On the optimal location of singularities arising in variational problems of nonlinear elasticity

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Abstract. Experiments on elastomers have shown that triaxial tension can induce a material to exhibit holes that were not previously evident. Analytic work in nonlinear elasticity has established that such cavity formation may indeed be an elastic phenomenon: sufficiently-large prescribed boundary deformations yield a hole-creating deformation as the energy minimizer whenever the elastic energy is of slow growth. One of the many unanswered problems is where such holes will form. In this paper we suggest a new method, which is based upon asymptotics and linear elasticity, that can be used to determine the optimal location for hole creation. Using this method we show that, under reasonable hypotheses, the centre is (locally) the best position for a solitary hole to form in an elastic ball.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be the bounded, simply connected domain occupied by a nonlinearly elastic body in its reference configuration. A **deformation** of the body is an injective (almost everywhere) mapping $\mathbf{u} : \Omega \to \mathbb{R}^n$ that is contained in the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^n)$. If the material is homogeneous and hyperelastic and there are no body forces then the total energy stored in a body that undergoes such a deformation is given by

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.1}$$

where $W : M_+^{n\times n} \to \mathbb{R}$ is the stored energy function and $M_+^{n\times n}$ denotes the set of real $n \times n$ matrices with positive determinant.

A simple example of the stored energy functions to which the analysis of this paper applies is given by the class

$$W(\mathbf{F}) = k||\mathbf{F}||^p + h(\det \mathbf{F}) \text{ for all } \mathbf{F} \in M_+^{n\times n}, \tag{1.2}$$

where $k > 0$, $p \in [1, n)$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ is C^2, strictly convex, and satisfies $h(\delta) \to \infty$ as $\delta \to 0^+$ and $h(\delta)/\delta \to \infty$ as $\delta \to \infty$. Here, and in what follows, $\det \mathbf{F}$ is the determinant of $\mathbf{F}$ while $||\cdot||$ denotes the Euclidean norm on $n \times n$ matrices, viz,

$$||\mathbf{F}|| = (\mathbf{F} : \mathbf{F})^{1/2}, \quad \mathbf{K} : \mathbf{L} = \mathrm{tr}(\mathbf{K}\mathbf{L}^T),$$

where $\mathrm{tr}(\mathbf{M})$ denotes the trace of an $n \times n$ matrix $\mathbf{M}$. For clarity of exposition we will assume throughout this introduction that W is of the above form, though we stress that our approach applies to much more general polyconvex energy functions of slow growth.

The equilibrium equations of nonlinear elasticity are the Euler-Lagrange equations for E. These can take a number of forms depending on the variations taken in the energy functional E. We list three forms of particular relevance in this paper. The first form is

$$(\mathrm{Div}_{\mathbf{x}}\, \mathbf{S}(\nabla \mathbf{u}(\mathbf{x})))_i = \frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla \mathbf{u}(\mathbf{x}))\right] = 0 \quad \text{for } i = 1, 2, .., n, \tag{1.3}$$

and $\mathbf{x} \in \Omega$, where we use the convention of summation over repeated indices. The tensor $\mathbf{S} = \partial W/\partial \mathbf{F}$ is called the Piola-Kirchhoff stress tensor. A second form of the equilibrium equations of elasticity is

$$(\mathrm{Div}_{\mathbf{y}}\, \mathbf{T})_i = \frac{\partial}{\partial y^\alpha}[T_{i\alpha}] = 0 \;\text{ in } \mathbf{u}(\Omega), \tag{1.4}$$

where

$$\mathbf{T}(\mathbf{y}) = \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}(\mathbf{x}))[\nabla \mathbf{u}(\mathbf{x})]^T(\det \nabla \mathbf{u}(\mathbf{x}))^{-1}$$

is called the Cauchy stress tensor and $\mathbf{x} = \mathbf{u}^{-1}(\mathbf{y})$. A third form of the equilibrium equations is given by

$$\mathrm{Div}_{\mathbf{x}}\left[W(\nabla \mathbf{u}(\mathbf{x}))\mathbf{I} - (\nabla \mathbf{u}(\mathbf{x}))^T \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}(\mathbf{x}))\right] = \mathbf{0}, \tag{1.5}$$

which in component form is

$$\frac{\partial}{\partial x^\alpha}\left[W(\nabla \mathbf{u}(\mathbf{x}))\delta^\beta_\alpha - \frac{\partial u^k}{\partial x^\beta}\frac{\partial W}{\partial F^k_\alpha}(\nabla \mathbf{u}(\mathbf{x}))\right] = 0, \quad \beta = 1, 2.., n. \tag{1.6}$$

The expression in brackets is often referred to as the energy-momentum tensor. (See e.g. [2], [3], [12], and [13] for further details of the derivations of (1.4)–(1.5).) Here, and in what follows, $\mathbf{I}$ denotes the $n \times n$ identity matrix.

In this paper we consider the **displacement boundary-value problem** in which we require that the deformations $\mathbf{u}$ satisfy the (linear) Dirichlet boundary condition

$$\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ for } \mathbf{x} \in \partial\Omega, \tag{1.7}$$

where $\mathbf{A} \in M^{n\times n}_{+}$. We note that each of the **homogeneous deformations** $\mathbf{u}^h$ defined by

$$\mathbf{u}^h(\mathbf{x}) \equiv \mathbf{A}\mathbf{x} \tag{1.8}$$

is a solution of (1.3)–(1.5) and satisfies the boundary condition (1.7).

As noted in Ball [1], the existence of discontinuous weak solutions of (1.3) with finite energy depends on the growth of the stored energy function W at infinity. In particular, if $p > n$ in (1.2) then any deformation with finite energy is in $W^{1,p}(\Omega; \mathbb{R}^n)$ and hence continuous by the Sobolev embedding theorem. In an innovative paper [1], Ball considers $\Omega = B$ (the unit ball in $\mathbb{R}^n$), $\mathbf{A} = \lambda\mathbf{I}$, $\lambda > 0$, and deformations that are *radial*, i.e. those of the form

$$\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{for } \mathbf{x} \in B, \tag{1.9}$$

where $r : [0, 1] \to [0, \infty)$. In this case he proves that for each $\lambda > 0$ there exists a unique minimiser $\mathbf{u}^{(r)}$ of the energy functional E among radial deformations in $W^{1,p}(\Omega; \mathbb{R}^n)$ satisfying $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ for $\mathbf{x} \in \partial B$. Moreover, there exists a critical value $\lambda_{\text{crit}} > 0$ with the property that

(i) If $\lambda \le \lambda_{\text{crit}}$ the unique radial minimiser $\mathbf{u}^{(r)}$ is the homogeneous map $\mathbf{u}^h(\mathbf{x}) \equiv \lambda\mathbf{x}$.

(ii) If $\lambda > \lambda_{\text{crit}}$ the unique radial minimiser $\mathbf{u}^{(r)}$ corresponds to a map of the form (1.9) satisfying $r(0) > 0$.

Thus for $\lambda > \lambda_{\text{crit}}$ the deformation $\mathbf{u}^{(r)}$ is discontinuous and produces a hole of radius $r(0)$ at the centre of the ball (this is the phenomenon of cavitation).

Remark. An intuitive idea as to why cavitation occurs for the energy functional in question (i.e. (1.1), (1.2)) is given by the following observation. The total energy of a deformation $\mathbf{u}$ consists of the sum of two parts:

$$\int_B k||\nabla\mathbf{u}||^p d\mathbf{x} \tag{1.10}$$

and

$$\int_B h(\det \nabla\mathbf{u})\, d\mathbf{x}. \tag{1.11}$$

The first functional (1.10) is convex, and hence by Jensen's inequality is minimised, for the Dirichlet problem, by the homogeneous deformation $\mathbf{u}^h$ given by (1.8) with $\mathbf{A} = \lambda\mathbf{I}$, for any value of λ. If $\lambda \ge d_0^{1/n}$, the second functional (1.11) is minimised by choosing

$$\tilde{\mathbf{u}}(\mathbf{x}) = [d_0|\mathbf{x}|^n + (\lambda^n - d_0)]^{\frac{1}{n}}\frac{\mathbf{x}}{|\mathbf{x}|},$$

where d_0 is the unique scalar satisfying $h(d_0) = \inf h$. This follows from observing that this discontinuous radial mapping $\tilde{\mathbf{u}}$ of B satisfies $\det \nabla\tilde{\mathbf{u}}(\mathbf{x}) = d_0$ almost everywhere. Moreover, $\tilde{\mathbf{u}} \in W^{1,p}(B;\mathbb{R}^n)$ for $1 \le p < n$.

The discontinuous radial minimisers $\mathbf{u}^{(r)}$ exhibits a combination of the properties of the two maps $\mathbf{u}^h$ and $\tilde{\mathbf{u}}$ above. In particular, $\det \nabla\mathbf{u}^{(r)}(\mathbf{x}) \to d_0$ as $|\mathbf{x}| \to 0^+$ (see Section 7.5 of [1]).

The similarity of $\mathbf{u}^{(r)}$ with $\mathbf{u}^h$ is highlighted in the next result taken from [14].

Proposition 1.1. *For each $\lambda > 0$ let $\mathbf{u}^{(r)}(\mathbf{x}, \lambda)$ denote the radial minimiser of E that satisfies $\mathbf{u}^{(r)}(\mathbf{x}, \lambda) = \lambda\mathbf{x}$ for $\mathbf{x} \in \partial B$ and let $\mathbf{u}^h_{\text{crit}}$ be the homogeneous map $\mathbf{u}^h_{\text{crit}}(\mathbf{x}) \equiv \lambda_{\text{crit}}\mathbf{x}$. Then for each $\delta \in (0, 1)$*

$$\mathbf{u}^{(r)}(\cdot, \lambda) \to \mathbf{u}^h_{\text{crit}} \textit{ in } C^2(B\backslash B_\delta(\mathbf{x}_0)) \textit{ as } \lambda \to \lambda_{\text{crit}}.$$

There has been much work following Ball's original work on the radial cavitation problem (see for example the references in the review article by Horgan and Polignone [6]). The first existence theorem in a class of deformations which allows cavitation to occur without the assumption of radial symmetry of the admissible deformations was given by Müller and Spector [12]: their approach was to add a surface energy term to (1.1) which penalised the formation of cavities and involved key ideas from geometric measure theory and fine properties of functions.

Adapting ideas from [12], the work in [13] demonstrates that (if $n-1 < p < n$) given **any** point $\mathbf{x}_0 \in B$ there exist minimisers, and hence weak equilibrium solutions, which produce a cavity at $\mathbf{x}_0$, provided λ is sufficiently large (in particular for $\lambda > \lambda_{\text{crit}}$). An elementary, though striking feature of problems where radial symmetry is not assumed is contained in the following lemma which demonstrates that, if the boundary condition (1.7) is such that it is energetically favourable to introduce a discontinuity at one point $\mathbf{x}_0 \in \Omega$, then it is energetically favourable to do so at **any** point $\mathbf{x} \in \Omega$.

Lemma 1.2. *Let $\mathbf{u}^h$ be given by (1.8). Suppose that $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^n)$ satisfies (1.7), has a discontinuity at $\mathbf{x}_0 \in \Omega$, and is such that $E(\mathbf{u}) < E(\mathbf{u}^h)$. Then given any $\mathbf{x}_1 \in \Omega$ there exists $\tilde{\mathbf{u}} \in W^{1,p}(\Omega;\mathbb{R}^n)$, which also satisfies (1.7), that has a discontinuity at $\mathbf{x}_1$ and is such that $E(\tilde{\mathbf{u}}) < E(\mathbf{u}^h)$.*

Proof. First assume, without loss of generality, that $\mathbf{x}_0 = \mathbf{0}$. Given $\mathbf{x}_1 \in \Omega$, define

$$\tilde{\mathbf{u}}(\mathbf{x}) = \begin{cases} \mathbf{A}\mathbf{x}_1 + \epsilon\mathbf{u}\left(\frac{\mathbf{x}-\mathbf{x}_1}{\epsilon}\right), & \text{if } \mathbf{x} \in \mathbf{x}_1 + \epsilon\Omega, \\ \mathbf{A}\mathbf{x}, & \text{otherwise.} \end{cases}$$

Then it is easily verified that, for small ϵ, $\tilde{\mathbf{u}} \in W^{1,p}(\Omega;\mathbb{R}^n)$ satisfies (1.7) and

$$\begin{aligned} E(\tilde{\mathbf{u}}) - E(\mathbf{u}^h) &= \int_{\mathbf{x}_1+\epsilon\Omega} \left[W\left(\nabla\mathbf{u}\left(\frac{\mathbf{x}-\mathbf{x}_1}{\epsilon}\right)\right) - W(\mathbf{A})\right] d\mathbf{x} \\ &= \epsilon^n \int_\Omega [W(\nabla\mathbf{u}(\mathbf{y})) - W(\mathbf{A})]\, d\mathbf{y} = \epsilon^n [E(\mathbf{u}) - E(\mathbf{u}^h)] < 0, \end{aligned}$$

as claimed. □

The main purpose of this paper is to suggest an approach to differentiating between the possible points of discontinuity and predicting the most energetically favourable point at which a discontinuity will form. We assume henceforth in this paper that in the boundary condition (1.7)

$$\mathbf{A} = \lambda\mathbf{I}, \tag{1.12}$$

where $\lambda > 0$. To present our approach we make hypotheses on the regularity and asymptotic behaviour of singular minimisers. The precise form of these hypotheses is given in Section 4.

We next illustrate the main features and consequences of these hypotheses in predicting the point of discontinuity of an energy minimiser. For $\mathbf{x}_0 \in \Omega$ and $\lambda > 0$ let $\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda)$ denote a minimiser of the energy that has a single discontinuity at $\mathbf{x}_0 \in \Omega$ and satisfies the boundary condition

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \lambda\mathbf{x} \quad \text{for} \quad \mathbf{x} \in \partial\Omega.$$

The existence of such minimisers follows from Theorem 4.1 of [13]. Our approach starts with a divergence identity, originally derived for smooth equilibria, to express the energy of a discontinuous minimiser

$$\mathbf{u}(\cdot, \mathbf{x}_0, \lambda) \in C^2(\Omega\backslash\{\mathbf{x}_0\};\mathbb{R}^n) \cap C^1(\overline{\Omega}\backslash\{\mathbf{x}_0\};\mathbb{R}^n)$$

in the form

$$nE(\mathbf{u}) = \int_{\partial\Omega} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)W(\nabla_{\mathbf{x}}\mathbf{u}) + (\mathbf{u} - (\nabla_{\mathbf{x}}\mathbf{u})[\mathbf{x}-\mathbf{x}_0])\cdot\frac{\partial W}{\partial\mathbf{F}}(\nabla_{\mathbf{x}}\mathbf{u})\mathbf{n}\, dS, \tag{1.13}$$

i.e. the energy of the discontinuous equilibrium is expressible as a boundary integral on the outer boundary $\partial\Omega$ away from the point of discontinuity $\mathbf{x}_0$ (see Section 3). Here, and in what follows, $\mathbf{n} = \mathbf{n}(\mathbf{x})$ denotes the outward unit normal to the boundary of the region.

