathbf{x}, \mathbf{F}, \lambda)}{\partial \lambda}\bigg|_{\lambda = \det \mathbf{F}}. \tag{3.3}$$

Let $n = 3$. For stored-energy functions that satisfy $(3.1)_2$ it follows that the Piola-Kirchhoff stress satisfies, for *a.e.* $\mathbf{x} \in \mathscr{B}$, every $\mathbf{F} \in \mathrm{Lin}_3^{\succ}$, and every $\mathbf{H} \in \mathrm{Lin}_3$,

$$\mathbf{S}(\mathbf{x}, \mathbf{F}) : \mathbf{H} = \mathbf{H} : \Big[\omega(\mathbf{x})|\mathbf{F}|^{p-2}\mathbf{F} + \mathbf{B}(\mathbf{x}, \mathbf{F}) + \Lambda(\mathbf{x}, \mathbf{F}) \operatorname{cof} \mathbf{F}\Big] + \mathbf{D}(\mathbf{x}, \mathbf{F}) : \mathbb{K}(\mathbf{F})[\mathbf{H}], \tag{3.4}$$

where

$$\begin{aligned} \mathbf{B}(\mathbf{x}, \mathbf{F}) &:= \frac{\partial \Phi^{(3)}(\mathbf{x}, \mathbf{F}, \mathbf{A}, \lambda)}{\partial \mathbf{F}}\bigg|_{\substack{\mathbf{A} = \operatorname{cof} \mathbf{F} \\ \lambda = \det \mathbf{F}}}, \qquad \mathbf{D}(\mathbf{x}, \mathbf{F}) := \frac{\partial \Phi^{(3)}(\mathbf{x}, \mathbf{F}, \mathbf{A}, \det \mathbf{F})}{\partial \mathbf{A}}\bigg|_{\mathbf{A} = \operatorname{cof} \mathbf{F}}, \\ \Lambda(\mathbf{x}, \mathbf{F}) &:= \frac{\partial \Phi^{(3)}(\mathbf{x}, \mathbf{F}, \operatorname{cof} \mathbf{F}, \lambda)}{\partial \lambda}\bigg|_{\lambda = \det \mathbf{F}}, \quad \mathbb{K}(\mathbf{F})[\mathbf{H}] := \frac{\mathrm{d}(\operatorname{cof} \mathbf{F})}{\mathrm{d}\mathbf{F}}[\mathbf{H}]. \end{aligned} \tag{3.5}$$

## 4. Uniqueness of Minimizers

4.1. **Equilibrium Solutions.** In this subsection we consider the displacement, traction, and mixed problems and obtain a uniqueness result that is valid for all of them. For the pure-displacement problem $\mathscr{D} = \partial\mathscr{B}$, $\mathrm{Var} = W_0^{1,n}(\mathscr{B}; \mathbb{R}^n) \cap C(\overline{\mathscr{B}}; \mathbb{R}^n)$, and we have the following identities (see, e.g., [2] or [30, pp. 28–31]), for all $\mathbf{z} \in W^{1,n}(\mathscr{B}; \mathbb{R}^n)$ and $\mathbf{w} \in W_0^{1,n}(\mathscr{B}; \mathbb{R}^n)$,

$$\int_{\mathscr{B}} \nabla \mathbf{z} : \operatorname{cof} \nabla \mathbf{w} \, \mathrm{d}\mathbf{x} = 0, \qquad \int_{\mathscr{B}} \det \nabla \mathbf{w} \, \mathrm{d}\mathbf{x} = 0. \tag{4.1}$$

For the mixed and traction problems $(4.1)_1$ is satisfied by all $\mathbf{w} \in \mathrm{Var}$ and $\mathbf{z} \in \mathrm{Trac_o}$, where[11]

$$\mathrm{Trac_o} := \begin{cases} \{\mathbf{z} \in W^{1,n}(\mathscr{B}; \mathbb{R}^n) : \ \mathbf{z} = \mathbf{0} \text{ on } \mathscr{S}\} & \text{if } \mathscr{S} \neq \varnothing \\ W^{1,n}(\mathscr{B}; \mathbb{R}^n) & \text{if } \mathscr{D} = \partial\mathscr{B}, \end{cases}$$

i.e., those mappings that are equal to zero on the portion of the boundary where dead-load tractions are prescribed.

**Lemma 4.1.** *Assume that $W$ is uniformly polyconvex. Let $\mathbf{u}_\mathrm{e}$ be a weak equilibrium solution. Then, for any $\mathbf{v} \in \mathscr{A}$, $\mathbf{z} \in \mathrm{Trac_o}$, $\sigma \in L^\infty(\mathscr{B}; [0, 1])$, and $\nu \in \mathbb{R}$ ($\nu = 0$ if $\mathscr{D} \neq \partial\mathscr{B}$)*

$$\begin{aligned} E(\mathbf{v}) \geq & E(\mathbf{u}_\mathrm{e}) + \int_{\mathscr{B}} \omega(\mathbf{x}) \Big( \frac{1}{2}[1 - \sigma(\mathbf{x})] |\nabla \mathbf{u}_\mathrm{e}(\mathbf{x})|^{p-2} |\nabla \mathbf{w}(\mathbf{x})|^2 + \frac{\kappa_p}{p} \sigma(\mathbf{x}) |\nabla \mathbf{w}(\mathbf{x})|^p \Big) \mathrm{d}\mathbf{x} \\ & + \int_{\mathscr{B}} \Big[\Lambda(\mathbf{x}, \nabla \mathbf{u}_\mathrm{e}(\mathbf{x})) - \nu\Big] \det \nabla \mathbf{w}(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \delta_{n3} \int_{\mathscr{B}} \mathbf{X}_{\mathrm{e},\mathbf{z}}(\mathbf{x}) : \operatorname{cof} \nabla \mathbf{w}(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \end{aligned} \tag{4.2}$$

[11]The equality on the boundary is to be taken in the sense of trace.

*where* $\mathbf{w} := \mathbf{v} - \mathbf{u}_{\rm e}$*,* $\kappa_p > 0$ *is given by Proposition A.1, and*

$$\mathbf{X}_{\rm e,\mathbf{z}}(\mathbf{x}) := \mathbf{D}\big(\mathbf{x}, \nabla\mathbf{u}_{\rm e}(\mathbf{x})\big) + \mathit{\Lambda}\big(\mathbf{x}, \nabla\mathbf{u}_{\rm e}(\mathbf{x})\big)\nabla\mathbf{u}_{\rm e}(\mathbf{x}) - \nabla\mathbf{z}(\mathbf{x}). \tag{4.3}$$

*Proof.* We prove the result when $n = 3$. The proof for $n = 2$ is similar. Suppose that $W$ is uniformly polyconvex. Let $\mathbf{F}, \mathbf{G} \in \mathrm{Lin}_3^{\succ}$ and define $\mathbf{H} := \mathbf{G} - \mathbf{F}$. For clarity of exposition we suppress the $\mathbf{x}$ in our calculation. We note that the convexity of $\mathit{\Phi}^{(3)}$ yields (for *a.e.* $\mathbf{x} \in \mathscr{B}$)

$$\widehat{\mathit{\Phi}}^{(3)}(\mathbf{G}) \geq \widehat{\mathit{\Phi}}^{(3)}(\mathbf{F}) + \mathbf{B}(\mathbf{F}) : \mathbf{H} + \mathbf{D}(\mathbf{F}) : \big[\,\mathrm{cof}\,\mathbf{G} - \mathrm{cof}\,\mathbf{F}\big] + \mathit{\Lambda}(\mathbf{F})\big[\det\mathbf{G} - \det\mathbf{F}\big], \tag{4.4}$$

where $\widehat{\mathit{\Phi}}^{(3)}(\mathbf{G}) := \mathit{\Phi}^{(3)}(\mathbf{G}, \mathrm{cof}\,\mathbf{G}, \det\mathbf{G})$. If we now multiply (A.1), with $\mathbf{a} = \mathbf{G}$ and $\mathbf{b} = \mathbf{F}$, by $\omega/p$ and add the result to (4.4) we find, with the aid of $(3.1)_2$, (3.4), (B.2), and (B.3), that, for any $\sigma \in [0, 1]$,

$$\begin{aligned} W(\mathbf{G}) \geq W(\mathbf{F}) + \mathbf{S}(\mathbf{F}) : \mathbf{H} + \omega\Big(\frac{1}{2}[1-\sigma]|\mathbf{F}|^{p-2}|\mathbf{H}|^2 + \frac{\kappa_p}{p}\sigma|\mathbf{H}|^p\Big) \\ + \mathbf{D}(\mathbf{F}) : \mathrm{cof}\,\mathbf{H} + \mathit{\Lambda}(\mathbf{F})\big(\det\mathbf{H} + \mathbf{F} : \mathrm{cof}\,\mathbf{H}\big). \end{aligned} \tag{4.5}$$

Next, let $\mathbf{u}_{\rm e}$ be a weak equilibrium solution, $\mathbf{v} \in \mathscr{A}$, and define $\mathbf{w} := \mathbf{v} - \mathbf{u}_{\rm e}$. Suppose that $\sigma \in L^\infty(\mathscr{B}; [0, 1])$. If we now take $\mathbf{G} = \nabla\mathbf{v}(\mathbf{x})$, $\mathbf{F} = \nabla\mathbf{u}_{\rm e}(\mathbf{x})$, and $\mathbf{H} = \nabla\mathbf{w}(\mathbf{x})$ in (4.5) and then integrate the result over $\mathscr{B}$ and subtract $(4.1)_1$, we conclude, with the aid of (2.2), (2.3), and (4.3), that (4.2) is satisfied with $\nu = 0$. Finally, if $\mathscr{D} = \partial\mathscr{B}$ then (4.2) follows upon subtracting $\nu$ times $(4.1)_2$ from (4.2) with $\nu = 0$. □

We now make use of Lemma 4.1 in order to establish the uniqueness of an energy minimizer subject to certain constraints. We first recall that a mapping $\mathbf{u}_{\rm e} \in W^{1,n}(\mathscr{B}; \mathbb{R}^n) \cap C(\overline{\mathscr{B}}; \mathbb{R}^n)$ is a *weak equilibrium solution* if $\det\nabla\mathbf{u}_{\rm e} > 0$ *a.e.*, $\mathbf{u}_{\rm e}$ is a weak solution of the equations of equilibrium (2.4) (see (2.3) on p. 5), and $\mathbf{u}_{\rm e}$ satisfies $\mathbf{u}_{\rm e} = \mathbf{d}$ on $\mathscr{D}$.

**Theorem 4.2** (Uniqueness of Energy Minimizers)**.** *Assume that* $W$ *is uniformly polyconvex. Let* $\mathbf{u}_{\rm e}$ *be a weak equilibrium solution.*

(a) *If* $n = 2$ *suppose that* $\mathbf{u}_{\rm e}$ *satisfies, for some* $\nu \in \mathbb{R}$ *(*$\nu = 0$ *if* $\mathscr{D} \neq \partial\mathscr{B}$*) and a.e.* $\mathbf{x} \in \mathscr{B}$*,*

$$\big|\mathit{\Lambda}\big(\mathbf{x}, \nabla\mathbf{u}_{\rm e}(\mathbf{x})\big) - \nu\big| \leq \omega(\mathbf{x})\big|\nabla\mathbf{u}_{\rm e}(\mathbf{x})\big|^{p-2}. \tag{4.6}$$

(b) *If* $n = 3$ *suppose that* $\mathbf{u}_{\rm e}$ *satisfies, for some* $\mathbf{z} \in \mathrm{Trac}_{\rm o}$*,* $\nu \in \mathbb{R}$ *(*$\nu = 0$ *if* $\mathscr{D} \neq \partial\mathscr{B}$*) and a.e.* $\mathbf{x} \in \mathscr{B}$*,*

$$2\Gamma_{\rm e,\mathbf{z}}(\mathbf{x}) + \beta_p\big|\mathit{\Lambda}\big(\mathbf{x}, \nabla\mathbf{u}_{\rm e}(\mathbf{x})\big) - \nu\big||\nabla\mathbf{u}_{\rm e}(\mathbf{x})| \leq \omega(\mathbf{x})|\nabla\mathbf{u}_{\rm e}(\mathbf{x})|^{p-2}. \tag{4.7}$$

*Then* $\mathbf{u}_{\rm e}$ *is an absolute minimizer of* $E$*. Moreover, if, in addition,* (4.6) *or* (4.7) *is a strict inequality on a set of positive measure, then* $\mathbf{u}_{\rm e}$ *is the unique absolute minimizer of* $E$*. Further, there are no other weak equilibrium solutions that satisfy* (4.6) *or* (4.7) *with strict inequality on a set of positive measure. Here* $\Gamma_{\rm e,\mathbf{z}}$ *is the largest principal stretch of* $\mathbf{X}_{\rm e,\mathbf{z}}$ *given by* (4.3)*,* $\kappa_p$ *is given by Proposition A.1, and*

$$\beta_p := \frac{2}{3^{3/2}}\left[\frac{p}{2\kappa_p}\right]^{1/(p-2)}. \tag{4.8}$$

*Proof.* We will prove the result for the pure-displacement problem. The result for the mixed and traction problems will follow from the same calculations with $\nu = 0$. We first show that $E(\mathbf{v}) \geq E(\mathbf{u}_{\rm e})$ for all $\mathbf{v} \in \mathscr{A}$. Fix $\mathbf{z} \in \mathrm{Trac}_{\rm o}$ and $\nu \in \mathbb{R}$. We first note that, in view of Lemma 4.1, it suffices to show that there exists a measurable function $\sigma : \mathscr{B} \to [0,1]$ such that, for every $\mathbf{H} \in \mathrm{Lin}_n$ and *a.e.* $\mathbf{x} \in \mathscr{B}$,

$$\omega|\mathbf{H}|^2\left[\frac{(1-\sigma)}{2}|\nabla\mathbf{u}_{\mathrm{e}}|^{p-2} + \frac{\kappa_p\sigma}{p}|\mathbf{H}|^{p-2}\right] + (\varLambda - \nu)\det\mathbf{H} + \delta_{n3}\Big(\mathbf{X}_{\mathrm{e},\mathbf{z}} : \mathrm{cof}\,\mathbf{H}\Big) \geq 0. \tag{4.9}$$

If $n = 2$ then (4.9) follows from (4.6), Hadamard's inequality:

$$2|\det\mathbf{H}| \leq |\mathbf{H}|^2,$$

and the choice $\sigma \equiv 0$. If $n = p = 3$ then (4.9) follows from (4.7), (4.8), Hadamard's inequality (see, e.g., [41, p. 408]):

$$3^{3/2}|\det\mathbf{H}| \leq |\mathbf{H}|^3,$$

the cofactor inequality (B.5), and the choice

$$\sigma(\mathbf{x}) = \frac{|\varLambda(\mathbf{x}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x})) - \nu|}{\omega(\mathbf{x})\kappa_3\sqrt{3}}.$$

Now assume that $n = 3$ and $p > 3$. Then Hadamard's inequality and (B.5) reduce (4.9) to showing that there exists a measurable function $\sigma : \mathscr{B} \to [0,1]$ such that, for every $\mathbf{H} \in \mathrm{Lin}_3$ and *a.e.* $\mathbf{x} \in \mathscr{B}$,

$$\omega\left[\frac{(1-\sigma)}{2}|\nabla\mathbf{u}_{\mathrm{e}}|^{p-2} + \frac{\kappa_p\sigma}{p}|\mathbf{H}|^{p-2}\right] - \frac{|\varLambda-\nu|}{3^{3/2}}|\mathbf{H}| - \Gamma_{\mathrm{e},\mathbf{z}} \geq 0, \tag{4.10}$$

where $\Gamma_{\mathrm{e},\mathbf{z}} \geq 0$ is the largest principal stretch of $\mathbf{X}_{\mathrm{e},\mathbf{z}}$ given by (4.3).

Before we determine $\sigma$ such that (4.7) implies (4.10), we first consider the implications of $\sigma(\mathbf{x}_{\mathrm{o}}) = 0$ at some $\mathbf{x}_{\mathrm{o}} \in \mathscr{B}$. We note that (4.10) with $\sigma(\mathbf{x}_{\mathrm{o}}) = 0$ is satisfied for every $\mathbf{H} \in \mathrm{Lin}_3$ if and only if

$$|\varLambda(\mathbf{x}_{\mathrm{o}}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x}_{\mathrm{o}})) - \nu| = 0 \qquad \text{and} \qquad 2\Gamma_{\mathrm{e},\mathbf{z}}(\mathbf{x}_{\mathrm{o}}) \leq \omega(\mathbf{x}_{\mathrm{o}})|\nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x}_{\mathrm{o}})|^{p-2}.$$

We therefore conclude that:

(i) If $|\varLambda(\mathbf{x}_{\mathrm{o}}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x}_{\mathrm{o}})) - \nu| = 0$ at some $\mathbf{x}_{\mathrm{o}} \in \mathscr{B}$, then (4.7) yields (4.10) with $\sigma(\mathbf{x}_{\mathrm{o}}) = 0$.
(ii) If $|\varLambda(\mathbf{x}_{\mathrm{o}}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x}_{\mathrm{o}})) - \nu| \neq 0$ for some $\mathbf{x}_{\mathrm{o}} \in \mathscr{B}$, then $\sigma(\mathbf{x}_{\mathrm{o}}) \neq 0$.

Next, since $\omega(\mathbf{x}) \geq \omega_{\mathrm{o}} > 0$ and $\det\nabla\mathbf{u}_{\mathrm{e}} > 0$ *a.e.*, we can fix $\mathbf{x}_{\mathrm{o}} \in \mathscr{B}$ and assume that

$$\omega(\mathbf{x}_{\mathrm{o}})|\nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x}_{\mathrm{o}})| > 0, \qquad |\varLambda(\mathbf{x}_{\mathrm{o}}, \nabla\mathbf{u}_{\mathrm{e}}(\mathbf{x}_{\mathrm{o}})) - \nu| > 0, \quad \text{and (hence)} \quad \sigma(\mathbf{x}_{\mathrm{o}}) \in (0,1]. \tag{4.11}$$

Define $t := |\mathbf{H}| \geq 0$. Then (4.10) can be viewed as

$$f(t) := at^{p-2} - bt + c \geq 0 \quad \text{for all } t \geq 0, \tag{4.12}$$

where, in view of (4.7), $c = f(0) \geq 0$, $b \geq 0$, and $a > 0$ (since $\sigma > 0$). A necessary and sufficient condition for (4.12) to be satisfied is that $f$ be nonnegative at the unique $t_{\mathrm{m}}$ that satisfies $f'(t_{\mathrm{m}}) = 0$, i.e., $a(p-2)t_{\mathrm{m}}^{p-3} = b$. If we substitute $t_{\mathrm{m}}$ into (4.12) we find that (4.10) is a consequence of

$$f(t_{\mathrm{m}}) = c - b\left(\frac{p-3}{p-2}\right)\left[\frac{b}{a(p-2)}\right]^{1/(p-3)} \geq 0$$

or, equivalently,

$$2\Gamma_{\mathrm{e},\mathbf{z}} + \delta_p\left[\frac{|\varLambda-\nu|^{p-2}}{\omega\sigma}\right]^{1/(p-3)} \leq \omega|\nabla\mathbf{u}_{\mathrm{e}}|^{p-2}(1-\sigma), \tag{4.13}$$

where

$$\delta_p := \frac{2(p-3)}{\big[3^{3/2}(p-2)\big]^{(p-2)/(p-3)}}\left[\frac{p}{\kappa_p}\right]^{1/(p-3)}. \tag{4.14}$$