We next assume that in the case that $\mathbf{x}_0 = \mathbf{0}$ and $\Omega = B$ the minimiser $\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda)$ is the radial minimiser. This hypothesis guarantees, by Lemma 1.2, that, when $\Omega = B$, $\lambda = \lambda_{\text{crit}}$ is the infimum of values of λ at which a single cavity will form at any $\mathbf{x}_0 \in B$.

A scaling argument then shows that this holds with B replaced by any bounded domain $\Omega \subset \mathbb{R}^n$. We also assume that the family of minimisers satisfies the following generalisation of Proposition 1.1.

Convergence Hypothesis. For each $\mathbf{x}_0 \in \Omega$ and $\delta \in (0,1)$

$$\mathbf{u}(\cdot, \mathbf{x}_0, \lambda) \to \mathbf{u}^h_{\text{crit}}(\cdot) \ \text{ in } \ C^2(\overline{\Omega} \backslash B_\delta(\mathbf{x}_0)) \text{ as } \lambda \to \lambda_{\text{crit}},$$

where $\mathbf{u}^h_{\text{crit}}(\mathbf{x}) = \lambda_{\text{crit}}\mathbf{x}$. We then write

$$\begin{aligned} \mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) &= \lambda\mathbf{x} + \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda), \\ \mathbf{u}^h(\mathbf{x}, \lambda) &= \lambda\mathbf{x}, \end{aligned} \tag{1.14}$$

and use (1.13)–(1.14) and the convergence hypothesis to expand the energy difference,

$$\Delta E := E(\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)) - E(\mathbf{u}^h(\cdot, \lambda))$$

between the discontinuous and homogeneous maps as a series in $(\lambda - \lambda_{\text{crit}})$ to obtain

$$\Delta E = -(\lambda - \lambda_{\text{crit}})^2 \frac{1}{2n} \int_{\partial\Omega} \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) \nabla_{\mathbf{x}} \dot{\mathbf{w}} : \mathbb{C}[\nabla_{\mathbf{x}} \dot{\mathbf{w}}] \, dS_{\mathbf{x}} + o(|\lambda - \lambda_{\text{crit}}|^2),$$

where

$$\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0) = \frac{\partial}{\partial\lambda} \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda)\bigg|_{\lambda = \lambda_{\text{crit}}} \qquad \text{and} \qquad \mathbb{C} = \frac{\partial^2 W}{\partial \mathbf{F}^2}(\lambda_{\text{crit}}\mathbf{I}) \tag{1.15}$$

is the elasticity tensor at the (stressed) configuration $\mathbf{u}^h_{\text{crit}}(\mathbf{x}) \equiv \lambda_{\text{crit}}\mathbf{x}$.

In order to predict the most energetically favourable point $\mathbf{x}_0$ at which to initiate a discontinuity we maximise the energy difference by maximising the coefficient of $-(\lambda - \lambda_{\text{crit}})^2$ in the above expansion, i.e.

$$\frac{1}{2n} \int_{\partial\Omega} \mathbf{n}(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0) \nabla_{\mathbf{x}} \dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0) : \mathbb{C}[\nabla_{\mathbf{x}} \dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0)] \, dS_{\mathbf{x}} \ , \tag{1.16}$$

over all choices of $\mathbf{x}_0 \in \Omega$.

We next attempt to evaluate the integral (1.16). To do this we first observe (in Lemma 5.2) that one consequence of the convergence hypothesis is that $\dot{\mathbf{w}}(\cdot, \mathbf{x}_0)$ satisfies the **linear** system of equations

$$\text{Div}_{\mathbf{x}} \, \mathbb{C}[\nabla_{\mathbf{x}} \dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0)] = \mathbf{0} \text{ for } \mathbf{x} \in \Omega \backslash \{\mathbf{x}_0\} \tag{1.17}$$

and the boundary condition

$$\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0) = \mathbf{0} \text{ for } \mathbf{x} \in \partial\Omega, \tag{1.18}$$

where $\mathbb{C}$ is the elasticity tensor given by $(1.15)_2$.

Thus, in order to evaluate (1.16) it suffices to determine a suitable (singular) solution of the equations of linear elasticity. In Section 5 we postulate an ansatz for $\mathbf{u}(\mathbf{x},\mathbf{x}_0,\lambda)$ (see (5.4)) that enables us to evaluate this integral (1.16) explicitly in Section 6 and hence show that

$$\Delta E = (\lambda-\lambda_{\text{crit}})^2\kappa_n\mu^2(\mathbf{x}_0)\,[\operatorname{div}_{\mathbf{x}}\mathbf{v}(\mathbf{x},\mathbf{x}_0)]|_{\mathbf{x}=\mathbf{x}_0} + o(|\lambda-\lambda_{\text{crit}}|^2)\ ,$$

where $\kappa_n > 0$ is a constant depending on the dimension n (see Theorem 6.1), $\mu(\mathbf{x}_0)$ is a constant depending on $\mathbf{x}_0$, and $\mathbf{v}(\mathbf{x},\mathbf{x}_0)$ is the unique *smooth* solution of (1.17) on *all* of Ω that satisfies the boundary condition

$$\mathbf{v}(\mathbf{x},\mathbf{x}_0) = -\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n}\ \text{for}\ \mathbf{x}\in\partial\Omega.$$

In Section 7 we treat the case $\Omega = B$, the unit ball in $\mathbb{R}^n$, where we are able to determine $\mathbf{v}$ explicitly as a series and hence evaluate $\operatorname{div}_{\mathbf{x}}\mathbf{v}(\mathbf{x},\mathbf{x}_0)|_{\mathbf{x}=\mathbf{x}_0}$. In this case, by symmetry, it is clear that $\mu(\mathbf{x}_0) = \tilde{\mu}(|\mathbf{x}_0|)$ and from this we deduce, in Proposition 7.5, sufficient conditions under which the radial minimiser is locally minimising. (i.e conditions under which $\mathbf{x}_0 = \mathbf{0}$ yields the greatest energy drop over all choices of $\mathbf{x}_0$ in a neighbourhood of $\mathbf{0}$.)

2. Radial Cavitation

In this section we gather together certain properties of the radial cavitation solutions found in Ball [1]. Recall that in the radial problem $\Omega = B$, the unit ball in $\mathbb{R}^n$, and $\mathbf{A} = \lambda\mathbf{I}$, $\lambda > 0$ in (1.7). We restrict attention to **radial** deformations $\mathbf{u}$, i.e. deformations of the form

$$\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}\quad \text{for}\ \mathbf{x}\in B, \tag{2.1}$$

where $r:[0,1]\to[0,\infty)$.

If W is frame indifferent and isotropic then it is well-known that there exists a symmetric function Φ such that

$$\Phi(v_1,v_2,...,v_n) = W(\mathbf{F})\ \text{for all}\ \mathbf{F}\in M_+^{n\times n},$$

where $v_1,v_2,..,v_n$ are the eigenvalues of $(\mathbf{F}^T\mathbf{F})^{\frac{1}{2}}$. Solutions $\mathbf{u}$ to the equilibrium equations (1.3) of the form (2.1) correspond to solutions $r(R)$ on $(0,1)$ of the radial equilibrium equation

$$\frac{d}{dR}\left[R^{n-1}\Phi_{,1}\right] = (n-1)R^{n-2}\Phi_{,2}\ , \tag{2.2}$$

where

$$\Phi_{,i} = \Phi_{,i}\left(r'(R), \frac{r(R)}{R}, ..., \frac{r(R)}{R}\right)$$

and $\Phi_{,i}(v_1, v_2, ..., v_n)$ denotes differentiation of Φ with respect to its i^{th}-argument (see Theorem 7.3 of [1]).

For ease of exposition we state the following theorem under the assumption that W is as in the Example (1.2) in the introduction. However, the theorem is known to hold for much more general stored energy functions and the interested reader is referred to the available literature on this (see e.g. [15], [16], [11], [6] and the references therein).

Proposition 2.1. *Let W be as in example (1.2) with $1 < p < n$. Then for each $\lambda > 0$ there exists a unique radial minimiser $\mathbf{u}^{(r)}(\mathbf{x}, \lambda)$ for E that satisfies the Dirichlet data*

$$\mathbf{u}^{(r)}(\mathbf{x}, \lambda) = \lambda\mathbf{x} \ for\ \mathbf{x} \in \partial B.$$

Moreover, there exists $\lambda_{\rm crit} > 0$ with the property that:

(a) (i) If $\lambda \leq \lambda_{\rm crit}$ then $\mathbf{u}^{(r)}(\mathbf{x}, \lambda) \equiv \lambda\mathbf{x}$ is this radial minimiser.

(ii) If $\lambda > \lambda_{\rm crit}$ then the radial minimiser $\mathbf{u}^{(r)}(\mathbf{x}, \lambda)$ corresponds to a map of the form (2.1) that satisfies the radial equilibrium equation on $(0, 1)$, $r(0) > 0$, and the natural boundary condition

$$T(R) := \left(\frac{R}{r(R)}\right)^{n-1} \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, ..., \frac{r(R)}{R}\right) \to 0 \quad as\ R \to 0^+.$$

(Thus the deformation $\mathbf{u}^{(r)}$ produces a hole of radius $r(0)$ at the centre of the ball. Since $T(R)$ is the radial component of the Cauchy stress, it follows that the surface of the cavity is stress free.)

(b) For each $\delta \in (0, 1)$

$$\mathbf{u}^{(r)}(\cdot, \lambda) \to \mathbf{u}^h_{\rm crit} \ in\ C^2(B\backslash B_\delta(\mathbf{x}_0))\ as\ \lambda \to \lambda_{\rm crit}\ ,$$

where $\mathbf{u}^h_{\rm crit}$ is the homogeneous map $\mathbf{u}^h_{\rm crit}(\mathbf{x}) \equiv \lambda_{\rm crit}\mathbf{x}$.

The proof of part (a) follows from Section 7 of [1] and part (b) follows from Proposition 2.1 and Lemma 2.2 of [14].

For later use in this paper we note the following property of $\mathbf{u}^{(r)}(\mathbf{x}, \lambda)$.

Lemma 2.2. *The radial minimiser minimiser* $\mathbf{u}^{(r)}(\mathbf{x},\lambda)$ *satisfies*

$$\frac{\partial \mathbf{u}^{(r)}}{\partial \lambda}(\mathbf{x},\lambda)\bigg|_{\lambda=\lambda_{\rm crit}} = \frac{\mathbf{x}}{|\mathbf{x}|^n} \quad \textit{for each} \quad \mathbf{x}\in B\backslash\{\mathbf{0}\}, \tag{2.3}$$

where the derivative is understood as a right-sided derivative.

Proof. First note that (by e.g. Proposition 1.6 of [15]) for each $\lambda > 0$, the corresponding function $r(\cdot,\lambda)$ in (2.1) can be extended to $(0,\infty)$ as a solution of the radial equilibrium equation (2.2). Consequently, the corresponding deformation $\mathbf{u}$ given by (2.1) is a solution of (1.3) on $\mathbb{R}^n\backslash\{\mathbf{0}\}$.

Next, by Proposition 1.6 in [15] the extended function satisfies $r'(R,\lambda) \nearrow \lambda_{\rm crit}$ and $\frac{r(R,\lambda)}{R} \searrow \lambda_{\rm crit}$ as $R\to\infty$ ($r'(R,\lambda) := \partial r(R,\lambda)/\partial R$). Therefore for $\lambda \geq \lambda_{\rm crit}$

$$R \mapsto (r(R,\lambda) - \lambda_{\rm crit} R)$$

is a monotonic decreasing function (since $r'(R,\lambda) - \lambda_{\rm crit} \leq 0$ for all R) and consequently

$$\phi(R) = \frac{\partial r(R,\lambda)}{\partial\lambda}\bigg|_{\lambda=\lambda_{\rm crit}} = \lim_{\lambda\to\lambda_{\rm crit}^+} \frac{r(R,\lambda)-\lambda_{\rm crit}R}{\lambda-\lambda_{\rm crit}} \tag{2.4}$$

is a monotonic decreasing function of R.

By standard results for ordinary differential equations, ϕ satisfies the equation (2.2) linearised around the solution $\lambda_{\rm crit}R$, i.e.

$$\frac{d}{dR}\left[R^{n-1}\Phi_{11}^{\rm crit}\phi'\right] - (n-1)R^{n-3}\Phi_{11}^{\rm crit}\phi = 0,$$

$\Phi_{11}^{\rm crit} := \Phi_{,11}(\lambda_{\rm crit},\lambda_{\rm crit},..,\lambda_{\rm crit})$, and since $r(1,\lambda)=\lambda$ it follows that $\phi(1)=1$. Here we have used the fact that Φ is symmetric in to obtain the linearization in this form.

An easy calculation yields

$$\phi(R) = aR + \frac{(1-a)}{R^{n-1}}$$

for some constant a. Since ϕ is bounded at infinity it follows that $a=0$. Hence

$$\phi(R) = \frac{1}{R^{n-1}} \text{ for } \ R\in[1,\infty)$$

and the lemma now follows from (2.4) and (2.1). □

Remark. Note that by (a)(i) of Proposition 2.1, the derivative (2.3) is not equal to the corresponding left-hand derivative, which is $\mathbf{x}$.