Next, define

$$\sigma := \frac{\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}-2\Gamma_{\rm e,\mathbf{z}}}{(p-2)\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}} > 0, \qquad 1-\sigma = \frac{2\Gamma_{\rm e,\mathbf{z}}+(p-3)\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}}{(p-2)\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}} > 0,$$

where $\sigma\in(0,1)$ follow from $\Gamma_{\rm e,\mathbf{z}}\geq 0$, (4.7), and (4.11). With this choice of $\sigma$ inequality (4.13) becomes

$$2\Gamma_{\rm e,\mathbf{z}} + \delta_p\left[\frac{(p-2)|\nabla\mathbf{u}_{\rm e}|^{p-2}|\mathit{\Lambda}-\nu|^{p-2}}{\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}-2\Gamma_{\rm e,\mathbf{z}}}\right]^{1/(p-3)} \leq \frac{2\Gamma_{\rm e,\mathbf{z}}+(p-3)\omega|\nabla\mathbf{u}_{\rm e}|^{p-2}}{(p-2)},$$

which, after some algebra, reduces to

$$\left[\delta_p\frac{(p-2)}{(p-3)}\right]^{(p-3)/(p-2)}(p-2)^{1/(p-2)}|\nabla\mathbf{u}_{\rm e}||\mathit{\Lambda}-\nu| \leq \omega|\nabla\mathbf{u}_{\rm e}|^{p-2}-2\Gamma_{\rm e,\mathbf{z}}. \tag{4.15}$$

However, (4.8) and (4.14) yield

$$(p-2)^{1/(p-2)}\left[\delta_p\frac{(p-2)}{(p-3)}\right]^{(p-3)/(p-2)} = \beta_p$$

which shows that (4.15) and (4.7) are identical.

We next note that it is clear that if (4.6) or (4.7) is a strict inequality on a set of positive measure then (4.2) and the above proof yield $E(\mathbf{v})>E(\mathbf{u}_{\rm e})$ unless $\nabla\mathbf{v}=\nabla\mathbf{u}_{\rm e}$ *a.e.* Since $\mathscr{B}$ is open and connected it follows that $\mathbf{v}=\mathbf{u}_{\rm e}+\mathbf{a}$ *a.e.* for some $\mathbf{a}\in\mathbb{R}^n$. If $\mathscr{D}\neq\varnothing$, then $\mathbf{u}_{\rm e}=\mathbf{v}=\mathbf{d}$ on the nonempty, relatively open set $\mathscr{D}$, while if $\mathscr{D}=\varnothing$, then (see (2.7))

$$\int_{\mathscr{B}}(\mathbf{u}_{\rm e}-\mathbf{v})\,{\rm d}\mathbf{x}=\mathbf{0}.$$

In either case $\mathbf{a}=\mathbf{0}$.

Finally, if we suppose that $\mathbf{v}_{\rm e}\not\equiv\mathbf{u}_{\rm e}$ is a weak equilibrium solution that satisfies (4.6) or (4.7) with strict inequality on a set of positive measure (and with $\mathbf{u}_{\rm e}$ replaced by $\mathbf{v}_{\rm e}$), then the above argument yields $E(\mathbf{u}_{\rm e})>E(\mathbf{v}_{\rm e})$, which is a contradiction. □

**Remark 4.3.** Suppose we replace the assumption that $(\mathbf{M},\lambda)\mapsto\mathit{\Phi}^{(n)}(\mathbf{x},\mathbf{M},\lambda)$ is (globally) convex with the weaker assumption that, for *a.e.* $\mathbf{x}\in\mathscr{B}$,

$$(\mathbf{M}_{\rm e}(\mathbf{x}),\det\nabla\mathbf{u}_{\rm e}(\mathbf{x}))\ \textit{is a point of convexity of}\ (\mathbf{M},\lambda)\mapsto\mathit{\Phi}^{(n)}(\mathbf{x},\mathbf{M},\lambda), \tag{4.16}$$

where $\mathbf{M}_{\rm e}(\mathbf{x})=\nabla\mathbf{u}_{\rm e}(\mathbf{x})$ if $n=2$ or $\mathbf{M}_{\rm e}(\mathbf{x})=(\nabla\mathbf{u}_{\rm e}(\mathbf{x}),\operatorname{cof}\nabla\mathbf{u}_{\rm e}(\mathbf{x}))$ if $n=3$. Then it is clear from the proof of Theorem 4.2 that any weak equilibrium solution $\mathbf{u}_{\rm e}$ that satisfies (4.16) and either (4.6) or (4.7) is an absolute minimizer of $E$. Theorem 4.2 therefore has implications for stored-energies that admit phase transitions (see Ball and James [5] or, e.g., Grabovsky and Truskinovsky [19] and the references therein); while one would not expect uniqueness of minimizers for such constitutive relations, Theorem 4.2 yields conditions under which an equilibrium solution is an absolute minimizer of the energy.

**Remark 4.4.** (a) Suppose that $p = n$ and, for *a.e.* $\mathbf{x}$,

$$|\Lambda(\mathbf{x}, \mathbf{F})| \geq \varphi(\mathbf{F}), \quad \text{where} \quad \varphi(\mathbf{F}) \to \infty \quad \text{as} \quad |\mathbf{F}| \to \infty. \tag{4.17}$$

Then any admissible deformation $\mathbf{u} \in \mathscr{A}$ that satisfies (4.6) or (4.7) must have additional regularity, i.e., $\mathbf{u} \in W^{1,\infty}(\mathscr{B}; \mathbb{R}^n)$.

(b) Suppose that $p \geq n$ and, for *a.e.* $\mathbf{x}$,

$$|\Lambda(\mathbf{x}, \mathbf{F})| \geq \varphi(\mathbf{F}), \quad \text{where} \quad \varphi(\mathbf{F}) \to \infty \quad \text{as} \quad \det \mathbf{F} \to 0^+. \tag{4.18}$$

Then any $\mathbf{u} \in \mathscr{A}$ that satisfies (4.6) or (4.7) will satisfy, for some $\epsilon > 0$,

$$\det \nabla \mathbf{u}(\mathbf{x}) > \epsilon \quad \text{for } a.e.\ \mathbf{x} \in \mathscr{B}.$$

The next result, which is an immediate consequence of Remark 2.7 and Theorem 4.2, shows that, for the genuine mixed problem on a homogeneous body with $\Phi^{(n)}$ continuously differentiable, the set of homogeneous deformations that satisfies (4.7) with strict inequality and $\mathbf{z} \equiv \mathbf{0}$ or (4.6) with strict inequality is an open subset of $\mathrm{Lin}_n^{\succ}$.

**Proposition 4.5.** *Let both $\mathscr{D}$ and $\mathscr{S}$ be nonempty. Assume that the body is homogeneous and that $W$ is uniformly polyconvex with $\Phi^{(n)}$ continuously differentiable.*

(a) *If $n = 2$ suppose that $\mathbf{F}_\mathrm{e} \in \mathrm{Lin}_2^{\succ}$ satisfies*

$$|\Lambda(\mathbf{F}_\mathrm{e})| < \omega_\mathrm{o} |\mathbf{F}_\mathrm{e}|^{p-2}. \tag{4.19}$$

(b) *If $n = 3$ suppose that $\mathbf{F}_\mathrm{e} \in \mathrm{Lin}_3^{\succ}$ satisfies*

$$2\Gamma_\mathrm{e} + \beta_p |\Lambda(\mathbf{F}_\mathrm{e})||\mathbf{F}_\mathrm{e}| < \omega_\mathrm{o} |\mathbf{F}_\mathrm{e}|^{p-2} \tag{4.20}$$

*where $\Gamma_\mathrm{e}$ is the largest principal stretch of $\mathbf{X}_\mathrm{e} := \mathbf{D}(\mathbf{F}_\mathrm{e}) + \Lambda(\mathbf{F}_\mathrm{e})\mathbf{F}_\mathrm{e}$.*

*Suppose further that $\mathbf{b} \equiv \mathbf{0}$, $\mathbf{d}$ is given by $\mathbf{d}(\mathbf{x}) := \mathbf{F}_\mathrm{e}\mathbf{x} + \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^n$ and all $\mathbf{x} \in \mathscr{D}$, and $\mathbf{s}$ is given by $\mathbf{s}(\mathbf{x}) := \mathbf{S}(\mathbf{F}_\mathrm{e})\mathbf{n}(\mathbf{x})$ for $\mathbf{x} \in \mathscr{S}$, where $\mathbf{n}(\mathbf{x})$ denotes the outward unit normal to the boundary (for $\mathscr{H}^{n-1}$-a.e. $\mathbf{x} \in \mathscr{S}$). Then the weak equilibrium solution $\mathbf{u}_\mathrm{e}(\mathbf{x}) := \mathbf{F}_\mathrm{e}\mathbf{x} + \mathbf{a}$ is the unique absolute minimizer of the energy $E$. Moreover, there does not exist a weak equilibrium solution $\mathbf{v}_\mathrm{e} \not\equiv \mathbf{u}_\mathrm{e}$ that satisfies (4.19) or (4.20) with $\mathbf{F}_\mathrm{e}$ replaced by $\nabla \mathbf{v}_\mathrm{e}(\mathbf{x})$.*

**Remark 4.6.** Proposition 4.5 can be viewed as a simple analogue, for the mixed problem, of the well-known result of Knops and Stuart [28]. Consider a homogeneous star-shaped body $\mathcal{T}$. Fix $\mathbf{F}_\mathrm{e} \in \mathrm{Lin}_n^{\succ}$ and consider the pure-displacement problem: $\mathscr{S} = \varnothing$ with $\mathrm{d}(\mathbf{x}) := \mathbf{F}_\mathrm{e}\mathbf{x}$ for $\mathbf{x} \in \partial\mathcal{T}$. Then results in [28] show that, in the absence of body forces, there is at most one deformation $\mathbf{u}_\mathrm{e} \in C^2(\mathcal{T}; \mathbb{R}^n) \cap C^1(\overline{\mathcal{T}}; \mathbb{R}^n) \cap \mathscr{A}$ that satisfies the equilibrium equations (2.4) *provided* that the stored-energy function $W$ is strictly quasiconvex at $\mathbf{F}_\mathrm{e}$ and (globally) rank-one convex. Similarly, Bevan [8] has shown that there is at most one weak relative minimizer $\mathbf{u}_\mathrm{m} \in C^1(\overline{\mathcal{T}}; \mathbb{R}^n) \cap \mathscr{A}$ for certain stored-energy functions $W(\mathbf{F}) = \Psi(\mathbf{F}) + h(\det \mathbf{F})$ with $h$ convex, $\Psi$ strictly quasiconvex, and both functions having appropriate growth.

4.2. **Weak Relative Minimizers.** We now assume that we are given an admissible deformation $\mathbf{u}_\mathrm{m}$ that is a weak relative minimizer (see Definition 2.8) rather than an equilibrium solution. Then the additional hypotheses of Lemma 2.9 allow us to conclude that $\mathbf{u}_\mathrm{m}$ is a weak equilibrium solution and so we can then apply the results from the previous subsection. The next result follows directly from Lemma 2.9 and Theorem 4.2.

**Proposition 4.7.** *Let* $\mathbf{u}_{\mathrm{m}} \in \mathscr{A} \cap W^{1,\infty}(\mathscr{B};\mathbb{R}^n)$ *be a weak relative minimizer of* $E$ *that satisfies* (2.8) *and either* (4.6) *or* (4.7) *with strict inequality on a set of positive measure. Suppose, in addition, that* $\mathbf{S}$ *satisfies* (2.9). *Then* $\mathbf{u}_{\mathrm{m}}$ *is a weak equilibrium solution that is the unique absolute minimizer of* $E$.

Remark 4.4 together with the above proposition then gives us the following result.

**Corollary 4.8.** *Assume that* $p = n$. *Let* $\mathbf{u}_{\mathrm{m}} \in \mathscr{A}$ *be a weak relative minimizer of* $E$ *that satisfies* (4.6) *or* (4.7) *with strict inequality on a set of positive measure. Suppose, in addition, that* $\mathbf{S}$ *satisfies* (2.9) *and that* $\Lambda$ *satisfies* (4.17) *and* (4.18). *Then* $\mathbf{u}_{\mathrm{m}}$ *is a weak equilibrium solution that is the unique absolute minimizer of* $E$.

## 5. Further results for the Displacement Problem: a Theorem of Zhang and a Theorem of John

5.1. **A Theorem of Zhang.** We now present a result of Zhang [48] who showed that, in 3-dimensions for the pure-displacement problem, there is at most one equilibrium solution $\mathbf{u}_{\mathrm{e}}$ that is uniformly close, in $W^{1,\infty}$, to a given homogeneous deformation and, moreover, that $\mathbf{u}_{\mathrm{e}}$ must then be the minimizer of the energy obtained by both the direct method of the calculus of variations and the implicit function theorem.

**Theorem 5.1** (Zhang [48])**.** *Suppose* $\mathscr{D} = \partial\mathscr{B}$ *and that* $W$ *is homogeneous and uniformly polyconvex.*[12] *Fix* $\mathbf{F}_{\mathrm{o}} \in \mathrm{Lin}^{\succ}$. *Assume that* $(\mathbf{M},\lambda) \mapsto \Phi^{(n)}(\mathbf{M},\lambda)$ *is continuously differentiable at* $(\mathbf{F}_{\mathrm{o}}, \det \mathbf{F}_{\mathrm{o}})$ *if* $n = 2$ *or* $(\mathbf{F}_{\mathrm{o}}, \operatorname{cof}\mathbf{F}_{\mathrm{o}}, \det \mathbf{F}_{\mathrm{o}})$ *if* $n = 3$. *Then there exists a* $\delta = \delta(\mathbf{F}_{\mathrm{o}}) > 0$ *such that any weak equilibrium solution* $\mathbf{u}_{\mathrm{e}}$ *that satisfies*

$$\left\|\nabla\mathbf{u}_{\mathrm{e}} - \mathbf{F}_{\mathrm{o}}\right\|_{L^\infty(\mathscr{B})} < \delta \tag{5.1}$$

*is a strict absolute minimizer of* $E$. *Consequently, there is at most one weak equilibrium solution that satisfies* (5.1).

**Remark 5.2.** The main difficulty is showing that there are *any* weak equilibrium solutions, especially solutions that satisfy (5.1). However, in this instance and with suitable additional assumptions one can make use of the implicit function theorem to get classical solutions of the equations of equilibrium (2.4) (see Zhang [48]). In particular, if one assumes that the boundary is sufficiently smooth, and one replaces $\mathbf{b}$ with $\epsilon\mathbf{b}$ and the boundary condition $\mathbf{u} = \mathbf{d}$ with

$$\mathbf{u}(\mathbf{x}) = \mathbf{F}_{\mathrm{o}}\mathbf{x} + \epsilon\mathbf{d}(\mathbf{x}) \quad \text{for } \mathbf{x} \in \partial\mathscr{B},$$

where $\mathbf{b}$ and $\mathbf{d}$ are sufficiently smooth, then results of Valent [46] (see, also, [9, Chapter 6] or [36, §20.9]) yield the existence of a classical solution of the equilibrium equations for small $\epsilon$.

*Proof of Theorem 5.1.* We prove the result when $n = 3$. The proof for $n = 2$ is similar. For clarity of exposition, we suppress the variable $\mathbf{x}$ as well as the "almost every $\mathbf{x}$" that should accompany most of our inequalities. Fix $\mathbf{F}_{\mathrm{o}} \in \mathrm{Lin}^{\succ}$ and assume the hypotheses of the theorem. We will show that, if $\delta < \frac{1}{2}|\mathbf{F}_{\mathrm{o}}|$ in (5.1), then the right-hand side of (4.7) is bounded away from

[12] Zhang [48] instead assumes that $W(\mathbf{F}) = a|\mathbf{F}|^p + b|\operatorname{cof}\mathbf{F}|^q + \Phi(\mathbf{F}, \operatorname{cof}\mathbf{F}, \det\mathbf{F})$ with $a > 0$, $b > 0$, $p \geq 2$, $q \geq p/(p-1)$, and $\Phi$ convex.

zero, while the left-hand side of (4.7) goes to zero as $\delta$ approaches zero. This will allow us to apply Theorem 4.2 to obtain the desired uniqueness.

Define
$$\varepsilon := \frac{\omega_{\mathrm{o}}|\mathbf{F}_{\mathrm{o}}|^{p-2}}{2^{p-1}\big[(2+\beta_p)|\mathbf{F}_{\mathrm{o}}|+1\big]}, \tag{5.2}$$
where $\beta_p$ is given by (4.8). Then in view of the continuity of $\mathbf{D}$ and $\Lambda$ at $\mathbf{F}_{\mathrm{o}}$ there exists a $\delta = \delta(\mathbf{F}_{\mathrm{o}}, \varepsilon) > 0$ such that, for $\mathbf{G} \in \mathrm{Lin}_3^{\succ}$,
$$|\mathbf{G} - \mathbf{F}_{\mathrm{o}}| < \delta \quad \Longrightarrow \quad |\Lambda(\mathbf{G}) - \Lambda(\mathbf{F}_{\mathrm{o}})| < \varepsilon, \quad |\mathbf{D}(\mathbf{G}) - \mathbf{D}(\mathbf{F}_{\mathrm{o}})| < \varepsilon. \tag{5.3}$$
Without loss of generality, we assume that
$$2\delta < |\mathbf{F}_{\mathrm{o}}|. \tag{5.4}$$

Let $\mathbf{u}_{\mathrm{e}}$ be a weak equilibrium solution that satisfies (5.1) with $\delta$ given in (5.3)–(5.4). We note that by the triangle inequality, (5.1), and (5.4)
$$\begin{aligned} 2|\mathbf{F}_{\mathrm{o}}| &\le 2|\mathbf{F}_{\mathrm{o}} - \nabla\mathbf{u}_{\mathrm{e}}| + 2|\nabla\mathbf{u}_{\mathrm{e}}| \le |\mathbf{F}_{\mathrm{o}}| + 2|\nabla\mathbf{u}_{\mathrm{e}}|, \\ 2|\nabla\mathbf{u}_{\mathrm{e}}| &\le 2|\nabla\mathbf{u}_{\mathrm{e}} - \mathbf{F}_{\mathrm{o}}| + 2|\mathbf{F}_{\mathrm{o}}| \le 3|\mathbf{F}_{\mathrm{o}}|, \end{aligned} \tag{5.5}$$
and hence, in particular,
$$\omega_{\mathrm{o}}|\mathbf{F}_{\mathrm{o}}|^{p-2} \le \omega_{\mathrm{o}} 2^{p-2} |\nabla\mathbf{u}_{\mathrm{e}}|^{p-2}. \tag{5.6}$$
Next, by the triangle inequality, (5.1), (5.3) with $\mathbf{G} = \nabla\mathbf{u}_{\mathrm{e}}$, and (5.5),
$$\begin{aligned} &\Big|\big[\mathbf{D}(\nabla\mathbf{u}_{\mathrm{e}}) + \Lambda(\nabla\mathbf{u}_{\mathrm{e}})\nabla\mathbf{u}_{\mathrm{e}}\big] - \big[\mathbf{D}(\mathbf{F}_{\mathrm{o}}) + \Lambda(\mathbf{F}_{\mathrm{o}})\nabla\mathbf{u}_{\mathrm{e}}\big]\Big| \\ &\le |\mathbf{D}(\nabla\mathbf{u}_{\mathrm{e}}) - \mathbf{D}(\mathbf{F}_{\mathrm{o}})| + |[\Lambda(\nabla\mathbf{u}_{\mathrm{e}}) - \Lambda(\mathbf{F}_{\mathrm{o}})]\nabla\mathbf{u}_{\mathrm{e}}| < \varepsilon(1 + 2|\mathbf{F}_{\mathrm{o}}|). \end{aligned} \tag{5.7}$$