3. The energy associated with a discontinuous minimiser

Existence

The results of [13] demonstrate in particular, that for a wide class of stored energy functions with slow growth, there exist singular weak solutions of the equilibrium equations of elasticity (1.4) or (1.6) that satisfy (1.7) and contain a discontinuity at any prescribed point $\mathbf{x}_0 \in \Omega$. More precisely, for $p \in (n-1, n)$ and $\mathbf{x}_0 \in \Omega$ let

$$\mathcal{A}(\mathbf{x}_0) = \big\{ \mathbf{u} \in W^{1,p}_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^n) : \mathbf{u} = \mathbf{u}^h \text{ on } \mathbb{R}^n \backslash \Omega, \ \det \nabla \mathbf{u} > 0 \text{ a.e.}, \\ \mathbf{u}^* \text{ satisfies (INV)}, \operatorname{Det} \nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^n + \alpha_{\mathbf{u}} \delta_{\mathbf{x}_0} \big\}, \quad (3.1)$$

where $\mathbf{u}^*$ denotes the precise representative of $\mathbf{u}$, condition (INV) is the "invertibility" condition defined in [12], $\operatorname{Det} \nabla \mathbf{u}$ denotes the distributional Jacobian of $\mathbf{u}$, $\mathcal{L}^n$ denotes n-dimensional Lebesgue measure, $\alpha_{\mathbf{u}} \geq 0$ is a scalar depending on the map $\mathbf{u}$, and $\delta_{\mathbf{x}_0}$ denotes the Dirac distribution with support at $\mathbf{x}_0$. Thus $\mathcal{A}(\mathbf{x}_0)$ contains maps $\mathbf{u}$ that produce a cavity of volume $\alpha_{\mathbf{u}}$ located at $\mathbf{x}_0 \in \Omega$. (Further details of condition (INV) and the above notions can be found in [12] and [13].)

The Energy

We next assume that $\mathbf{u} \in C^2(\Omega \backslash \{\mathbf{x}_0\}) \cap C^1(\overline{\Omega} \backslash \{\mathbf{x}_0\})$ is a solution of (1.3) and derive an expression for $E(\mathbf{u})$ as a boundary integral on $\partial\Omega$. To do this we first recall the following identity, due originally to Green [5], and used by Knops and Stuart [8] in their proof of uniqueness of smooth solutions to the equilibrium equations of elasticity. (The radial version of this identity was used by Ball in [1] in the study of radial cavitation.)

Lemma 3.1. *Let $\mathbf{u} \in C^2(\Omega;\mathbb{R}^n)$ be a solution of (1.3). Then*

$$\operatorname{div}_{\mathbf{x}} \left[W(\nabla \mathbf{u})\mathbf{x} + \left[\frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}) \right]^T \Big(\mathbf{u} - (\nabla \mathbf{u})\mathbf{x} \Big) \right] = nW(\nabla \mathbf{u}) \quad (3.2)$$

for all $\mathbf{x} \in \Omega$.

Proof. This is a straightforward calculation (see [5] or [8]). □

Remark. If we replace $\mathbf{x}$ in (3.2) by $\mathbf{x} - \mathbf{x}_0$, where $\mathbf{x}_0 \in \Omega$, we find that

$$\operatorname{div}_{\mathbf{x}} \left[W(\nabla \mathbf{u})(\mathbf{x} - \mathbf{x}_0) + \left[\frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u}) \right]^T \Big(\mathbf{u} - (\nabla \mathbf{u})[\mathbf{x} - \mathbf{x}_0] \Big) \right] = nW(\nabla \mathbf{u}) \quad (3.3)$$

for $\mathbf{x} \in \Omega$. Finally, we observe for later use that the above expression holds with $\mathbf{u}$ replaced by $\mathbf{u} + \mathbf{c}$ for any constant vector $\mathbf{c} \in \mathbb{R}^n$.

In order to treat the case of discontinuous equilibria we now introduce constitutive hypotheses on the stored energy function W.

Hypotheses on W

Throughout this section we assume that $W \in C^2(M_+^{n\times n})$ and we will refer to the following hypotheses on W:

(**W1**) W is strongly elliptic: i.e. for any $\mathbf{F} \in M_+^{n\times n}$

$$\frac{\partial^2 W(\mathbf{F})}{\partial F^i_\alpha \partial F^j_\beta} a^i b^\alpha a^j b_\beta > 0 \tag{3.4}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b}$.

(**W2**) There exists $\varepsilon > 0$ and a constant C such that for any $\mathbf{A} \in M_+^{n\times n}$ with $\|\mathbf{A} - \mathbf{I}\| < \varepsilon$ we have

$$\left\| \mathbf{F}^T \frac{\partial W}{\partial \mathbf{F}}(\mathbf{FA}) \right\| \leq C[W(\mathbf{F}) + 1] \quad \text{for all } \mathbf{F} \in M_+^{n\times n}.$$

(**W3**) There exists $\varepsilon > 0$ and a constant C such that for any $\mathbf{A} \in M_+^{n\times n}$ with $\|\mathbf{A} - \mathbf{I}\| < \varepsilon$ we have

$$\left\| \frac{\partial W}{\partial \mathbf{F}}(\mathbf{AF}) \mathbf{F}^T \right\| \leq C[W(\mathbf{F}) + 1] \quad \text{for all } \mathbf{F} \in M_+^{n\times n} \ .$$

Remark. Hypotheses (**W2**) and (**W3**) have been used to prove that minimisers of E satisfy the weak forms of the equilibrium equations (1.4) and (1.6) (see [2], [3], [12], [13]).

Lemma 3.2. *Let W satisfy (W2) and (W3). Suppose that $\mathbf{x}_0 \in \Omega$ and that $\tilde{\mathbf{u}} \in C^2(\Omega\backslash\{\mathbf{x}_0\}) \cap C^1(\overline{\Omega}\backslash\{\mathbf{x}_0\})$ is a minimiser of E on $\mathcal{A}(\mathbf{x}_0)$. Then*

$$nE(\tilde{\mathbf{u}}) = \int_{\partial\Omega} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)W(\nabla\tilde{\mathbf{u}}) + (\tilde{\mathbf{u}} - (\nabla\tilde{\mathbf{u}})[\mathbf{x}-\mathbf{x}_0]) \cdot \frac{\partial W}{\partial \mathbf{F}}(\nabla\tilde{\mathbf{u}})\mathbf{n}\, dS, \tag{3.5}$$

where $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. Expression (3.5) also holds with $\tilde{\mathbf{u}}$ replaced by $\tilde{\mathbf{u}} + \mathbf{c}$ for any constant vector $\mathbf{c} \in \mathbb{R}^n$.

Proof. The proof of (3.5) will follow from the divergence theorem and (3.3). First note that $\tilde{\mathbf{u}}$ satisfies (3.3) on $\Omega\backslash\{\mathbf{x}_0\}$. Next, integrate the divergence expression in (3.3) over $\Omega\backslash B_\delta(\mathbf{x}_0)$, where $\delta > 0$ is sufficiently small and $B_\delta = B_\delta(\mathbf{x}_0)$. The claim (3.5) will now follow from the divergence theorem once we demonstrate that

$$\int_{\partial B_{\delta_n}} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)W(\nabla\tilde{\mathbf{u}}) + (\tilde{\mathbf{u}} - (\nabla\tilde{\mathbf{u}})[\mathbf{x}-\mathbf{x}_0]) \cdot \frac{\partial W}{\partial \mathbf{F}}(\nabla\tilde{\mathbf{u}})\mathbf{n}\, dS \to 0 \tag{3.6}$$

for some sequence $\delta_n \to 0^+$ as $n \to \infty$.

In order to prove (3.6) we split the integral into three terms and treat each separately.

Step 1. First consider

$$\int_{\partial B_\delta} \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) W(\nabla \tilde{\mathbf{u}}) \, dS = \int_{\partial B_\delta} \delta \; W(\nabla \tilde{\mathbf{u}}) \, dS. \tag{3.7}$$

Suppose for a contradiction that (3.7) does not tend to zero for *any sequence* $\delta_n \to 0^+$. Then by the non-negativity of the integrand there exists $\eta > 0$ such that

$$\int_{\partial B_\delta} \delta \; W(\nabla \tilde{\mathbf{u}}) \, dS \geq \eta \quad \text{for all} \quad \delta > 0 \quad \text{and sufficiently small.}$$

Consequently,

$$\int_{\delta=0}^{\delta_0} \int_{\partial B_\delta} W(\nabla \tilde{\mathbf{u}}) \, dS d\delta \; = \; \int_{B_{\delta_0}} W(\nabla \tilde{\mathbf{u}}) \, d\mathbf{x} \geq \int_0^{\delta_0} \frac{\eta}{\delta} \, d\delta. \tag{3.8}$$

Since the right hand side of (3.8) is infinite this contradicts the fact that $\tilde{\mathbf{u}}$ has finite energy. Thus (3.7) converges to zero for some sequence $\delta_n \to 0^+$ and hence, by the monotone convergence theorem, for *every* such sequence.

Step 2. We next consider the term

$$\int_{\partial B_\delta} ((\nabla \tilde{\mathbf{u}})[\mathbf{x} - \mathbf{x}_0]) \cdot \frac{\partial W}{\partial \mathbf{F}} (\nabla \tilde{\mathbf{u}}) \mathbf{n} \, dS. \tag{3.9}$$

To see the expression (3.9) converges to zero for any sequence $\delta_n \to 0^+$, observe that by (W2) and Hölder's inequality

$$\left| \int_{\partial B_\delta} ((\nabla \tilde{\mathbf{u}})[\mathbf{x} - \mathbf{x}_0]) \cdot \frac{\partial W}{\partial \mathbf{F}} (\nabla \tilde{\mathbf{u}}) \mathbf{n} \, dS \right| \; \leq \; \int_{\partial B_\delta} \delta \left\| (\nabla \tilde{\mathbf{u}})^T \frac{\partial W}{\partial \mathbf{F}} (\nabla \tilde{\mathbf{u}}) \right\| \, dS \tag{3.10}$$

$$\leq \; \delta \int_{\partial B_\delta} C[W(\nabla \tilde{\mathbf{u}}) + 1] \, dS. \tag{3.11}$$

An exactly analogous argument to that used in the proof of Step 1 now demonstrates that (3.11) and hence the first integral in (3.10) converges to zero as $\delta \to 0^+$.

Step 3. Finally we treat the remaining term

$$\int_{\partial B_\delta} \tilde{\mathbf{u}} \cdot \frac{\partial W}{\partial \mathbf{F}} (\nabla \tilde{\mathbf{u}}) \mathbf{n} \, dS. \tag{3.12}$$

By (W3) and the proof of Theorem 6.1 in [12] it follows that $\tilde{\mathbf{u}}$ satisfies

$$\int_\Omega \frac{\partial W}{\partial \mathbf{F}}(\nabla\tilde{\mathbf{u}}) : [\nabla \mathbf{v}(\tilde{\mathbf{u}}(\mathbf{x}))\nabla\tilde{\mathbf{u}}(\mathbf{x})]\, d\mathbf{x} = 0 \tag{3.13}$$

for any $\mathbf{v} \in C_0^1(\mathbb{R}^n;\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n\backslash\mathbf{u}^h(\Omega)$. (The proof of this result follows from considering "outer" variations of the form $\Phi_\epsilon(\tilde{\mathbf{u}}) = \tilde{\mathbf{u}} + \epsilon\mathbf{v}(\tilde{\mathbf{u}})$, and is a consequence of setting $\frac{d}{d\epsilon}E(\Phi_\epsilon)\big|_{\epsilon=0} = 0$, since $\tilde{\mathbf{u}}$ is a minimiser.) Note also that by (W3) it follows that the integral in (3.13) is in $L^1(\Omega)$. By the smoothness of $\tilde{\mathbf{u}}$ on $\Omega\backslash\{\mathbf{x}_0\}$, we obtain, for small $\delta > 0$,

$$\int_{\Omega\backslash B_\delta} \frac{\partial W}{\partial \mathbf{F}}(\nabla\tilde{\mathbf{u}}) : [\nabla \mathbf{v}(\tilde{\mathbf{u}})\nabla\tilde{\mathbf{u}}]\, d\mathbf{x} = \int_{\Omega\backslash B_\delta} \operatorname{div}\left(\left[\frac{\partial W}{\partial \mathbf{F}}(\nabla\tilde{\mathbf{u}})\right]^T [\mathbf{v}(\tilde{\mathbf{u}})]\right) d\mathbf{x}. \tag{3.14}$$

Here we have used the fact that $\tilde{\mathbf{u}}$ satisfies (1.3) : since by Theorem 6.1 in [12] $\tilde{\mathbf{u}}$ satisfies (1.6) and hence (1.3) on $\Omega\backslash\{\mathbf{x}_0\}$.

Thus by the divergence theorem we find that (3.14) is equal to

$$\int_{\partial B_\delta} \mathbf{v}(\tilde{\mathbf{u}}) \cdot \frac{\partial W}{\partial \mathbf{F}}(\nabla\tilde{\mathbf{u}})\mathbf{n}\, dS. \tag{3.15}$$

By the dominated convergence theorem (3.14) converges to the left-hand side of (3.13) as $\delta \to 0^+$ and hence expression (3.15) converges to zero as $\delta \to 0^+$.

In order to complete the proof of step 3, we choose $\tilde{\mathbf{v}} \in C_0^1(\mathbb{R}^n;\mathbb{R}^n)$ that satisfies $\tilde{\mathbf{v}} = \mathbf{0}$ on $\mathbb{R}^n\backslash\mathbf{u}^h(\Omega)$ and $\tilde{\mathbf{u}}(\mathbf{y}) = \mathbf{y}$ for any $\mathbf{y} \in \tilde{\mathbf{u}}(B_{\varepsilon_0}(\mathbf{x}_0)\backslash\{\mathbf{x}_0\})$ where $\varepsilon_0 \in (0,1)$ is fixed and sufficiently small so that $B_{\varepsilon_0}(\mathbf{x}_0) \subset \Omega$. Then setting $\mathbf{v} = \tilde{\mathbf{v}}$ in (3.15) we obtain expression (3.12), which must therefore converge to zero as $\delta \to 0^+$ as required.

The final claim of the lemma follows upon observing that the energy $E(\tilde{\mathbf{u}})$ is invariant when $\tilde{\mathbf{u}}$ is replaced by $\tilde{\mathbf{u}} + \mathbf{c}$ for any constant vector $\mathbf{c} \in \mathbb{R}^n$ and noting that $\tilde{\mathbf{u}} + \mathbf{c}$ minimises E on $\mathcal{A}(\mathbf{x}_0)$ (given by (3.1)) with $\mathbf{u}^h$ replaced by $\mathbf{u}^h + \mathbf{c}$. □

Remark. We remark that it is a consequence of (3.5) of Lemma 3.2 and the arguments of [8] that if $\tilde{\mathbf{u}}$ satisfies the hypotheses of the Lemma, $\tilde{\mathbf{u}} = \mathbf{u}^h$ (given by (1.8)), on $\partial\Omega$, and W is strongly elliptic, then

$$E(\tilde{\mathbf{u}}) < E(\mathbf{u}^h). \tag{3.16}$$

Hence a discontinuous minimiser $\tilde{\mathbf{u}}$ has strictly less energy than the corresponding homogeneous map $\mathbf{u}^h$. (Note that (3.16) holds with $<$ replaced by $\leq$ trivially since $\mathbf{u}^h \in \mathcal{A}(\mathbf{x}_0)$.)