Finally, define
$$\nu := \Lambda(\mathbf{F}_{\mathrm{o}}), \qquad \mathbf{z}(\mathbf{x}) := \mathbf{D}(\mathbf{F}_{\mathrm{o}})\mathbf{x} + \Lambda(\mathbf{F}_{\mathrm{o}})\mathbf{u}_{\mathrm{e}}(\mathbf{x}) \tag{5.8}$$
and note that $(5.7)_1$ is the norm of $\mathbf{X}_{\mathrm{e},\mathbf{z}}$ given by (4.3). Therefore, (5.1), (5.3) with $\mathbf{G} = \nabla\mathbf{u}_{\mathrm{e}}$, (5.5), (5.7), and (5.8), together with (B.5) ($\Gamma_{\mathrm{e},\mathbf{z}} \le |\mathbf{X}_{\mathrm{e},\mathbf{z}}|$), yield
$$2\Gamma_{\mathrm{e},\mathbf{z}} + \beta_p|\Lambda(\nabla\mathbf{u}_{\mathrm{e}}) - \nu||\nabla\mathbf{u}_{\mathrm{e}}| < 2\varepsilon\big[(2+\beta_p)|\mathbf{F}_{\mathrm{o}}| + 1\big]. \tag{5.9}$$
The desired uniqueness now follows (5.2), (5.6), (5.9), and Theorem 4.2. □

5.2. **A Theorem of John.** We next show that our results also imply a result of John [26] who showed that, in 3-dimensions, there is at most one solution of the pure-displacement problem with small *strain* $\mathbf{E} := \frac{1}{2}[(\nabla\mathbf{u})^{\mathrm{T}}\nabla\mathbf{u} - \mathbf{I}]$.

We first recall that Šilhavý [37] identifies a particular $\Phi^{(n)}$ that satisfies (see (3.2))
$$\Phi^{(n)}(\mathbf{QM}, \lambda) = \Phi^{(n)}(\mathbf{M}, \lambda) \quad \text{for every } \mathbf{M} \in \mathscr{E}_n^{\succ},\ \mathbf{Q} \in \mathrm{Orth}_n^{\succ},\ \lambda \in \mathbb{R}^{\succ}, \tag{5.10}$$
when $W$ is homogeneous. Suppose that this $\Phi^{(n)}$ is used in (3.1). It follows from (5.10) that the derivatives of $\Phi^{(n)}$ (see $(3.3)_2$ and $(3.5)_{3,2}$) satisfy, for every $\mathbf{F} \in \mathrm{Lin}_n^{\succ}$ and $\mathbf{Q} \in \mathrm{Orth}_n^{\succ}$,
$$\Lambda(\mathbf{QF}) = \Lambda(\mathbf{F}), \qquad \mathbf{D}(\mathbf{QF}) = \mathbf{QD}(\mathbf{F}).$$
Standard representation theorems (see, e.g., [9, Theorems 3.3-1 and 4.2-1] or [20, §25, §27] and [36, Theorem 8.3.3]) then yield functions[13] $\Lambda^* : \mathrm{Psym}_n \to \mathbb{R}$ and $\mathbf{D}^* : \mathrm{Psym}_3 \to \mathrm{Sym}_3$ that

[13] We write $\mathrm{Sym}_n$ for those $\mathbf{H} \in \mathrm{Lin}_n$ that satisfy $\mathbf{H} = \mathbf{H}^{\mathrm{T}}$; $\mathrm{Psym}_n$ denotes those $\mathbf{H} \in \mathrm{Sym}_n$ that are strictly positive definite.

satisfy, for every $\mathbf{F} \in \mathrm{Lin}_n^{\succ}$,

$$\Lambda(\mathbf{F}) = \Lambda^*(\mathbf{F}^{\mathrm{T}}\mathbf{F}), \qquad \mathbf{D}(\mathbf{F}) = \mathbf{F}\mathbf{D}^*(\mathbf{F}^{\mathrm{T}}\mathbf{F}). \tag{5.11}$$

John's theorem then follows from (5.11) and our proof of Zhang's theorem.

**Theorem 5.3** (John [26])**.** *Let $\mathscr{D} = \partial\mathscr{B}$ and let $W$ be homogeneous and uniformly polyconvex. Suppose that $\Lambda$ satisfies $(5.11)_1$ with $\Lambda^*$ continuous at $\mathbf{I}$. If $n = 3$ suppose, in addition, that $\mathbf{D}$ satisfies $(5.11)_2$ and that $\mathbf{D}^*$ is continuous at $\mathbf{I}$ with $\mathbf{D}^*(\mathbf{I}) = \xi\mathbf{I}$ for some $\xi \in \mathbb{R}$. Then there exists a $\delta > 0$ such that there is at most one weak equilibrium solution $\mathbf{u}_{\mathrm{e}}$ that satisfies*

$$\left\|(\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}} - \mathbf{I}\right\|_{L^\infty(\mathscr{B})} < \delta. \tag{5.12}$$

*Moreover, if such a $\mathbf{u}_{\mathrm{e}}$ exists it is a strict absolute minimizer of $E$.*

**Remark 5.4.** The corresponding theorem in [26] does not assume polyconvexity.[14] Instead, John assumes that the stored-energy function $W \in C^3(\mathrm{Lin}_3;\mathbb{R})$ satisfies[15]

$$W(\mathbf{F}) = \mu|\mathbf{E}|^2 + \tfrac{1}{2}\lambda(\mathrm{tr}\,\mathbf{E})^2 + O(|\mathbf{E}|^3), \tag{5.13}$$

where $\mu > 0$ and $\lambda > 0$ denote the Lamé moduli and $\mathbf{E} := \frac{1}{2}[\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}]$. The proof in [26] makes use of the properties of BMO developed by John and Nirenberg [27] rather than the elementary techniques used herein.

**Remark 5.5.** Let $n = 3$ and suppose that $W$ is isotropic. Then results of Šilhavý [37] yield a $\Phi^{(3)}$ that is isotropic. It follows that the corresponding $\mathbf{D}$ is isotropic and hence, by the representation theorem for isotropic tensor-valued functions, $\mathbf{D}(\mathbf{F}) = \phi_1(\mathbf{B})\mathbf{I} + \phi_2(\mathbf{B})\mathbf{B} + \phi_3(\mathbf{B})\mathbf{B}^2$, where $\mathbf{B} := \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\phi_i : \mathrm{Psym}_3 \to \mathbb{R}$. Thus, $\mathbf{D}^*(\mathbf{I}) = \mathbf{D}(\mathbf{I}) = \xi\mathbf{I}$ follows from isotropy. Whether or not $W$ is isotropic, one can show that the assumption $\mathbf{S}(\mathbf{I}) = \mathbf{0}$, which is implicit in (5.13), yields $\mathbf{B}(\mathbf{I}) - \mathbf{D}(\mathbf{I}) = \eta\mathbf{I}$ for some $\eta \in \mathbb{R}$.

*Proof of Theorem 5.3.* We prove the result when $n = 3$. The proof for $n = 2$ is similar. Assume the hypotheses of the theorem. We will show that, if $\delta < \frac{1}{2}$ in (5.12), then the right-hand side of (4.7) is bounded away from zero, while the left-hand side of (4.7) goes to zero as $\delta$ approaches zero. This will allow us to once again apply Theorem 4.2 to obtain the desired uniqueness.

Define

$$\varepsilon := \frac{\omega_{\mathrm{o}}}{2(4 + \beta_p)}, \tag{5.14}$$

where $\beta_p$ is given by (4.8). Then, in view of the continuity of $\Lambda^*$ and $\mathbf{D}^*$ at $\mathbf{I}$ and the fact that $\mathbf{D}^*(\mathbf{I}) = \xi\mathbf{I}$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for $\mathbf{C} \in \mathrm{Psym}_3$,

$$|\mathbf{C} - \mathbf{I}| < \delta \quad \Longrightarrow \quad |\Lambda^*(\mathbf{C}) - \Lambda^*(\mathbf{I})| < \varepsilon, \quad |\mathbf{D}^*(\mathbf{C}) - \xi\mathbf{I}| < \varepsilon. \tag{5.15}$$

Without loss of generality, we assume that

$$2\delta < 1. \tag{5.16}$$

[14]The proof in [26, Eqns. (8)–(11)] is also not compatible with $W(\mathbf{F}) = +\infty$ when $\det\mathbf{F} = 0$.

[15]More generally, it is not difficult to show that the results in [26] are valid for any $W \in C^3(\mathrm{Lin}_n;\mathbb{R})$ with $\mathbf{S}(\mathbf{I}) = \mathbf{0}$ and $\mathbb{C}(\mathbf{I})$ strongly elliptic, where $\mathbb{C}(\mathbf{F}) := \partial\mathbf{S}/\partial\mathbf{F}$ here denotes the elasticity tensor. In particular, for a $W$ that satisfies (5.13), $\mu > 0$ and $\mu + \lambda > 0$ suffice. See, e.g., [9, Theorem 4.10.2] for examples of polyconvex energies that are consistent with (5.13).

Let $\mathbf{u}_e$ be a weak equilibrium solution that satisfies (5.12) with $\delta$ given in (5.15)–(5.16). We note that, by the triangle inequality, (5.12), (5.16), and the Cauchy-Schwarz inequality,

$$\begin{aligned} 2\sqrt{3} = 2|\mathbf{I}| &\leq 2|\mathbf{I} - (\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e| + 2|(\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e| < 1 + 2|\nabla\mathbf{u}_e|^2, \\ |\nabla\mathbf{u}_e|^2 = \mathbf{I} : \big[(\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e - \mathbf{I} + \mathbf{I}\big] &\leq \sqrt{3}\,|(\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e - \mathbf{I}| + 3 < 4, \end{aligned} \tag{5.17}$$

and hence, in particular,

$$\omega_o < \omega_o|\nabla\mathbf{u}_e|^{p-2}. \tag{5.18}$$

Next, by the triangle inequality, (5.12), (5.15) with $\mathbf{C} = (\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e$, and (5.17),

$$\begin{aligned} &\Big|\big[\nabla\mathbf{u}_e\mathbf{D}^*((\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e) + \Lambda^*((\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e)\nabla\mathbf{u}_e\big] - \big[\xi\nabla\mathbf{u}_e + \Lambda^*(\mathbf{I})\nabla\mathbf{u}_e\big]\Big| \\ &\qquad \leq \Big|\nabla\mathbf{u}_e\big[\mathbf{D}^*((\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e) - \xi\mathbf{I}\big]\Big| + \Big|\big[\Lambda^*((\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e) - \Lambda^*(\mathbf{I})\big]\nabla\mathbf{u}_e\Big| < 4\varepsilon. \end{aligned} \tag{5.19}$$

Finally, define

$$\nu := \Lambda^*(\mathbf{I}), \qquad \mathbf{z}(\mathbf{x}) := \big[\xi + \Lambda^*(\mathbf{I})\big]\mathbf{u}_e(\mathbf{x}) \tag{5.20}$$

and note that, in view of (5.11) and the fact that $\mathbf{D}^*(\mathbf{I}) = \xi\mathbf{I}$, $(5.19)_1$ is the norm of $\mathbf{X}_{e,\mathbf{z}}$ given by (4.3). Then (5.11), (5.12), (5.15) with $\mathbf{C} = (\nabla\mathbf{u}_e)^{\mathrm{T}}\nabla\mathbf{u}_e$, (5.17), (5.19), and (5.20), together with (B.5) ($\Gamma_{e,\mathbf{z}} \leq |\mathbf{X}_{e,\mathbf{z}}|$), yield

$$2\Gamma_{e,\mathbf{z}} + \beta_p|\Lambda(\nabla\mathbf{u}_e) - \nu||\nabla\mathbf{u}_e| < 2\varepsilon(4 + \beta_p). \tag{5.21}$$

The desired uniqueness now follows (5.14), (5.18), (5.21), and Theorem 4.2. $\square$

## 6. Examples

6.1. **An Application of Theorem 5.1 to an Annular Region.** Fix $b > a > 0$ and let $\mathscr{B} := \{\mathbf{x} \in \mathbb{R}^n : a < |\mathbf{x}| < b\}$, where $n = 2$ or $n = 3$. Suppose that the deformation $\mathbf{u}$ is prescribed on both the inner and outer boundary, i.e., $\mathbf{u}(\mathbf{x}) = \nu\mathbf{x}$ when $|\mathbf{x}| = a$ and $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ when $|\mathbf{x}| = b$, where $0 < \nu a < \lambda b$. Then for a large class of homogeneous, isotropic, stored-energy functions it has been shown (see [3, 39]) that there is a unique minimizer of the energy among all *radial deformations*, i.e., deformations that satisfy

$$\mathbf{u}_r(\mathbf{x}, \nu, \lambda) = \frac{r(R, \nu, \lambda)}{R}\mathbf{x}, \quad R := |\mathbf{x}|$$

for some absolutely continuous, strictly increasing function $R \mapsto r(R, \nu, \lambda)$. Moreover, it turns out that $\mathbf{u}_r$ is a classical solution of the equilibrium equations. In addition, the continuous dependence of a solution (and its derivative) of a second-order ordinary differential equation on its initial data can be used to show that

$$\mathbf{u}_r(\nu, \lambda, \mathbf{x}) \to \nu\mathbf{x} \quad \text{uniformly in } C^1(\overline{\mathscr{B}}; \mathbb{R}^2) \text{ as } \lambda \to \nu.$$

*Therefore such solutions satisfy* (5.1) with $\mathbf{F}_o = \nu\mathbf{I}$. Consequently, Theorem 5.1 implies that when the radial minimizer of the energy, $\mathbf{u}_r$, is sufficiently close (in $C^1$) to $\nu\mathbf{x}$, it is the unique absolute minimizer of the energy among all admissible deformations, not just the radial ones.

For compressible neo-Hookean materials (see (6.1)) in 2-dimensions, a more general result has been obtained by Iwaniec and Onninen [24] (see also [41]) who proved that the radial minimizer, $\mathbf{u}_r$, is the unique absolute minimizer of the energy among all admissible deformations, whether or not $\mathbf{u}_r$ is close to a homogeneous deformation. However, results of Post and Sivaloganathan [32], which verified a long standing conjecture of John [25, 26], demonstrated that, for compressible neo-Hookean materials in 2-dimensions, there are (at least) a countable number of distinct, nonradial, classical solutions of the equilibrium equations. The solutions in [32, §4] are of the form

$$\mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\begin{bmatrix}\cos(\Theta(R)) & -\sin(\Theta(R))\\ \sin(\Theta(R)) & \cos(\Theta(R))\end{bmatrix}\mathbf{x}, \quad R := |\mathbf{x}|,$$

where $r(a) = \nu a$, $r(b) = \lambda b$, $\Theta(a) = 0$, $\Theta(b) = 2n\pi$ *for any nonzero integer* $n$ ($n = 0$ yields the radial solution). An extension of this result to all even dimensions can be found in [35].

6.2. **Examples in 2-Dimensions.** Let $n = 2$. We will herein restrict our attention to *compressible neo-Hookean materials*, i.e., constitutive relations of the form:

$$W(\mathbf{F}) = \frac{\omega_o}{2}|\mathbf{F}|^2 + h(\det\mathbf{F}), \qquad \mathbf{S}(\mathbf{F}) = \omega_o\mathbf{F} + h'(\det\mathbf{F})\operatorname{cof}\mathbf{F}, \tag{6.1}$$

where $\omega_o > 0$ denotes a constant and $h \in C^2(\mathbb{R}^{\succ};\mathbb{R}^{\succeq})$ satisfies

$$h'(1) = -\omega_o, \qquad h''(1) > 0, \qquad h'' \geq 0, \qquad \lim_{t\to 0^+} h(t) = +\infty. \tag{6.2}$$

We will use Proposition 4.5 to construct two explicit examples, one for a mixed problem and one for a pure-traction problem, of homogeneous equilibrium solutions that are each the unique absolute minimizer of the energy when the stored-energy density is given by (6.1) and (6.2).

Let the body, in its reference configuration, occupy the rectangle $\mathscr{B} = (-R, R) \times (-L, L)$ for some $R > 0$ and $L > 0$. Fix $\lambda > 1$ and consider homogeneous deformations of the form

$$\mathbf{u}_{\mathrm{h}}(x, y) = \begin{bmatrix}\alpha x\\ \lambda y\end{bmatrix}, \tag{6.3}$$

where $\alpha > 0$. Consider the functional

$$\mathcal{E}(\alpha) = \frac{\omega_o}{2}(\alpha^2 + \lambda^2) + h(\alpha\lambda). \tag{6.4}$$

Our hypotheses on $h$ imply that $\mathcal{E} \in C(\mathbb{R}^{\succ};\mathbb{R}^{\succ})$ blows up at 0 and $+\infty$ and hence that $\mathcal{E}$ achieves its infimum at some $\alpha = \alpha(\lambda) > 0$. Since

$$\mathcal{E}'(\alpha) = \omega_o\alpha + \lambda h'(\alpha\lambda) = 0, \qquad \mathcal{E}''(\alpha) = \omega_o + \lambda^2 h''(\alpha\lambda) > 0, \tag{6.5}$$

we conclude from the strict convexity of $\mathcal{E}$ that $\alpha = \alpha(\lambda)$ is unique and satisfies $(6.5)_1$ (which is the condition that the sides of the rectangle $x = \pm R$ are free of tractions). We now have the following result.

**Proposition 6.1.** *Let* $\lambda > 1$, $\mathscr{B} = (-R, R) \times (-L, L) \subset \mathbb{R}^2$, *and*

$$E(\mathbf{u}) := \int_{-L}^{L}\int_{-R}^{R} W(\nabla\mathbf{u}(x, y))\,\mathrm{d}x\,\mathrm{d}y,$$

*where* $W$ *is given by* (6.1)–(6.2). *Then the unique absolute minimizer of* $E$ *among deformations* $\mathbf{u} \in \mathscr{A}$ *with*

$$\mathbf{d}(x, -L) := \begin{bmatrix}\alpha x\\ -\lambda L\end{bmatrix}, \qquad \mathbf{d}(x, L) := \begin{bmatrix}\alpha x\\ \lambda L\end{bmatrix}, \quad -R < x < R, \tag{6.6}$$

*is the homogeneous deformation* $\mathbf{u}_\mathrm{h}$ *given by* (6.3) *with* $\alpha = \alpha(\lambda)$ *given by* $(6.5)_1$. *Moreover, there are no other weak equilibrium solutions* $\mathbf{v}_\mathrm{e}$ *that satisfy*

$$|h'(\det \nabla \mathbf{v}_\mathrm{e})| < \omega_\mathrm{o} \ a.e.$$

**Remark 6.2.** The existence theory of Ball and Murat [6] can be applied to (6.1)–(6.2). Thus, their minimizer is $\mathbf{u}_\mathrm{h}$.