4. Families of minimisers with one point of discontinuity

Given $\mathbf{x}_0 \in \Omega$ and $\lambda > 0$, it follows from Theorem 4.1 of [13] that there exists a minimiser $\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda)$ of E on $\mathcal{A}(\mathbf{x}_0)$ (given by (3.1), (1.8), (1.12)) that satisfies the boundary condition $\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \lambda\mathbf{x}$ for $\mathbf{x} \in \partial\Omega$. Throughout this section we make the following hypotheses on this set of minimisers.

Hypotheses on the minimisers

(M1) For each $\mathbf{x}_0 \in \Omega$ and $\lambda > 0$ the minimiser $\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)$ of E on $\mathcal{A}(\mathbf{x}_0)$ (given by (3.1)) is **unique** and this family of minimisers satisfies:

(a) For each $\mathbf{x}_0 \in \Omega$ and $\lambda > 0$

$$\mathbf{u}(\cdot, \mathbf{x}_0, \lambda) \in C^2(\Omega \backslash \{\mathbf{x}_0\}) \cap C^1(\overline{\Omega} \backslash \{\mathbf{x}_0\}),$$

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \lambda\mathbf{x} \text{ for } \mathbf{x} \in \partial\Omega.$$

(b) For each $\mathbf{x}_0 \in \Omega$

$$\mathbf{u}(\cdot, \mathbf{x}_0, \cdot) \in C^3\big((\overline{\Omega} \backslash \{\mathbf{x}_0\}) \times [\lambda_{\text{crit}}, \infty)\big).$$

(M2) For each $\mathbf{x}_0 \in \Omega$

$$\mathbf{u}(\cdot, \mathbf{x}_0, \lambda) \to \mathbf{u}^h_{\text{crit}}(\cdot) \quad \text{in} \quad C^2(\overline{\Omega} \backslash B_\delta(\mathbf{x}_0)) \quad \text{as} \quad \lambda \to \lambda_{\text{crit}}$$

for any, sufficiently small, $\delta > 0$, where

$$\mathbf{u}^h_{\text{crit}}(\mathbf{x}) := \lambda_{\text{crit}}\mathbf{x}.$$

(M3) In the case $\Omega = B$, $\mathbf{x}_0 = \mathbf{0}$, the radial minimiser, whose existence is given in Proposition 2.1, is the unique minimiser of E on $\mathcal{A}(\mathbf{x}_0)$ (given by (3.1)).

Remark. Condition (M1)(a) is a regularity hypothesis on $\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda)$ for fixed $\mathbf{x}_0$ and λ. Condition (M1)(b) is a joint differentiality hypothesis in $\mathbf{x}$ and λ. Condition (M2) is the generalisation to the case $\mathbf{x}_0 \neq \mathbf{0}$ of the result given in Theorem 2.1(b) for radial minimisers.

Lemma 3.2 applied to each member of our family $\mathbf{u}(\cdot) = \mathbf{u}(\cdot, \mathbf{x}_0, \lambda)$ yields

$$nE(\mathbf{u}) = \int_{\partial\Omega} \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) W(\nabla_{\mathbf{x}}\mathbf{u}) + (\mathbf{u} + \mathbf{c} - (\nabla_{\mathbf{x}}\mathbf{u})[\mathbf{x} - \mathbf{x}_0]) \cdot \frac{\partial W}{\partial \mathbf{F}}(\nabla_{\mathbf{x}}\mathbf{u})\mathbf{n}\, dS_{\mathbf{x}}. \tag{4.1}$$

Next observe that since, for each $\mathbf{x}_0 \in \Omega$ and $\lambda > 0$,

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \lambda \mathbf{x} \text{ for } \mathbf{x} \in \partial\Omega,$$

it follows that

$$\nabla_{\mathbf{x}}[\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) - \lambda \mathbf{x}] = \frac{\partial[\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) - \lambda \mathbf{x}]}{\partial \mathbf{n}} \otimes \mathbf{n}(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \partial\Omega, \tag{4.2}$$

where $\mathbf{n}(\mathbf{x})$ is the (outward) unit normal to $\partial\Omega$ at $\mathbf{x}$. (See, e.g. [8] for a proof of (4.2).)

Lemma 4.1. *Let (M1)–(M3) hold and for each $\mathbf{x}_0 \in \Omega$ and $\lambda > 0$ let*

$$\begin{aligned} \mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) &= \lambda \mathbf{x} + \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda), \\ \mathbf{u}^h(\mathbf{x}, \lambda) &= \lambda \mathbf{x}. \end{aligned} \tag{4.3}$$

Then, for $\lambda > \lambda_{\text{crit}}$ the difference in energies between the maps $\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)$ and $\mathbf{u}^h(\cdot, \lambda)$

$$\Delta E := E(\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)) - E(\mathbf{u}^h(\cdot, \lambda))$$

is given by

$$\Delta E = -(\lambda - \lambda_{\text{crit}})^2 \frac{1}{2n} \int_{\partial\Omega} \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) \nabla_{\mathbf{x}} \dot{\mathbf{w}} : \mathbb{C}[\nabla_{\mathbf{x}} \dot{\mathbf{w}}] \, dS_{\mathbf{x}} + o(|\lambda - \lambda_{\text{crit}}|^2), \tag{4.4}$$

where

$$\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0) = \frac{\partial}{\partial \lambda} \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda) \Big|_{\lambda = \lambda_{\text{crit}}} \qquad \text{and} \qquad \mathbb{C} = \frac{\partial^2 W}{\partial \mathbf{F}^2}(\lambda_{\text{crit}} \mathbf{I}), \tag{4.5}$$

is the elasticity tensor at the (stressed) configuration $\mathbf{u}^h(\mathbf{x}, \lambda) \equiv \lambda_{\text{crit}} \mathbf{x}$.

Proof. First note that from the definition of $\mathbf{w}$, (4.2) and Hypotheses (M1) and (M2) it follows that, for each $\mathbf{x}_0 \in \Omega$,

$$\begin{aligned} &(i)\ \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda) = \mathbf{0} \quad \text{for all } \mathbf{x} \in \partial\Omega \text{ and } \lambda > 0, \\ &(ii)\ \nabla_{\mathbf{x}} \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda) = \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \otimes \mathbf{n}, \ \text{for all } \mathbf{x} \in \partial\Omega \text{ and } \lambda > 0, \\ &(iii)\ \mathbf{w}(\mathbf{x}, \mathbf{x}_0, \lambda_{\text{crit}}) \equiv \mathbf{0} \text{ for } \mathbf{x} \in \overline{\Omega} \backslash \{\mathbf{x}_0\}. \end{aligned} \tag{4.6}$$

By (4.1) and (4.3)

$$\begin{aligned} n\Delta E = \int_{\partial\Omega} & \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) \left[W(\lambda \mathbf{I} + \nabla_{\mathbf{x}} \mathbf{w}) - W(\lambda \mathbf{I}) \right] + \\ & (\mathbf{w} - (\nabla_{\mathbf{x}} \mathbf{w})[\mathbf{x} - \mathbf{x}_0]) \cdot \frac{\partial W}{\partial \mathbf{F}}(\lambda \mathbf{I} + \nabla_{\mathbf{x}} \mathbf{w}) \mathbf{n} + \\ & (\mathbf{c} + \lambda \mathbf{x}_0) \cdot \left[\frac{\partial W}{\partial \mathbf{F}}(\lambda \mathbf{I} + \nabla_{\mathbf{x}} \mathbf{w}) - \frac{\partial W}{\partial \mathbf{F}}(\lambda \mathbf{I}) \right] \mathbf{n} \, dS_{\mathbf{x}} \end{aligned}$$

and hence, by (4.6)(i) and the choice $\mathbf{c} = -\lambda\mathbf{x}_0$, we obtain

$$\begin{aligned} n\Delta E = \int_{\partial\Omega} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)\,[W(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}) - W(\lambda\mathbf{I})] \\ - ((\nabla_{\mathbf{x}}\mathbf{w})[\mathbf{x}-\mathbf{x}_0])\cdot\frac{\partial W}{\partial\mathbf{F}}(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w})\mathbf{n}\,dS_{\mathbf{x}}. \end{aligned} \tag{4.7}$$

Therefore, by (4.6)(ii), Equation (4.7) can be rewritten as

$$n\Delta E = \int_{\partial\Omega} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)\left[W(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}) - W(\lambda\mathbf{I}) - \frac{\partial W}{\partial\mathbf{F}}(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}) : \left(\frac{\partial\mathbf{w}}{\partial\mathbf{n}}\otimes\mathbf{n}\right)\right] dS. \tag{4.8}$$

In order to finish the proof, for each $\mathbf{x}_0 \in \Omega$ and $\mathbf{x} \in \overline{\Omega}\backslash\{\mathbf{x}_0\}$ define

$$\Phi(\lambda) = W(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}) - W(\lambda\mathbf{I}) - \nabla_{\mathbf{x}}\mathbf{w} : \frac{\partial W}{\partial\mathbf{F}}(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}).$$

Then

$$\begin{aligned} \Phi'(\lambda) = \frac{\partial W}{\partial\mathbf{F}}(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}) : (\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}') - \nabla_{\mathbf{x}}\mathbf{w}' : \frac{\partial W}{\partial\mathbf{F}}(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}) \\ - \frac{\partial W}{\partial\mathbf{F}}(\lambda\mathbf{I}) : \mathbf{I} - \nabla_{\mathbf{x}}\mathbf{w} : \frac{\partial^2 W}{\partial\mathbf{F}^2}(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w})[\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}'] \end{aligned}$$

and hence

$$\Phi'(\lambda) = \left[\frac{\partial W}{\partial\mathbf{F}}(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}) - \frac{\partial W}{\partial\mathbf{F}}(\lambda\mathbf{I})\right] : \mathbf{I} - \nabla_{\mathbf{x}}\mathbf{w} : \frac{\partial^2 W}{\partial\mathbf{F}^2}(\lambda\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w})[\mathbf{I}+\nabla_{\mathbf{x}}\mathbf{w}'],$$

where "prime" denotes the derivative with respect to λ. It is clear from (4.6)(iii) that, for $\mathbf{x} \in \Omega\backslash\{\mathbf{x}_0\}$,

$$\Phi(\lambda_{\text{crit}}) = 0, \qquad \Phi'(\lambda_{\text{crit}}) = 0.$$

In addition, it is not difficult to see that

$$\Phi''(\lambda_{\text{crit}}) = -\nabla_{\mathbf{x}}\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0) : \mathbb{C}[\nabla_{\mathbf{x}}\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0)],$$

where $\mathbb{C}$ is given by $(4.5)_2$ and $\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0) = \mathbf{w}'(\mathbf{x},\mathbf{x}_0,\lambda_{\text{crit}})$. Therefore, for each $\mathbf{x}_0 \in \Omega$ and $\mathbf{x} \in \overline{\Omega}\backslash\{\mathbf{x}_0\}$

$$\Phi(\lambda) = -\frac{1}{2}(\lambda-\lambda_{\text{crit}})^2\nabla_{\mathbf{x}}\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0) : \mathbb{C}[\nabla_{\mathbf{x}}\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0)] + o(|\lambda-\lambda_{\text{crit}}|^2).$$

Finally, from Hypothesis (M2) it follows that $\mathbf{w}(\cdot,\mathbf{x}_0,\lambda) \to \mathbf{0}$ in $C^2(\overline{\Omega}\backslash B_\delta(\mathbf{x}_0))$ as $\lambda \to \lambda_{\text{crit}}^+$ for any sufficiently small $\delta > 0$. Thus, in particular, the above expansion is uniform for $\mathbf{x} \in \partial\Omega$. Equation (4.4) now follows from the above expansion, (4.8), and (4.6)(ii). □

Remark. It follows from (4.6)(ii) that an alternative form for (4.4) is

$$\Delta E = \frac{-(\lambda-\lambda_{\text{crit}})^2}{2n}\int_{\partial\Omega} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)\left[\frac{\partial\dot{\mathbf{w}}}{\partial\mathbf{n}}\otimes\mathbf{n}\right] : \mathbb{C}\left[\frac{\partial\dot{\mathbf{w}}}{\partial\mathbf{n}}\otimes\mathbf{n}\right] dS + o(|\lambda-\lambda_{\text{crit}}|^2).$$

5. Asymptotically radial maps

In view of Lemma 4.1 we have an expansion for the energy drop ΔE due to the introduction of a hole at the point $\mathbf{x}_0 \in \Omega$, which to lowest order depends on a boundary integral that involves $\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0)$. In this section we derive an explicit form for $\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0)$ for $\mathbf{x}$ near $\partial\Omega$, based on the postulate that for each $\mathbf{x}_0 \in \Omega$ the family of maps $\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda)$ is asymptotically radial. Roughly speaking, this postulate asserts that if $\mathbf{x}_0 \in \Omega$ is fixed then for λ arbitrarily close to λ_{crit} and for $\mathbf{x}$ close to $\mathbf{x}_0$, the map $\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda)$ behaves like one of the radial cavitation solutions (described in Section 2) centred on $\mathbf{x}_0$.

To present the postulate, first recall from Section 2 that $\mathbf{u}^{(r)}(\mathbf{x}, \lambda)$ denotes the radial minimiser that satisfies

$$\mathbf{u}^{(r)}(\mathbf{x}, \lambda) = \lambda\mathbf{x} \text{ for } \mathbf{x} \in \partial B.$$

Next note that by [1] or [15], for $\lambda > \lambda_{\text{crit}}$, $\mathbf{u}^{(r)}(\cdot, \lambda)$ may be extended as an equilibrium solution to all of $\mathbb{R}^n \backslash \{\mathbf{0}\}$. Notice also that, for each $\beta > 0$,

$$\beta\mathbf{u}^{(r)}\left(\frac{\mathbf{x} - \mathbf{x}_0}{\beta}, \lambda\right)$$

is a solution of the equilibrium equations (1.3) for $\mathbf{x} \neq \mathbf{x}_0$, and every radial minimiser in (a)(ii) of Proposition 2.1 is of this form for some choice of β (see e.g. [1] or part (iv) of Theorem 1.11 in [15]). Since $\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda)$ forms a single hole at $\mathbf{x}_0$ in the material, it is not unreasonable to assume that one can decompose such a deformation into a radial cavitating deformation centred at $\mathbf{x}_0$ followed by a deformation that introduces no new hole in the body, i.e. for $\mathbf{x} \neq \mathbf{x}_0$

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \mathbf{f}\left(\beta\mathbf{u}^{(r)}\left(\frac{\mathbf{x} - \mathbf{x}_0}{\beta}, \lambda\right), \beta, \lambda, \mathbf{x}_0\right) \tag{5.1}$$

for some map $\mathbf{f} = \mathbf{f}(\mathbf{z}, \beta, \lambda, \mathbf{x}_0)$.