*Proof of Proposition 6.1.* It is clear that $\mathbf{u}_\mathrm{h} \in \mathscr{A}$ is a classical equilibrium solution that satisfies the displacement boundary condition $\mathbf{u}_\mathrm{h} = \mathbf{d}$, where $\mathbf{d}$ is given by (6.6). Therefore, in view of Proposition 4.5, $(6.2)_1$, and $(6.5)_1$ all we need to show is that

$$\alpha(\lambda) < \lambda \ \text{ for all } \ \lambda > 1. \tag{6.7}$$

If we differentiate $(6.5)_1$ with respect to $\lambda$ we find, with the aid of (6.1)–(6.2), that

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\lambda} = -\frac{h'(\alpha\lambda) + \alpha\lambda h''(\alpha\lambda)}{\omega_\mathrm{o} + \lambda^2 h''(\alpha\lambda)}. \tag{6.8}$$

Consequently,

$$\theta(\lambda) := \lambda^2 \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\alpha}{\lambda}\right] = -\frac{2\alpha\lambda h''(\alpha\lambda)}{\omega_\mathrm{o} + \lambda^2 h''(\alpha\lambda)} \le 0, \tag{6.9}$$

where we have made use of (6.1)–(6.2) and $(6.5)_1$. Moreover, when $\lambda = 1$ it follows that $\alpha = 1$ and so, in view of (6.1)–(6.2) and (6.9), $\theta(1) < 0$, which together with (6.9) yields (6.7). ☐

Next, instead of the displacement boundary condition on the top and bottom of the rectangle we apply tractions. For the deformation $\mathbf{u}_\mathrm{h}$ (with $\alpha = \alpha(\lambda)$) we will specify (cf. (6.4))

$$\mathcal{S}(\lambda) = \omega_\mathrm{o}\lambda + \alpha h'(\lambda\alpha(\lambda)) > 0. \tag{6.10}$$

We note that $\mathcal{S}(1) = \omega_\mathrm{o} + h'(1) = 0$. Then we differentiate (6.10) with respect to $\lambda$ and make use of (6.8), $(6.5)_1$, and (6.7) to conclude that

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\lambda} &= \omega_\mathrm{o} + \alpha^2 h'' + \frac{\mathrm{d}\alpha}{\mathrm{d}\lambda}[h' + \alpha\lambda h''] \\
&= \frac{(\omega_\mathrm{o} + \alpha^2 h'')(\omega_\mathrm{o} + \lambda^2 h'') - (h' + \alpha\lambda h'')^2}{\omega_\mathrm{o} + \lambda^2 h''} \\
&= \frac{\omega_\mathrm{o}^2(\lambda^2 - \alpha^2) + \omega_\mathrm{o}\lambda^2(3\alpha^2 + \lambda^2)h''}{\lambda^2(\omega_\mathrm{o} + \lambda^2 h'')} > 0,
\end{aligned}$$

where we have written $h'$ for $h'(\alpha\lambda)$ and $h''$ for $h''(\alpha\lambda)$. Thus, we can consider $\lambda$ as a function of $\mathcal{S}$, i.e., for each $\mathcal{S} > 0$ there is a unique $\lambda = \lambda(\mathcal{S}) > 1$ that satisfies (6.10). Proposition 4.5 and the proof of Proposition 6.1 then yield the following result.

**Proposition 6.3.** *Let* $\mathcal{S} > 0$, $\mathscr{B} = (-R, R) \times (-L, L) \subset \mathbb{R}^2$, *and*

$$E(\mathbf{u}) := \int_{-L}^{L}\int_{-R}^{R} W(\nabla\mathbf{u}(x,y))\,\mathrm{d}x\,\mathrm{d}y - \mathcal{S}\int_{-R}^{R}[u_2(x,L) + u_2(x,-L)]\,\mathrm{d}x,$$

*where* $W$ *is given by* (6.1)–(6.2). *Then the unique absolute minimizer of* $E$ *among deformations* $\mathbf{u} \in \mathscr{A}$ *that satisfy*

$$\int_{-L}^{L}\int_{-R}^{R} \mathbf{u}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \mathbf{0}$$

*is the homogeneous deformation* $\mathbf{u}_{\rm h}$ *given by* (6.3)*, where* $\alpha = \alpha(\lambda)$ *is given by* $(6.5)_1$ *and* $\lambda = \lambda(\mathcal{S})$ *is the unique solution of* (6.10)*. Moreover, there are no other weak equilibrium solutions* $\mathbf{v}_{\rm e}$ *that satisfy*

$$|h'(\det \nabla \mathbf{v}_{\rm e})| < \omega_{\rm o} \ a.e. \tag{6.11}$$

**Remark 6.4.** There is a second homogeneous equilibrium solution, which does not satisfy (6.11), to this traction problem. The solution can be obtained by starting with the homogeneous deformation $\mathbf{v}_{\rm h}(x, y) = [-\alpha x, -\lambda y]^{\rm T}$, where $\alpha = \alpha(\lambda)$ is still given by $(6.5)_1$, but $\lambda \in (0, 1)$ instead satisfies

$$-\mathcal{S} = \omega_{\rm o}\lambda + \alpha h'(\alpha\lambda) < 0.$$

There may also be equilibrium solutions that exhibit buckling.

## 7. An Alternate Approach

Recently, Gao, Neff, Roventa, and Thiel [16] have established an interesting alternative approach to uniqueness. Assuming the stored-energy density, $W$, is homogeneous and invariant under a change in observer, a standard result yields a function[16] $\widehat{W} \in C^1(\mathrm{Psym}_n)$ that satisfies $\widehat{W}(\mathbf{C}) = W(\mathbf{F})$, where $\mathbf{C} := \mathbf{F}^{\rm T}\mathbf{F}$. The *second Piola-Kirchhoff stress tensor* $\mathbf{K} : \mathrm{Psym}_n \to \mathrm{Sym}_n$ is defined by

$$\mathbf{K}(\mathbf{C}) := 2\frac{{\rm d}\widehat{W}}{{\rm d}\mathbf{C}}(\mathbf{C}) = \mathbf{F}^{-1}\mathbf{S}(\mathbf{F}).$$

The proof in [16, Proposition 2.1] implies[17] the following:

**Proposition 7.1.** *Let* $\mathbf{u}_{\rm e} \in \mathscr{A}$ *be a weak solution of the equilibrium equations. Define* $\mathbf{C}_{\rm e}(\mathbf{x}) := [\nabla\mathbf{u}_{\rm e}(\mathbf{x})]^{\rm T}[\nabla\mathbf{u}_{\rm e}(\mathbf{x})]$*. Suppose that, for a.e.* $\mathbf{x} \in \mathscr{B}$*,*

(1) $\mathbf{C}_{\rm e}(\mathbf{x})$ *is a point of convexity of* $\widehat{W}$*; and*
(2) $\mathbf{K}(\mathbf{C}_{\rm e}(\mathbf{x}))$ *is positive semi-definite.*

*Then* $\mathbf{u}_{\rm e}$ *is an absolute minimizer of* $E$*. Suppose, in addition, that* $\mathscr{D} \neq \varnothing$*, for a.e.* $\mathbf{x} \in \mathscr{B}$*,* $\mathbf{C}_{\rm e}(\mathbf{x})$ *is a point of strict convexity of* $\widehat{W}$*, and* $\mathbf{u}_{\rm e} \in C^1(\mathscr{B}; \mathbb{R}^n)$ *satisfies* $\det \nabla\mathbf{u}_{\rm e} > 0$ *on* $\mathscr{B}$*, then* $\mathbf{u}_{\rm e}$ *is the unique absolute minimizer of* $E$*.*

In particular, when $\widehat{W}$ is convex on its entire domain, $\mathrm{Psym}_n$, then one only needs to consider the sign of the eigenvalues of the second Piola-Kirchhoff stress tensor $\mathbf{K}$. The main difficulty with Proposition 7.1 is the required convexity of the function $\mathbf{C} \mapsto \widehat{W}(\mathbf{C})$. Such an assumption is independent of the polyconvexity of $W$; it is neither necessary nor sufficient for the existence[18] of minimizers. A minor additional problem is the smoothness assumption on $\mathbf{u}_{\rm e}$ that is required for uniqueness. However, when $\widehat{W}$ is in fact convex, Proposition 7.1 may yield results that are better than those produced by Theorem 4.2.

[16] Recall that $\mathrm{Psym}_n$ denotes the set of strictly positive-definite, symmetric $\mathbf{C} \in \mathrm{Lin}_n$.
[17] The uniqueness of the absolute minimizer also requires a result of Ciarlet and Mardare [10, Theorem 2.1].
[18] However, see [11] where the implicit function theorem is used to obtain existence for small data.

## Appendix A. An Optimal Constant

We have made use of a variant of a result of Evans [14, pp. 248–250].

**Proposition A.1.** *Let $p \in [2, \infty)$. Then there exists a constant $\kappa_p \in [2^{2-p}, p2^{1-p}]$ such that*

$$|\mathbf{a}|^p \geq |\mathbf{b}|^p + p|\mathbf{b}|^{p-2}\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) + \sigma\kappa_p|\mathbf{a} - \mathbf{b}|^p + (1 - \sigma)\frac{p}{2}|\mathbf{b}|^{p-2}|\mathbf{a} - \mathbf{b}|^2 \tag{A.1}$$

*for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $\sigma \in [0, 1]$.*

**Remark A.2.** If $p = 2$ then (A.1) reduces to $|\mathbf{a}|^2 = |\mathbf{b}|^2 + 2\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) + |\mathbf{a} - \mathbf{b}|^2$; thus $\kappa_2 = 1$. Also, it follows from [29] that $\kappa_3 = 2 - \sqrt{2}$.

*Proof of Proposition A.1 (cf. the proof of Proposition A.1 in [29]).* In view of the previous remark we assume that $p > 2$. We note that when $\sigma = 1$ this result can be found in Müller et al. [29, Appendix]. We will prove the result when $\sigma = 0$. Inequality (A.1) will then follow upon taking a convex combination of the resulting inequalities.

Let $\sigma = 0$ and $p > 2$. If $\mathbf{b} = \mathbf{0}$ then (A.1) is clear. By homogeneity we may assume $|\mathbf{b}| = 1$. Therefore, (A.1) will follow if we show that, for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $|\mathbf{b}| = 1$,

$$|\mathbf{a}|^p \geq 1 + p\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) + \frac{p}{2}|\mathbf{a} - \mathbf{b}|^2. \tag{A.2}$$

Suppose now that $n = 1$ and define $t := \operatorname{sgn}(b)(a - b)$. Then (A.2) reduces to the inequality

$$\phi(t) := |t + 1|^p - 1 - pt - \frac{p}{2}t^2 \geq 0. \tag{A.3}$$

If we differentiate $\phi$ we find that, for $t \neq -1$,

$$\phi'(t) = p(t + 1)\big[\operatorname{sgn}(t + 1)|t + 1|^{p-2} - 1\big].$$

The only possible solutions of $\phi'(t) = 0$ are $t = -1$ and $t = 0$. Since $\phi(0) = 0$, $\phi(-1) = (p - 2)/2 > 0$, and $\phi$ is a continuous function that blows up at $\pm\infty$ inequality (A.3) follows.