We next show that if there exists a *smooth* map $\mathbf{f}$ such that $\mathbf{u}$ has the decomposition (5.1) then $\mathbf{u}$ does indeed have the same asymptotics as a radial minimizer. In order to show this we first set $\lambda = \lambda_{\text{crit}}$ in (5.1) to obtain

$$\lambda_{\text{crit}}\mathbf{x} = \mathbf{f}(\lambda_{\text{crit}}(\mathbf{x} - \mathbf{x}_0), \beta, \lambda_{\text{crit}}, \mathbf{x}_0)$$

and hence

$$\lambda_{\text{crit}}\mathbf{I} = \nabla_{\mathbf{z}}\mathbf{f}(\lambda_{\text{crit}}(\mathbf{x} - \mathbf{x}_0), \beta, \lambda_{\text{crit}}, \mathbf{x}_0)\lambda_{\text{crit}}\mathbf{I},$$

where $\nabla_{\mathbf{z}}\mathbf{f}$ denotes the gradient of $\mathbf{f}(\mathbf{z}, \beta, \lambda, \mathbf{x}_0)$ with respect to its first argument. Therefore

$$\mathbf{I} = \nabla_{\mathbf{z}}\mathbf{f}(\lambda_{\text{crit}}(\mathbf{x} - \mathbf{x}_0), \beta, \lambda_{\text{crit}}, \mathbf{x}_0). \tag{5.2}$$

Next differentiate (5.1) with respect to λ and use Proposition 2.1, Lemma 2.2, and (5.2) to obtain

$$\dot{\mathbf{u}}(\mathbf{x}, \mathbf{x}_0, \lambda_{\text{crit}}) = \beta^n \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} + \frac{\partial \mathbf{f}}{\partial \lambda}(\lambda_{\text{crit}}(\mathbf{x} - \mathbf{x}_0), \beta, \lambda_{\text{crit}}, \mathbf{x}_0). \tag{5.3}$$

Whence

$$\dot{\mathbf{u}}(\mathbf{x}, \mathbf{x}_0, \lambda_{\text{crit}}) = \mu \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} + \mathbf{r}(\mathbf{x}, \mathbf{x}_0, \mu) \quad \text{for } \ \mathbf{x} \neq \mathbf{x}_0,$$

where

$$\mathbf{r}(\mathbf{x}, \mathbf{x}_0, \mu) = \frac{\partial \mathbf{f}}{\partial \lambda}(\lambda_{\text{crit}}(\mathbf{x} - \mathbf{x}_0), \mu^{\frac{1}{n}}, \lambda_{\text{crit}}, \mathbf{x}_0), \quad \mu = \beta^n.$$

Suppose now that, given $\mathbf{x}_0 \in \Omega$, $\beta = \beta(\mathbf{x}_0)$ (and hence $\mu = \mu(\mathbf{x}_0)$) can be chosen so that $\mathbf{r}(\cdot, \mathbf{x}_0, \mu)$ is well-defined and $\mathbf{x} \mapsto \mathbf{r}(\mathbf{x}, \mathbf{x}_0, \mu)$ is a *smooth* function on **all** of Ω. Then by (5.1) and (5.3) we have that

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \lambda_{\text{crit}}\mathbf{x} + (\lambda - \lambda_{\text{crit}}) \left[\mu(\mathbf{x}_0) \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} + \mathbf{r}(\mathbf{x}, \mathbf{x}_0, \mu(\mathbf{x}_0)) \right] + o(|\lambda - \lambda_{\text{crit}}|^2).$$

The above derivation motivates our hypothesis concerning the asymptotic behaviour of our family of minimisers.

Hypothesis (M4). We assume that the family $\mathbf{u}(\cdot, \mathbf{x}_0, \lambda)$ is **asymptotically radial** i.e. that there exist functions $\mu \in C^2(\Omega)$ and $\tilde{\mathbf{v}} : \overline{\Omega} \times \Omega \to \mathbb{R}^n$, with $\tilde{\mathbf{v}}(\cdot, \mathbf{x}_0) \in C^2(\Omega; \mathbb{R}^n) \cap C(\overline{\Omega}; \mathbb{R}^n)$ such that for any compact set $S \subset \Omega$, and for $(\lambda - \lambda_{\text{crit}}) > 0$, the expansion

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda) = \lambda_{\text{crit}}\mathbf{x} + (\lambda - \lambda_{\text{crit}}) \left[\mu(\mathbf{x}_0) \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} + \tilde{\mathbf{v}}(\mathbf{x}, \mathbf{x}_0) \right] + o(|\lambda - \lambda_{\text{crit}}|^2) \tag{5.4}$$

holds uniformly for $(\mathbf{x}, \mathbf{x}_0) \in \overline{\Omega} \backslash B_\delta(\mathbf{x}_0) \times S$ for any $\delta > 0$ and sufficiently small.

Remark 5.1. 1. It is a straightforward consequence of (4.3) and our Hypothesis (M4), on differentiating (5.4) with respect to λ and setting $\lambda = \lambda_{\text{crit}}$, that

$$\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0) = \mu(\mathbf{x}_0) \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} + \tilde{\mathbf{v}}(\mathbf{x}, \mathbf{x}_0) - \mathbf{x}. \tag{5.5}$$

It is this expression that will eventually allow us to compute $\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0)$.

2. Note also that in the case when $\Omega = B$ and $\mathbf{x}_0 = \mathbf{0}$ it is a consequence of Hypothesis (M3) and Lemma 2.2 that $\mu(\mathbf{0}) = 1$.

The next two lemmas together demonstrate that, under Hypotheses (M1)–(M4), $\tilde{\mathbf{v}}(\cdot, \mathbf{x}_0)$ must be a smooth solution of (1.17) throughout Ω and satisfy the boundary condition (1.18).

Lemma 5.2. *Let (M1)–(M3) hold. Then for $\mathbf{x}_0 \in \Omega$, the right-sided derivative*

$$\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0) = \frac{\partial \mathbf{w}}{\partial \lambda}(\mathbf{x}, \mathbf{x}_0, \lambda)\bigg|_{\lambda=\lambda_{\text{crit}}}$$

satisfies $\dot{\mathbf{w}}(\cdot, \mathbf{x}_0) \in C^2(\Omega \backslash \{\mathbf{x}_0\}) \cap C(\overline{\Omega} \backslash \{\mathbf{x}_0\})$, the linear system of equations

$$\text{Div}_{\mathbf{x}}\, \mathbb{C}[\nabla_{\mathbf{x}} \dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0)] = \mathbf{0} \tag{5.6}$$

for $\mathbf{x} \in \Omega \backslash \{\mathbf{x}_0\}$, and the boundary condition

$$\dot{\mathbf{w}}(\mathbf{x}, \mathbf{x}_0) = \mathbf{0} \ \textit{for} \ \mathbf{x} \in \partial\Omega, \tag{5.7}$$

where $\mathbb{C}$ is the elasticity tensor given by (4.5).

Proof. First, note that (5.6) follows from (4.6)(ii) upon differentiation with respect to λ. To obtain (5.6), for $\mathbf{x} \neq \mathbf{x}_0$, from Hypotheses (M1) and (M2) we first evaluate (1.3) at $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{x}_0, \lambda)$, and then differentiate with respect to λ and set $\lambda = \lambda_{\text{crit}}$ to obtain

$$\mathbf{0} = \frac{\partial}{\partial \lambda} \text{Div}_{\mathbf{x}} \left[\frac{\partial W}{\partial \mathbf{F}}(\lambda \mathbf{I} + \nabla_{\mathbf{x}} \mathbf{w})\right]\bigg|_{\lambda=\lambda_{\text{crit}}}$$

and hence

$$\text{Div}_{\mathbf{x}} \left[\frac{\partial^2 W}{\partial \mathbf{F}^2}(\lambda_{\text{crit}} \mathbf{I})[\mathbf{I} + \nabla_{\mathbf{x}} \dot{\mathbf{w}}]\right] = \text{Div}_{\mathbf{x}}\, \mathbb{C}[\nabla_{\mathbf{x}} \dot{\mathbf{w}}] = \mathbf{0}$$

as required. □

Theorem 5.3. *Let (M1)–(M4) hold, then $\tilde{\mathbf{v}}(\cdot, \mathbf{x}_0)$ given in (5.5) satisfies the linear equations (5.6) throughout Ω and the boundary condition*

$$\tilde{\mathbf{v}}(\mathbf{x}, \mathbf{x}_0) = \mathbf{x} - \mu(\mathbf{x}_0) \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} \ \textit{for} \ \mathbf{x} \in \partial\Omega. \tag{5.8}$$

Proof. First, note that (5.8) follows from (5.5) and (5.7). Thus, the proof of this result will follow from (5.5) and Lemma 5.2 once we observe that

$$\mathbf{x} - \mu(\mathbf{x}_0) \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^n} \tag{5.9}$$

is a solution of (5.6) in $\Omega\backslash\{\mathbf{x}_0\}$. To see this first observe that $\mathbf{g}(\mathbf{x}) \equiv \mathbf{x}$ is clearly a solution of (5.6), hence it remains to consider the second term in (5.9). Next note that by Theorem 9.1 of the Appendix the elasticity tensor satisfies

$$\mathbb{C}[\mathbf{H}] = a\mathbf{H} + b\mathbf{H}^T + c(\operatorname{tr}\mathbf{H})\mathbf{I}, \tag{5.10}$$

where a, b, c are constants.

Define

$$\zeta(\mathbf{x}) = \begin{cases} \frac{1}{2-n}|\mathbf{x}-\mathbf{x}_0|^{2-n}, & \text{if } n > 2 \\ \log|\mathbf{x}-\mathbf{x}_0|, & \text{if } n = 2 \end{cases} \tag{5.11}$$

and note that

$$\nabla\zeta(\mathbf{x}) = \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n}, \quad \Delta\zeta(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \neq \mathbf{x}_0. \tag{5.12}$$

Consequently, by (5.10) and (5.12),

$$\operatorname{Div}\mathbb{C}[\nabla^2\zeta] = (a+b)\operatorname{Div}(\nabla^2\zeta) + c\operatorname{Div}((\Delta\zeta)\mathbf{I}) = (a+b)\nabla(\Delta\zeta) = \mathbf{0}$$

and therefore the claim of the lemma follows from the definition of $\tilde{\mathbf{v}}$. □

For later use we rewrite (5.5) in the form

$$\dot{\mathbf{w}}(\mathbf{x},\mathbf{x}_0) = \mu(\mathbf{x}_0)\left[\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n} + \mathbf{v}(\mathbf{x},\mathbf{x}_0)\right], \tag{5.13}$$

where

$$\mathbf{v}(\mathbf{x},\mathbf{x}_0) = \frac{1}{\mu(\mathbf{x}_0)}(\tilde{\mathbf{v}}(\mathbf{x},\mathbf{x}_0) - \mathbf{x}). \tag{5.14}$$

The next result follows from applying the above theorem to our definition of $\tilde{\mathbf{v}}$ given by (5.14)

Corollary 5.4. *Let (M1)–(M4) hold. Then* $\mathbf{v}(\cdot,\mathbf{x}_0)$ *satisfies the linear system of equations*

$$\operatorname{Div}_{\mathbf{x}}\mathbb{C}[\nabla_{\mathbf{x}}\mathbf{v}] = \mathbf{0} \ \textit{for } \mathbf{x} \in \Omega, \tag{5.15}$$

and the boundary condition

$$\mathbf{v}(\mathbf{x},\mathbf{x}_0) = -\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n} \ \textit{for } \mathbf{x} \in \partial\Omega. \tag{5.16}$$

Proof. This follows from applying the last theorem to our definition of $\mathbf{v}$ given by (5.14). Note in addition, that if our ellipticity hypothesis on the stored energy function (W1) holds then we have uniqueness of smooth solutions to the Dirichlet boundary value problem for (5.6) posed on all of Ω. □

In the next section we derive a precise form in terms of $\mathbf{v}$ for the coefficient of $(\lambda - \lambda_{\text{crit}})^2$ in the expansion of the energy difference ΔE given in Lemma 4.1.

6. Evaluation of the coefficient of $(\lambda-\lambda_{\text{crit}})^2$ in the expansion of ΔE

In this section we evaluate the coefficient of $(\lambda-\lambda_{\text{crit}})^2$ in the expansion of the energy difference ΔE as defined in Lemma 4.1. We write ∇ to denote $\nabla_{\mathbf{x}}$ and first observe that

$$\operatorname{div}(\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}](\mathbf{x}-\mathbf{x}_0)) = n\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}] + (\mathbf{x}-\mathbf{x}_0)\cdot\nabla(\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}])$$

and hence, by Lemma 4.1, Lemma 5.2 and the divergence theorem, it follows that the coefficient of $-\frac{1}{2n}(\lambda-\lambda_{\text{crit}})^2$ in the expansion of ΔE is given by

$$\begin{aligned} Q(\dot{\mathbf{w}}) = \int_{\partial\Omega}\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}]\,dS &= -\int_{\partial B_\epsilon}\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}]\,dS \\ &\quad + n\int_{\Omega\setminus B_\epsilon}\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}]\,d\mathbf{x} \\ &\quad + \int_{\Omega\setminus B_\epsilon}(\mathbf{x}-\mathbf{x}_0)\cdot\nabla(\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}])\,d\mathbf{x}, \end{aligned} \tag{6.1}$$

where $B_\epsilon = B_\epsilon(\mathbf{x}_0)$ is the (open) ball of radius ϵ centred at $\mathbf{x}_0$. Next we note that, since W is C^2

$$C^{jk}_{\beta\gamma} = C^{kj}_{\gamma\beta} = \mathbf{e}_\gamma\otimes\mathbf{e}_k:\mathbb{C}[\mathbf{e}_\beta\otimes\mathbf{e}_j] \tag{6.2}$$

for any orthonormal basis $\mathbf{e}_i$ of $\mathbb{R}^n$. Moreover, by Lemma 5.2,

$$0 = (\operatorname{Div}\mathbb{C}[\nabla\dot{\mathbf{w}}])_j = C^{jk}_{\beta\gamma}\dot{w}^k{}_{,\gamma\beta} \quad \text{on } \Omega\setminus\{\mathbf{x}_0\}. \tag{6.3}$$

Now, in view of (6.2), (6.3), and the divergence theorem,

$$\int_{\Omega\setminus B_\epsilon}\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}]\,d\mathbf{x} = \int_{\partial\Omega\cup\partial B_\epsilon}\dot{\mathbf{w}}\cdot\mathbb{C}[\nabla\dot{\mathbf{w}}]\mathbf{n}\,dS \tag{6.4}$$

and

$$\begin{aligned} \int_{\Omega\setminus B_\epsilon}(\mathbf{x}-\mathbf{x}_0)\cdot\nabla(\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}])\,d\mathbf{x} &= \int_{\Omega\setminus B_\epsilon}(x^\alpha-x_0^\alpha)[\dot{w}^j{}_{,\beta}\,C^{jk}_{\beta\gamma}\dot{w}^k{}_{,\gamma}]_{,\alpha}\,d\mathbf{x} \\ &= 2\int_{\Omega\setminus B_\epsilon}[(x^\alpha-x_0^\alpha)\dot{w}^j C^{jk}_{\beta\gamma}\dot{w}^k{}_{,\gamma\alpha}]_{,\beta}\,d\mathbf{x} \\ &= 2\int_{\partial\Omega\cup\partial B_\epsilon}(x^\alpha-x_0^\alpha)\dot{w}^j C^{jk}_{\beta\gamma}\dot{w}^k{}_{,\gamma\alpha}\,n^\beta dS. \end{aligned} \tag{6.5}$$

Hence by (6.1), (6.4), (6.5) and since $\dot{\mathbf{w}}$ vanishes on $\partial\Omega$, it follows that $Q(\dot{\mathbf{w}})$ is equal to

$$\int_{\partial B_\epsilon}\left[-\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}] + n\dot{\mathbf{w}}\cdot\mathbb{C}[\nabla\dot{\mathbf{w}}]\mathbf{n} + 2(x^\alpha-x_0^\alpha)\dot{w}^j C^{jk}_{\beta\gamma}\dot{w},^k_{\gamma\alpha}\,n^\beta\right]dS. \tag{6.6}$$

Theorem 6.1. *Let (M1)–(M4) hold. Then the coefficient of $(\lambda-\lambda_{\rm crit})^2$ in the expansion of ΔE (see (4.4)) is given by*

$$-\frac{1}{2n}Q(\dot{\mathbf{w}}) = -\frac{1}{2n}\int_{\partial\Omega}\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)\nabla\dot{\mathbf{w}}:\mathbb{C}[\nabla\dot{\mathbf{w}}]\,dS = \mu^2\kappa_n\left[\operatorname{div}_{\mathbf{x}}\mathbf{v}(\mathbf{x},\mathbf{x}_0)\right]\big|_{\mathbf{x}=\mathbf{x}_0}, \tag{6.7}$$

where $\mathbf{v}$ is given by Lemma 5.2, (5.13), (5.14), $\kappa_n = n\omega_n(a+b+c)/2$, the tensor $\mathbb{C}$ the constants a, b, c are related as in Theorem 9.1, and ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Proof. Our approach to obtaining (6.7) is to use (5.13), (5.14) and set

$$\dot{\mathbf{w}} = \mu[\mathbf{v}+\nabla\zeta] \tag{6.8}$$

in the right-hand side (6.6) (where ζ is as in (5.11)), expand the quadratic terms in $\dot{\mathbf{w}}$ in terms of $\mathbf{v}$ and ζ and evaluate $Q(\dot{\mathbf{w}})$ by evaluating the right-hand side of the integral (6.6) in the limit $\epsilon\to 0^+$. In evaluating the limiting value of this integral it is clear that the terms which are quadratic in $\mathbf{v}$ converge to zero as $\epsilon\to 0^+$ by the smoothness of $\mathbf{v}$. Moreover, the most singular terms in the expansion of the right-hand side of (6.6) are those which are quadratic in ζ (and its derivatives) and are given by

$$\int_{\partial B_\epsilon}\left[-\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)\nabla^2\zeta:\mathbb{C}[\nabla^2\zeta]+n\nabla\zeta\cdot\mathbb{C}[\nabla^2\zeta]\mathbf{n}+2(x^\alpha-x_0^\alpha)\zeta,_j\,C^{jk}_{\beta\gamma}\zeta,_{k\gamma\alpha}\,n^\beta\right]dS. \tag{6.9}$$

The derivatives of ζ for $\mathbf{x}\in\partial B_\epsilon(\mathbf{x}_0)$ satisfy

$$\nabla\zeta(\mathbf{x}) = \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n} = O\left(\frac{1}{\epsilon^{n-1}}\right), \tag{6.10}$$

$$\nabla^2\zeta(\mathbf{x}) = \frac{1}{|\mathbf{x}-\mathbf{x}_0|^n}\left(\mathbf{I}-\frac{n(\mathbf{x}-\mathbf{x}_0)\otimes(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^2}\right) = O\left(\frac{1}{\epsilon^n}\right), \tag{6.11}$$

$$\zeta,_{\alpha\beta\gamma}(\mathbf{x}) = O\left(\frac{1}{\epsilon^{n+1}}\right)$$

as $\epsilon\to 0^+$, where

$$\begin{aligned}\zeta,_{\alpha\beta\gamma}(\mathbf{x}) = \frac{-n}{|\mathbf{x}-\mathbf{x}_0|^{n+2}}\Big(&\delta^\alpha_\beta(x^\gamma-x_0^\gamma)+\delta^\gamma_\beta(x^\alpha-x_0^\alpha)+\delta^\alpha_\gamma(x^\beta-x_0^\beta)\\ &-(n+2)\frac{(x^\alpha-x_0^\alpha)(x^\beta-x_0^\beta)(x^\gamma-x_0^\gamma)}{|\mathbf{x}-\mathbf{x}_0|^2}\Big).\end{aligned} \tag{6.12}$$

From these equations it is clear that the singular integral terms in (6.9) are each of order ϵ^{-n} and hence their integrals over $\partial B(\mathbf{x}_0)$ are each of order ϵ^{-1} as $\epsilon \to 0^+$. The sum of these terms contribute nothing to the integral in the limit $\epsilon \to 0^+$ as can be directly verified or by observing that the original integral (the left-hand side of (6.1)) from which (6.6) (and hence (6.9)) was obtained is clearly finite and independent of ϵ.

Thus it remains to use (6.8) and evaluate the quadratic terms in the expansion of the right-hand side of (6.6) that include both ζ and $\mathbf{v}$. On noting that, for the domain $\Omega\backslash B_\epsilon$, the (outward) pointing normal $\mathbf{n}$ on ∂B_ϵ is given by $\mathbf{n} = -\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|}$, and the fact that $\mathbb{C}$ is symmetric it follows that the cross terms in question are given by

$$\begin{aligned} \mu^2 \int_{\partial B_\epsilon} & 2\epsilon\nabla^2\zeta : \mathbb{C}[\nabla\mathbf{v}] - n\left(\mathbf{v}\cdot\mathbb{C}[\nabla^2\zeta]\hat{\mathbf{n}} + \nabla\zeta\cdot\mathbb{C}[\nabla\mathbf{v}]\hat{\mathbf{n}}\right) \\ & - 2(x^\alpha - x_0^\alpha)[v^j C^{jk}_{\beta\gamma}\zeta_{,k\gamma\alpha}\,\hat{n}^\beta + \zeta_{,j}\, C^{jk}_{\beta\gamma} v,^{k}_{\gamma\alpha}\,\hat{n}^\beta]\, dS, \end{aligned} \tag{6.13}$$

where

$$\hat{\mathbf{n}} = -\mathbf{n} = \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|} \tag{6.14}$$

is the *outward* unit normal to the boundary of the ball $B_\epsilon(\mathbf{x}_0)$.

We next calculate the contributions from each of the five terms listed in (6.13).

Term (I). We first note that by (6.11) and (6.14)

$$\nabla^2\zeta : \mathbb{C}[\nabla\mathbf{v}] = |\mathbf{x}-\mathbf{x}_0|^{-n}\left[\mathbf{I} : \mathbb{C}[\nabla\mathbf{v}] - \frac{n}{|\mathbf{x}-\mathbf{x}_0|}\hat{\mathbf{n}}\cdot(\mathbb{C}[\nabla\mathbf{v}])^T(\mathbf{x}-\mathbf{x}_0)\right],$$

while the identity $\operatorname{div}(\mathbf{M}^T\mathbf{r}) = \mathbf{M} : \nabla\mathbf{r} + \mathbf{r}\cdot\operatorname{Div}\mathbf{M}$ and the fact that $\mathbf{v}$ satisfies the linearized elasticity equations (6.3) implies

$$\operatorname{div}\left[(\mathbb{C}[\nabla\mathbf{v}])^T(\mathbf{x}-\mathbf{x}_0)\right] = \mathbf{I} : \mathbb{C}[\nabla\mathbf{v}]. \tag{6.15}$$

Consequently, if we make use of the last two equations and the divergence theorem we find that

$$\begin{aligned} \int_{\partial B_\epsilon} \epsilon\nabla^2\zeta : \mathbb{C}[\nabla\mathbf{v}]\, dS &= \frac{1}{\epsilon^{n-1}}\left[\int_{\partial B_\epsilon} \mathbf{I} : \mathbb{C}[\nabla\mathbf{v}]\, dS - \frac{n}{\epsilon}\int_{\partial B_\epsilon} \hat{\mathbf{n}}\cdot(\mathbb{C}[\nabla\mathbf{v}])^T(\mathbf{x}-\mathbf{x}_0)\, dS\right] \\ &= \frac{1}{\epsilon^{n-1}}\int_{\partial B_\epsilon} \mathbf{I} : \mathbb{C}[\nabla\mathbf{v}]\, dS - n\frac{1}{\epsilon^n}\int_{B_\epsilon} \mathbf{I} : \mathbb{C}[\nabla\mathbf{v}]\, dS \\ &\to n\omega_n(\mathbf{I} : \mathbb{C}[\nabla\mathbf{v}(\mathbf{x}_0,\mathbf{x}_0)]) - n\omega_n(\mathbf{I} : \mathbb{C}[\nabla\mathbf{v}(\mathbf{x}_0,\mathbf{x}_0)]) = 0 \end{aligned}$$

as $\epsilon \to 0^+$. (Note that $\nabla\mathbf{v}(\mathbf{x}_0,\mathbf{x}_0) = \nabla_{\mathbf{x}}\mathbf{v}(\mathbf{x},\mathbf{x}_0)|_{\mathbf{x}=\mathbf{x}_0}$.)

Term (II). Using (6.11), (6.14), Theorem 9.1 of the Appendix, and the divergence theorem we obtain

$$\begin{aligned}\int_{\partial B_\epsilon} \mathbf{v}\cdot\mathbb{C}[\nabla^2\zeta]\hat{\mathbf{n}}\,dS &= \int_{\partial B_\epsilon} \mathbf{v}\cdot\left((a+b)[\nabla^2\zeta]\left[\frac{(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|}\right]\right)dS\\ &= (a+b)(1-n)\frac{1}{\epsilon^n}\int_{\partial B_\epsilon}\mathbf{v}\cdot\hat{\mathbf{n}}\,dS\\ &= (a+b)(1-n)\frac{1}{\epsilon^n}\int_{B_\epsilon}\operatorname{div}\mathbf{v}\,d\mathbf{x}\\ &\to (1-n)(a+b)\omega_n\operatorname{div}\mathbf{v}(\mathbf{x}_0)\end{aligned}$$

as $\epsilon\to 0^+$, where

$$\operatorname{div}\mathbf{v}(\mathbf{x}_0) = \operatorname{div}_{\mathbf{x}}\mathbf{v}(\mathbf{x},\mathbf{x}_0)\big|_{\mathbf{x}=\mathbf{x}_0}.$$

Term (III). In view of (6.10), (6.14), (6.15) and the divergence theorem

$$\begin{aligned}\int_{\partial B_\epsilon}\nabla\zeta\cdot\mathbb{C}[\nabla\mathbf{v}]\hat{\mathbf{n}}\,dS &= \frac{1}{\epsilon^n}\int_{\partial B_\epsilon}\hat{\mathbf{n}}\cdot(\mathbb{C}[\nabla\mathbf{v}])^T(\mathbf{x}-\mathbf{x}_0)\,dS\\ &= \frac{1}{\epsilon^n}\int_{B_\epsilon}\mathbf{I}:\mathbb{C}[\nabla\mathbf{v}]\,d\mathbf{x}\\ &\to \omega_n\mathbf{I}:\mathbb{C}[\nabla\mathbf{v}(\mathbf{x}_0,\mathbf{x}_0)]\end{aligned}$$

as $\epsilon\to 0^+$.

Term (IV). By Theorem 9.1 and $(5.12)_2$

$$C^{jk}_{\beta\gamma}\zeta,_{k\gamma\alpha} = (a\zeta,_{j\beta}+b\zeta,_{\beta j}+c(\Delta\zeta)\delta^j_\beta),_\alpha = (a+b)\zeta,_{j\beta\alpha}$$

while by (6.12) and (6.14),

$$(x^\alpha - x^\alpha_0)\zeta,_{j\beta\alpha}\hat{n}^\beta = \frac{n(n-1)}{|\mathbf{x}-\mathbf{x}_0|^{n+1}}(x^j - x^j_0) = \frac{n(n-1)}{|\mathbf{x}-\mathbf{x}_0|^n}\hat{n}^j.$$

Consequently, if we make use of the last two equations, and (6.14), we conclude that, as in Term (II),

$$\begin{aligned}\int_{\partial B_\epsilon}(x^\alpha - x^\alpha_0)v^j\left(C^{jk}_{\beta\gamma}\zeta,_{k\gamma\alpha}\right)\hat{n}^\beta dS &= n(n-1)(a+b)\frac{1}{\epsilon^n}\int_{\partial B_\epsilon}\mathbf{v}\cdot\hat{\mathbf{n}}\,dS\\ &\to n(n-1)(a+b)\omega_n\operatorname{div}\mathbf{v}(\mathbf{x}_0)\end{aligned}$$

as $\epsilon\to 0^+$.