Next, let $\sigma = 0$, $p > 2$, $|\mathbf{b}| = 1$, and $n > 1$. We write $\mathbf{a} = \mathbf{b} + t\mathbf{e}$ and $\alpha = \mathbf{e} \cdot \mathbf{b}$, where $|\mathbf{e}| = 1$ and $\alpha \in [-1, 1]$. Then (A.2) reduces to

$$\theta(t, \alpha) := \big[1 + 2\alpha t + t^2\big]^{p/2} - 1 - p\alpha t - \frac{p}{2}t^2 \geq 0 \tag{A.4}$$

for all $t \in \mathbb{R}$ and $\alpha \in [-1, 1]$. When $\alpha = \pm 1$ the vectors $\mathbf{a}$ and $\mathbf{b}$ are colinear; (A.4) then follows from the above argument with $n = 1$. For fixed $t$ we differentiate $\theta$ with respect to $\alpha$ and set the result equal to zero to conclude that $\alpha = -t/2$. However, $\theta(t, -t/2) = 0$, which establishes (A.4) and completes the proof of (A.1). □

## Appendix B. Determinants and Cofactors

B.1. **The Determinant of a Sum.** A standard identity[19] is for every $\mathbf{F}, \mathbf{H} \in \mathrm{Lin}_n$,

$$\det(\mathbf{H} + \mathbf{F}) = \begin{cases} \det \mathbf{F} + \mathbf{H} : \operatorname{cof} \mathbf{F} + \det \mathbf{H}, & \text{if } n = 2, \\ \det \mathbf{F} + \mathbf{H} : \operatorname{cof} \mathbf{F} + \mathbf{F} : \operatorname{cof} \mathbf{H} + \det \mathbf{H}, & \text{if } n = 3. \end{cases} \tag{B.1}$$

In particular, the choice $\mathbf{H} = \mathbf{G} - \mathbf{F}$ yields

$$\det \mathbf{G} = \begin{cases} \det \mathbf{F} + [\mathbf{G} - \mathbf{F}] : \operatorname{cof} \mathbf{F} + \det(\mathbf{G} - \mathbf{F}), & \text{if } n = 2, \\ \det \mathbf{F} + [\mathbf{G} - \mathbf{F}] : \operatorname{cof} \mathbf{F} + \mathbf{F} : \operatorname{cof}[\mathbf{G} - \mathbf{F}] + \det(\mathbf{G} - \mathbf{F}), & \text{if } n = 3. \end{cases} \tag{B.2}$$

[19] See, e.g., [9, p. 51]. Alternatively, one can derive (B.1) from the characteristic polynomial.

B.2. **The Derivative of the Cofactor in 3-Dimensions.**

**Lemma B.1.** *Let* $\mathbf{F}, \mathbf{H} \in \mathrm{Lin}_3$. *Then*

$$\mathbb{K}(\mathbf{F})[\mathbf{H}] := \frac{\mathrm{d}(\operatorname{cof}\mathbf{F})}{\mathrm{d}\mathbf{F}}[\mathbf{H}] = \operatorname{cof}(\mathbf{F}+\mathbf{H}) - \operatorname{cof}\mathbf{F} - \operatorname{cof}\mathbf{H}. \tag{B.3}$$

*Proof.* Fix $\mathbf{F}, \mathbf{H} \in \mathrm{Lin}_3$. Then a simple computation shows that $\operatorname{cof}(\mathbf{F}+t\mathbf{H})$ is a quadratic polynomial in $t \in \mathbb{R}$, i.e., there exist $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathrm{Lin}_3$ such that, for all $t \in \mathbb{R}$,

$$\operatorname{cof}(\mathbf{F}+t\mathbf{H}) = \mathbf{A} + \mathbf{B}t + \mathbf{C}t^2. \tag{B.4}$$

At $t=0$ we get $\operatorname{cof}\mathbf{F} = \mathbf{A}$. If we divide (B.4) by $t^2$ and let $t \to \infty$ we find that $\operatorname{cof}\mathbf{H} = \mathbf{C}$. If we differentiate (B.4) with respect to $t$ and then let $t=0$ we conclude that

$$\frac{\mathrm{d}(\operatorname{cof}\mathbf{F})}{\mathrm{d}\mathbf{F}}[\mathbf{H}] = \mathbf{B}.$$

The desired result follows from (B.4) at $t=1$ together with our formulae for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. ☐

**Remark B.2.** It is clear from the definition that $\mathbb{K}(\mathbf{F}) : \mathrm{Lin}_3 \to \mathrm{Lin}_3$ is a linear map for every $\mathbf{F} \in \mathrm{Lin}_3$. If we interchange $\mathbf{F}$ and $\mathbf{H}$ in (B.3) we find that $\mathbb{K}(\mathbf{F})[\mathbf{H}] = \mathbb{K}(\mathbf{H})[\mathbf{F}]$; consequently, $\mathbf{F} \mapsto \mathbb{K}(\mathbf{F})[\mathbf{H}] : \mathrm{Lin}_3 \to \mathrm{Lin}_3$ is also linear. Thus, $(\mathbf{F}, \mathbf{H}) \mapsto \mathbb{K}(\mathbf{F})[\mathbf{H}]$ is bilinear.

B.3. **An Upper Bound on the Cofactor in 3-Dimensions.**

**Lemma B.3.** *Let* $\mathbf{X}, \mathbf{H} \in \mathrm{Lin}_3$. *Then*

$$|\mathbf{X} : \operatorname{cof}\mathbf{H}| \le \gamma |\mathbf{H}|^2 \le |\mathbf{X}||\mathbf{H}|^2, \tag{B.5}$$

*where* $\gamma$ *denotes the largest eigenvalue of* $\sqrt{\mathbf{X}^{\mathrm{T}}\mathbf{X}}$.

*Proof.* We first note that $\mathbf{V} := \sqrt{\mathbf{X}^{\mathrm{T}}\mathbf{X}}$ is symmetric and positive semi-definite and hence, by the spectral theorem, has eigenvalues $\gamma \ge \beta \ge \alpha \ge 0$. Thus,

$$|\mathbf{X}|^2 = |\mathbf{V}|^2 = \alpha^2 + \beta^2 + \gamma^2 \ge \gamma^2,$$

which establishes the second inequality in (B.5).

We now apply the polar decomposition theorem to conclude $\mathbf{H} = \mathbf{Q}\mathbf{U}$ and $\mathbf{X} = \mathbf{R}\mathbf{V}$, where $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive semi-definite and $\mathbf{Q}$ and $\mathbf{R}$ are orthogonal. Next, by the spectral theorem, there exists an orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ and real numbers $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge 0$ such that

$$\begin{aligned}
\mathbf{H} = \mathbf{Q}\mathbf{U} &= \lambda_1 \mathbf{Q}\mathbf{f}_1 \otimes \mathbf{f}_1 + \lambda_2 \mathbf{Q}\mathbf{f}_2 \otimes \mathbf{f}_2 + \lambda_3 \mathbf{Q}\mathbf{f}_3 \otimes \mathbf{f}_3,\\
\operatorname{cof}\mathbf{H} = \mathbf{Q}\operatorname{cof}\mathbf{U} &= \lambda_2\lambda_3 \mathbf{Q}\mathbf{f}_1 \otimes \mathbf{f}_1 + \lambda_1\lambda_3 \mathbf{Q}\mathbf{f}_2 \otimes \mathbf{f}_2 + \lambda_1\lambda_2 \mathbf{Q}\mathbf{f}_3 \otimes \mathbf{f}_3,\\
|\mathbf{H}|^2 = |\mathbf{Q}\mathbf{U}|^2 &= |\mathbf{U}|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2.
\end{aligned} \tag{B.6}$$

Consequently,

$$\begin{aligned}
(\operatorname{cof}\mathbf{H}) : \mathbf{X} &= \lambda_2\lambda_3 \mathbf{Q}\mathbf{f}_1 \cdot \mathbf{R}\mathbf{V}\mathbf{f}_1 + \lambda_1\lambda_3 \mathbf{Q}\mathbf{f}_2 \cdot \mathbf{R}\mathbf{V}\mathbf{f}_2 + \lambda_1\lambda_2 \mathbf{Q}\mathbf{f}_3 \cdot \mathbf{R}\mathbf{V}\mathbf{f}_3\\
&= \lambda_2\lambda_3 \mathbf{e}_1 \cdot \mathbf{V}\mathbf{f}_1 + \lambda_1\lambda_3 \mathbf{e}_2 \cdot \mathbf{V}\mathbf{f}_2 + \lambda_1\lambda_2 \mathbf{e}_3 \cdot \mathbf{V}\mathbf{f}_3,
\end{aligned} \tag{B.7}$$

where $\mathbf{e}_i := \mathbf{R}^{\mathrm{T}}\mathbf{Q}\mathbf{f}_i$, $i = 1, 2, 3$, denote unit vectors. However,

$$|\mathbf{e}_i \cdot \mathbf{V}\mathbf{f}_i| \le |\mathbf{e}_i|\,|\mathbf{V}\mathbf{f}_i| \le \gamma, \tag{B.8}$$

where $\gamma$ denotes the largest eigenvalue of $\mathbf{V} = \sqrt{\mathbf{X}^{\mathrm{T}}\mathbf{X}}$. The desired result, (B.5), now follows from (B.6)–(B.8), together with the inequality $2\lambda_i\lambda_j \leq \lambda_i^2 + \lambda_j^2$. □

## Acknowledgements

The authors thank Roger Fosdick for his helpful comments.

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Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
*E-mail address*: masjs@bath.ac.uk

Department of Mathematics, Southern Illinois University, Carbondale, IL 62901, USA
*E-mail address*: sspector@siu.edu