Term (V). A straightforward estimate shows that

$$2\int_{\partial B_\epsilon}(x^\alpha - x_0^\alpha)[\zeta,_j C^{jk}_{\beta\gamma} v^k,_{\gamma\alpha}\hat{n}^\beta]\, dS \ \to 0 \ \text{ as } \ \epsilon \to 0^+.$$

Finally, we note that by Theorem 9.1,

$$\mathbf{I} : \mathbb{C}[\nabla \mathbf{v}(\mathbf{x}_0, \mathbf{x}_0)] = (a + b + nc)\,\mathrm{div}\,\mathbf{v}(\mathbf{x}_0)$$

and hence, using (6.13) and our computations for Terms (I)–(V), we conclude that

$$-\int_{\partial\Omega} \mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)\nabla\dot{\mathbf{w}} : \mathbb{C}[\nabla\dot{\mathbf{w}}]\, dS = 2\mu^2 n^2(a+b+c)\,\mathrm{div}\,\mathbf{v}(\mathbf{x}_0).$$

Consequently,

$$\frac{1}{2n}Q(\dot{\mathbf{w}}) = \frac{1}{2}\mu^2 n(a+b+c)\,\mathrm{div}\,\mathbf{v}(\mathbf{x}_0),$$

and (6.7) follows as claimed. □

7. An Example

In this section we apply the results we have developed to study the case when $\Omega = B$ the unit ball in $\mathbb{R}^3$. In this case we are able to explicitly solve for the function $\mathbf{v}$ in Corollary 5.4 and hence evaluate the coefficient (6.7) in the expansion of the energy drop ΔE explicitly as a series.

Proposition 7.1. *Fix $\mathbf{x}_0 \in B$ and consider the linear system of equations*

$$\mathrm{Div}\,\mathbb{C}[\nabla\mathbf{v}(\mathbf{x})] = \mathbf{0} \quad \text{for} \quad \mathbf{x} \in B, \tag{7.1}$$

where

$$\mathbb{C}[\nabla\mathbf{v}] = a\nabla\mathbf{v} + b(\nabla\mathbf{v})^T + c(\mathrm{tr}\,\nabla\mathbf{v})\mathbf{I}, \tag{7.2}$$

together with the boundary condition

$$\mathbf{v}(\mathbf{x}) = -\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^3} \quad \text{for} \quad \mathbf{x} \in \partial B. \tag{7.3}$$

Choose a rectangular coordinate system (x_1, x_2, x_3) on $\mathbb{R}^3$ so that $\mathbf{x}_0 \in B$ satisfies $\mathbf{x}_0 = (0, 0, |\mathbf{x}_0|)$ and define spherical coordinates ρ, ϕ, and θ by

$$x_1 = \rho(\cos\theta)(\sin\phi), \ \ x_2 = \rho(\sin\theta)(\sin\phi), \ \ x_3 = \rho(\cos\phi).$$

Then the solution of (7.1)–(7.3) is given by

$$\mathbf{v} = \sum_{k=1}^{\infty} \mathbf{v}_k^{(1)} + (1-\rho^2)\sum_{k=2}^{\infty} \mathbf{v}_k^{(2)}, \tag{7.4}$$

where

$$\mathbf{v}_k^{(1)} = \rho^k |\mathbf{x}_0|^{k-1}\big(P_k'(\cos\phi)\cos\theta, P_k'(\cos\phi)\sin\theta, kP_k(\cos\phi)\big), \tag{7.5}$$

P_k *denotes the* k^{th}*-Legendre polynomial, and*

$$\mathbf{v}_k^{(2)} = \rho^k \tau_k |\mathbf{x}_0|^{k-1}(0,0,P_k(\cos\phi)), \qquad \tau_k = \frac{1}{2}\frac{(2k+1)(b+c)}{(2a+b+c)k-(a+b+c)}. \tag{7.6}$$

Whence

$$\begin{aligned} \psi(|\mathbf{x}_0|) := [\operatorname{div}_{\mathbf{x}} \mathbf{v}(\mathbf{x},\mathbf{x}_0)]\big|_{\mathbf{x}=\mathbf{x}_0} &= \sum_{k=1}^{\infty} k[2(k-1)\tau_k - (2k+1)]|\mathbf{x}_0|^{2k-2} \\ &= \sum_{k=2}^{\infty} 2k(k-1)\tau_k|\mathbf{x}_0|^{2k-2} - \frac{3+|\mathbf{x}_0|^2}{(1-|\mathbf{x}_0|^2)^3}. \end{aligned} \tag{7.7}$$

Remarks. 1. Each of the functions $\mathbf{v}_k^{(i)}$ are solutions of the vector Laplace equation.

2. It is clear from $(7.7)_1$ that $\psi < 0$.

Proof. Since $\mathbb{C}$ is strongly elliptic standard results from elliptic theory yield, for each $\mathbf{x}_0 \in B$, the existence of a unique solution $\mathbf{v}$ to (7.1)–(7.3) and moreover $\mathbf{v}(\cdot,\mathbf{x}_0) \in C^\infty(\overline{B};\mathbb{R}^3)$. However, the identity

$$\operatorname{Div}(\nabla\mathbf{v})^T = \nabla(\operatorname{div}\mathbf{v})$$

implies that (7.1) and (7.2) are equivalent to

$$\operatorname{Div}\,\mathbb{L}[\nabla\mathbf{v}(\mathbf{x})] = \mathbf{0} \quad \text{for} \quad \mathbf{x} \in B, \tag{7.8}$$

where

$$\mathbb{L}[\nabla\mathbf{v}] = a\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right) + \frac{1}{2}(b+c-a)\left(\operatorname{tr}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\right)\mathbf{I}, \tag{7.9}$$

which are the equations of linear elasticity (at a homogeneous, isotropic, stress-free reference configuration).

The series solution (7.4)–(7.6) to (7.3), (7.8), and (7.9) is due to Lord Kelvin (Thomson [17]) and can be found, for example, in Sections 181–183 of [9] or Chapter 8 in [10]. The expression (7.7) follows from (7.4)–(7.6), the identity

$$\frac{3+|\mathbf{x}_0|^2}{(1-|\mathbf{x}_0|^2)^3} = \sum_{k=1}^{\infty} k(2k+1)|\mathbf{x}_0|^{2k-2}, \tag{7.10}$$

and a straightforward calculation. □

Example 7.2. In the case $b+c=0$, the system (7.1) reduces to the vector Laplace equation and the corresponding solution $\mathbf{v}$ satisfies

$$[\operatorname{div}_{\mathbf{x}} \mathbf{v}(\mathbf{x}, \mathbf{x}_0)]|_{\mathbf{x}=\mathbf{x}_0} = -\frac{3+|\mathbf{x}_0|^2}{(1-|\mathbf{x}_0|^2)^3}\ .$$

Recall from Theorem 6.1 and Lemma 4.1 that the right-hand side of (6.7) is the coefficient of $(\lambda - \lambda_{\text{crit}})^2$ in the expansion of the energy drop ΔE. The next result concerns properties of this coefficient.

Corollary 7.3. *Let* $\mathbf{e}$ *be a unit vector. Define*

$$\psi(t) := [\operatorname{div}_{\mathbf{x}} \mathbf{v}(\mathbf{x}, t\mathbf{e})]|_{\mathbf{x}=t\mathbf{e}}\,. \tag{7.11}$$

Then

(i) $\psi(0) = -3$*;*

(ii) $\dot{\psi}(0) = 0$*;*

(iii) $\ddot{\psi}(0) = -60a/(3a+b+c)$*;*

(iv) ψ *and all of its derivatives are negative on* $(0,1)$*;*

(v) $\psi(t) \le -C(1-t)^{-3}$ *on* $[0,1)$ *for some constant* $C>0$*.*

Here a*,* b*,* c *are as given in Theorem 9.1, and "dot" denotes differentiation with respect to* t*.*

Proof. Equations (i)–(iii) follow directly from $(7.7)_1$. To prove (iv) we note that, by the strong ellipticity of $\mathbb{C}$, $a>0$ and $a+b+c>0$, which together with $(7.6)_2$ implies that each of the terms in the power series $(7.7)_1$ is negative. To obtain property (v), first suppose that $b+c \le 0$. Then $\tau_k \le 0$ and hence by $(7.7)_2$

$$\psi(t) \le -\frac{3+t^2}{(1-t^2)^3},$$

which implies (v). If instead $b+c>0$ then we first note that the coefficient of $|\mathbf{x}_0|^{2k-2}$ in the power series $(7.7)_1$ is

$$\begin{aligned} k(2k+1)\left[\frac{(k-1)(b+c)}{(2k-1)a+(k-1)(b+c)}-1\right] &= -k(2k+1)\frac{a}{a+\frac{k-1}{2k-1}(b+c)} \\ &\le -k(2k+1)\frac{a}{a+b+c} \end{aligned}$$

and hence we find, with the aid of (7.10), that

$$\frac{a+b+c}{a}\psi(t)\le -\sum_{k=1}^{\infty}k(2k+1)t^{2k-2}=-\frac{3+t^2}{(1-t^2)^3},$$

which once again yields (v). □

Remark 7.4. An alternative approach to Corollary 7.3(i)–(iii), which does not require the explicit solution to (7.1)–(7.3) and which therefore also easily generalizes to n-dimensions, is available. Let B be the unit ball in $\mathbb{R}^n$, $\mathbf{x}_0=t\mathbf{e}$, where $\mathbf{e}$ is a fixed unit vector, and define $\mathbf{g}(\mathbf{x},t)=\mathbf{v}(\mathbf{x},t\mathbf{e})$. Then if we let $\mathbf{x}_0=t\mathbf{e}$ in (5.15) we find, upon differentiation with respect to t that each of the functions $\mathbf{g}(\cdot,0)$, $\dot{\mathbf{g}}(\cdot,0)$ and $\ddot{\mathbf{g}}(\cdot,0)$ is a solution of (5.15). In addition, if we let $\mathbf{x}_0=t\mathbf{e}$ in (5.16) a similar computation shows that the following functions satisfy the boundary conditions

$$\left.\begin{aligned} &\mathbf{g}(\mathbf{x},0)=-\mathbf{x}, \qquad \dot{\mathbf{g}}(\mathbf{x},0)=\mathbf{e}-n(\mathbf{x}\cdot\mathbf{e})\mathbf{x}, \\ &\ddot{\mathbf{g}}(\mathbf{x},0)=2n(\mathbf{x}\cdot\mathbf{e})\mathbf{e}+n\mathbf{x}-n(n+2)(\mathbf{x}\cdot\mathbf{e})^2\mathbf{x} \end{aligned}\right\}\text{ for } \mathbf{x}\in\partial B.$$

A straightforward (though tedious) computation shows that the (unique) solutions of (5.15) that satisfy these boundary conditions are:

$$\begin{aligned} &\mathbf{g}(\mathbf{x},0)=-\mathbf{x}, \qquad \dot{\mathbf{g}}(\mathbf{x},0)=\mathbf{e}-n(\mathbf{x}\cdot\mathbf{e})\mathbf{x}+\frac{\xi_1}{2}(|\mathbf{x}|^2-1)\mathbf{e}, \\ &\ddot{\mathbf{g}}(\mathbf{x},0)=2n(\mathbf{x}\cdot\mathbf{e})\mathbf{e}+n\mathbf{x}-n(n+2)(\mathbf{x}\cdot\mathbf{e})^2\mathbf{x}+\frac{\xi_2}{2}(|\mathbf{x}|^2-1)(\mathbf{x}\cdot\mathbf{e})\mathbf{e}+\frac{\xi_3}{2}(|\mathbf{x}|^2-1)\mathbf{x}, \end{aligned}$$

where

$$\xi_1=\frac{n[2a+(n+1)(b+c)]}{an+b+c},\ \xi_2=\frac{2n(n+2)[2a+(n+2)(b+c)]}{a(n+2)+2(b+c)},\ \xi_3=\frac{2na-(b+c)\xi_2}{(n+2)(a+b+c)}.$$

Consequently, when B is the unit ball in $\mathbb{R}^n$ the function ψ in Corollary 7.3 satisfies

$$\psi(0)=-n, \qquad \dot{\psi}(0)=0, \qquad \ddot{\psi}(0)=-\frac{2n(n+2)(n-1)a}{an+b+c}.$$

Now, if (M1)-(M4) hold, then by Theorem 6.1 and Lemma 4.1 it follows that

$$\Delta E = (\lambda - \lambda_{\rm crit})^2 \kappa_3 \mu(|\mathbf{x}_0|)^2 \left[\mathrm{div}_{\mathbf{x}}\, \mathbf{v}(\mathbf{x}, \mathbf{x}_0)\right]\big|_{\mathbf{x}=\mathbf{x}_0} + o(|\lambda - \lambda_{\rm crit}|^2),$$

where we have used the symmetry of the domain B and the boundary condition to write $\mu(\mathbf{x}_0) = \mu(|\mathbf{x}_0|)$ in the expansion (5.4). If we set $\mathbf{x}_0 = t\mathbf{e}$ and $\psi(t) = \left[\mathrm{div}_{\mathbf{x}}\, \mathbf{v}(\mathbf{x}, t\mathbf{e})\right]\big|_{\mathbf{x}=t\mathbf{e}}$ and make use of (7.11) we obtain

$$\Delta E = (\lambda - \lambda_{\rm crit})^2 \kappa_3 \mu(t)^2 \psi(t) + o(|\lambda - \lambda_{\rm crit}|^2). \tag{7.12}$$

From the results of this section we are able to use (7.12) to prove the following local stability result that the energetically optimal location for a hole to form near the centre of the ball B is at the centre.

Proposition 7.5. *Suppose that (M1)–(M4) hold and suppose further that $\mu(t)$ satisfies either*

(i) $\dot{\mu}(0) < 0$ *or*

(ii) $\dot{\mu}(0) = 0$ *and* $\ddot{\mu}(0) < -10a/(3a + b + c)$.

Then the energy drop ΔE in Lemma 4.1 has a relative maximum at the origin.

Proof. The proof of this result follows in a straightforward manner from (7.12) on calculating the derivatives of $\Psi(t) = \mu^2(t)\psi(t)$ to obtain

(1) $\Psi(0) = \mu^2(0)\phi(0) = -3,$

(2) $\dot{\Psi}(0) = 2\mu(0)\dot{\mu}(0)\psi(0) + \mu^2(0)\dot{\psi}(0) = -6\dot{\mu}(0),$

where we have used Proposition 7.3. Similarly,

(3) $\ddot{\Psi}(0) = -6[\dot{\mu}^2(0) + \ddot{\mu}(0)] - 60a/(3a + b + c).$

The claim of the proposition now follows from (1)–(3) on observing that a sufficient condition for Ψ to have a (negative) local minimum at $t = 0$ is that either $\dot{\Psi}(0) > 0$ or $\dot{\Psi}(0) = 0$ and $\ddot{\Psi}(0) > 0$. (We have also used the observation in Remark 5.1.2 that $\mu(0) = 1$.) □

8. Concluding Remarks and Results

The results in this paper are we feel a first step towards unifying recent studies in nonlinear variational problems on the existence of discontinuous minimisers with classical engineering approaches to modelling fracture using linear fracture mechanics. The main difficulty in applying the approach to finding initiation points for discontinuities is that the function $\mu(\mathbf{x}_0)$ is not explicitly known.

At present the only information we have obtained concerning $\mu(\mathbf{x}_0)$ is the following preliminary estimate, which is a consequence of the scaling property of homogeneous elastic materials.

Proposition 8.1. *Let B be the unit ball in $\mathbb{R}^3$, $\mathbf{x}_0 \in B$, and set $\mathbf{x}_0 = t\mathbf{e}$, where $\mathbf{e}$ is a unit vector. Suppose that Hypotheses (M1)–(M4) are satisfied. Define*

$$\Psi(t) := \mu(t)^2\psi(t), \tag{8.1}$$

where ψ is given by (7.7) and $\mu(t) = \mu(t\mathbf{e})$ is independent of $\mathbf{e}$ by the symmetry of the domain B. Then

$$t \quad \mapsto \quad \frac{\Psi(t)}{(1+t)^3} \qquad \textit{is monotone increasing on } [0,1), \tag{8.2}$$

$$t \quad \mapsto \quad \frac{\Psi(t)}{(1-t)^3} \qquad \textit{is monotone decreasing on } [0,1). \tag{8.3}$$

Consequently,

$$\frac{3(1-t)^3}{-\psi(t)} \leq [\mu(t)]^2 \leq \frac{3(1+t)^3}{-\psi(t)} \tag{8.4}$$

and hence, in particular,

$$\mu(t) = O(|1-t|^{3/2}) \quad \text{as} \quad t \to 1^-. \tag{8.5}$$

Proof. We first observe that (8.4) follows from (8.2) and (8.3) since, as noted in Remark 5.1.2, $\mu(\mathbf{0}) = 1$ and hence, by Corollary 7.3(i), $\Psi(0) = \mu(0)^2\psi(0) = -3$. Also, (8.5) follows directly from (8.4) and Corollary 7.3(v).

In order to obtain (8.2) and (8.3) we let $\mathbf{x}_0 = t\mathbf{e} \in B \subset \mathbb{R}^3$, where $\mathbf{e}$ is a unit vector. For $\sigma \in [0,1)$ and $\lambda > \lambda_{\rm crit}$ let $\mathbf{u}_\sigma(\mathbf{x},\lambda)$ denote the (unique) global minimiser of E in $\mathcal{A}(\sigma\mathbf{e})$ that satisfies $\mathbf{u}_\sigma(\mathbf{x},\lambda) \equiv \mathbf{u}^h(\mathbf{x},\lambda) := \lambda\mathbf{x}$ on ∂B. Then in view of (7.7), (8.1), Lemma 4.1, and Theorem 6.1

$$E(\mathbf{u}_\sigma(\cdot,\lambda)) - E(\mathbf{u}^h(\cdot,\lambda)) = \Psi(\sigma)\kappa_3(\lambda-\lambda_{\rm crit})^2 + o(|\lambda-\lambda_{\rm crit}|^2), \tag{8.6}$$

where $\kappa_3 = \frac{3}{2}\omega_3(a+b+c) > 0$, in view of the strong ellipticity of $\mathbb{C}$.

For $s \in [t, 1)$ consider the rescaled deformations

$$\mathbf{u}(\mathbf{x}, s, \lambda) := \begin{cases} \frac{t+1}{s+1}\mathbf{u}_s(\frac{s+1}{t+1}\mathbf{x} + \frac{s-t}{t+1}\mathbf{e}, \lambda) - \frac{s-t}{s+1}\mathbf{e}, & \text{if} \quad |\mathbf{x} + \lambda\frac{s-t}{s+1}\mathbf{e}| \leq \frac{t+1}{s+1}, \\ \lambda\mathbf{x}, & \text{otherwise.} \end{cases}$$

Clearly, $\mathbf{u}(\cdot, s, \lambda) \in \mathcal{A}(t\mathbf{e})$. Moreover, the change of variables

$$\mathbf{z} := \frac{s+1}{t+1}\mathbf{x} + \frac{s-t}{t+1}\mathbf{e}$$

yields

$$\begin{aligned} \int_B [W(\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x}, s, \lambda)) - W(\lambda\mathbf{I})]\, d\mathbf{x} &= \int_{B(-\frac{s-t}{s+1}\mathbf{e}, \frac{t+1}{s+1})} [W(\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x}, s, \lambda)) - W(\lambda\mathbf{I})]\, d\mathbf{x} \\ &= \frac{(t+1)^3}{(s+1)^3}\int_B [W(\nabla_{\mathbf{z}}\mathbf{u}_s(\mathbf{z}, \lambda)) - W(\lambda\mathbf{I})]\, d\mathbf{z} \end{aligned}$$

and hence, in view of (8.6) with $\sigma = s$,

$$E(\mathbf{u}(\cdot, s, \lambda)) - E(\mathbf{u}^h(\cdot, \lambda)) = \frac{(t+1)^3}{(s+1)^3}\Psi(s)\kappa_3(\lambda - \lambda_{\text{crit}})^2 + o(|\lambda - \lambda_{\text{crit}}|^2). \tag{8.7}$$

Now, by hypothesis, for every $\lambda > \lambda_{\text{crit}}$, $\mathbf{u}_t$ is the minimizer of E in $\mathcal{A}(t\mathbf{e})$. Consequently,

$$E(\mathbf{u}_t(\cdot, \lambda)) - E(\mathbf{u}^h(\cdot, \lambda)) \leq E(\mathbf{u}(\cdot, s, \lambda)) - E(\mathbf{u}^h(\cdot, \lambda)) \tag{8.8}$$

for any $s \in [t, 1)$. Therefore, by (8.6) with $\sigma = t$, (8.7), and (8.8)

$$\Psi(t) \leq \frac{(t+1)^3}{(s+1)^3}\Psi(s) \quad \text{for} \quad t \leq s$$

or equivalently (8.2) is satisfied.

Similarly, if $t > 0$ and $s \in [0, t)$ the rescaled deformations

$$\mathbf{u}(\mathbf{x}, s, \lambda) := \begin{cases} \frac{1-t}{1-s}\mathbf{u}_s(\frac{1-s}{1-t}\mathbf{x} + \frac{s-t}{1-t}\mathbf{e}, \lambda) - \lambda\frac{s-t}{1-s}\mathbf{e}, & \text{if} \quad |\mathbf{x} + \frac{s-t}{1-s}\mathbf{e}| \leq \frac{1-t}{1-s}, \\ \lambda\mathbf{x}, & \text{otherwise} \end{cases}$$

can be used to show that

$$\Psi(t) \leq \frac{(1-t)^3}{(1-s)^3}\Psi(s) \quad \text{for} \quad s \leq t$$

or equivalently (8.3) is satisfied. □

9. Appendix

In this section we prove the following result

Theorem 9.1. *Let $W \in C^2(M_+^{n\times n};\mathbb{R})$ be an isotropic, frame indifferent, stored energy function and let $\lambda > 0$. Define*

$$\mathbb{C} = \mathbb{A}(\lambda\mathbf{I}), \quad \mathbb{A}(\mathbf{F}) = \frac{\partial^2 W}{\partial \mathbf{F}^2}(\mathbf{F}).$$

Then there are real constants a, b, and c such that

$$\mathbb{C}[\mathbf{H}] = a\mathbf{H} + b\mathbf{H}^T + c(\operatorname{tr}\mathbf{H})\mathbf{I} \tag{9.1}$$

for all $\mathbf{H} \in M^{n\times n}$. Moreover, $\mathbb{C}$ satisfies the strong ellipticity condition

$$\mathbf{m}\otimes\mathbf{n} : \mathbb{C}[\mathbf{m}\otimes\mathbf{n}] > 0,$$

for all $\mathbf{m}, \mathbf{n} \in \mathbb{R}^n$ with $\mathbf{m} \neq \mathbf{0}$ and $\mathbf{n} \neq \mathbf{0}$, if and only if

$$a > 0 \quad \textit{and} \quad a + b + c > 0.$$

The proof of the theorem uses the following well-known result (see, e.g. [7]).

Proposition 9.2. *Let $\operatorname{Sym}^{n\times n}$ denote those $n \times n$ matrices that are symmetric and suppose that $\mathbb{B} : \operatorname{Sym}^{n\times n} \to \operatorname{Sym}^{n\times n}$ is a linear function that is* symmetric, *viz*

$$\mathbf{E} : \mathbb{B}[\mathbf{F}] = \mathbf{F} : \mathbb{B}[\mathbf{E}] \quad \textit{for all} \quad \mathbf{E}, \mathbf{F} \in \operatorname{Sym}^{n\times n}$$

and isotropic, *viz*

$$\mathbb{B}[\mathbf{Q}\mathbf{E}\mathbf{Q}^T] = \mathbf{Q}\mathbb{B}[\mathbf{E}]\mathbf{Q}^T \quad \textit{for all} \quad \mathbf{E} \in \operatorname{Sym}^{n\times n},\ \mathbf{Q} \in \mathrm{SO}(n),$$

the special orthogonal matrices ($\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$, $\det\mathbf{Q} = 1$). Then there are real constants ν and μ such that

$$\mathbb{B}[\mathbf{E}] = 2\mu\mathbf{E} + \nu(\operatorname{tr}\mathbf{E})\mathbf{I} \quad \textit{for all} \quad \mathbf{E} \in \operatorname{Sym}^{n\times n}.$$

Proof of Theorem 9.1 (cf. [4]). From the frame indifference and isotropy of W we have, respectively,

$$W(\mathbf{F}) = W(\mathbf{Q}\mathbf{F}), \quad W(\mathbf{F}) = W(\mathbf{F}\mathbf{Q}^T) \tag{9.2}$$

for all $\mathbf{F} \in M_+^{n\times n}$ and $\mathbf{Q} \in \mathrm{SO}(n)$. If we differentiate (9.2), with respect to $\mathbf{F}$, in the direction $\mathbf{H} \in M^{n\times n}$ we find that

$$\mathbf{S}(\mathbf{F}) : \mathbf{H} = \mathbf{S}(\mathbf{QF}) : \mathbf{QH}, \quad \mathbf{S}(\mathbf{F}) : \mathbf{H} = \mathbf{S}(\mathbf{FQ}^T) : \mathbf{HQ}^T$$

and hence that

$$\mathbf{S}(\mathbf{QF}) = \mathbf{QS}(\mathbf{F}), \quad \mathbf{S}(\mathbf{FQ}^T) = \mathbf{S}(\mathbf{F})\mathbf{Q}^T \tag{9.3}$$

for all $\mathbf{F} \in M_+^{n\times n}$ and $\mathbf{Q} \in \mathrm{SO}(n)$, where

$$\mathbf{S}(\mathbf{F}) = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}).$$

If we then differentiate (9.3), with respect to $\mathbf{F}$, in the direction $\mathbf{H} \in M^{n\times n}$ we discover that

$$\mathbb{A}(\mathbf{QF})[\mathbf{QH}] = \mathbf{Q}\mathbb{A}(\mathbf{F})[\mathbf{H}], \quad \mathbb{A}(\mathbf{FQ}^T)[\mathbf{HQ}^T] = \mathbb{A}(\mathbf{F})[\mathbf{H}]\mathbf{Q}^T$$

and, consequently, if we combine these equations and set $\mathbf{F} = \lambda\mathbf{I}$ we conclude that

$$\mathbb{C}[\mathbf{QHQ}^T] = \mathbf{Q}\mathbb{C}[\mathbf{H}]\mathbf{Q}^T \quad \text{for all} \quad \mathbf{H} \in M^{n\times n}, \ \mathbf{Q} \in \mathrm{SO}(n). \tag{9.4}$$

Next, let $\mathbf{W}$ be a skew-symmetric $n\times n$ matrix and let $\mathbf{Q}(t)$ be the one-parameter family of special orthogonal matrices that satisfies

$$\mathbf{Q}'(t) = \tfrac{d}{dt}\mathbf{Q}(t) = \mathbf{WQ}(t), \quad \mathbf{Q}(0) = \mathbf{0}, \quad \text{and hence} \quad \mathbf{Q}'(0) = \mathbf{W}. \tag{9.5}$$

Then, if we set $\mathbf{Q} = \mathbf{Q}(t)$ in $(9.2)_1$, differentiate with respect to t, and take $t = 0$ we find, with the aid of (9.5), that $\mathbf{S}(\mathbf{F})\mathbf{F}^T : \mathbf{W} = 0$ for every such $\mathbf{W}$. Consequently,

$$\mathbf{S}(\mathbf{F})\mathbf{F}^T \in \mathrm{Sym}^{n\times n} \tag{9.6}$$

and hence, differentiating with respect to $\mathbf{F}$ in the direction $\mathbf{H} \in M^{n\times n}$,

$$\mathbb{A}(\mathbf{F})[\mathbf{H}]\mathbf{F}^T + \mathbf{S}(\mathbf{F})\mathbf{H}^T \in \mathrm{Sym}^{n\times n}. \tag{9.7}$$

Similarly, if we set $\mathbf{Q} = \mathbf{Q}(t)$ in $(9.3)_1$, differentiate with respect to t, and take $t = 0$ we find, with the aid of (9.5), that

$$\mathbb{A}(\mathbf{F})[\mathbf{WF}] = \mathbf{WS}(\mathbf{F}). \tag{9.8}$$

We note that, by (9.2), $\mathbf{S}(\lambda\mathbf{I}) = \mathbf{S}(\lambda\mathbf{QQ}^T) = \mathbf{QS}(\lambda\mathbf{I})\mathbf{Q}^T$; thus $\mathbf{S}(\lambda\mathbf{I})$ commutes with every element of $\mathrm{SO}(n)$. Moreover, by (9.6), $\mathbf{S}(\lambda\mathbf{I}) \in \mathrm{Sym}^{n\times n}$. A standard result from linear algebra then yields a real scalar p such that

$$\mathbf{S}(\lambda\mathbf{I}) = p\mathbf{I}. \tag{9.9}$$

Therefore, if we set $\mathbf{F} = \lambda\mathbf{I}$ in (9.7) and (9.8) we conclude, with the aid of (9.9), that

$$\begin{aligned} &\mathbb{C}[\mathbf{H}] + \tfrac{p}{\lambda}\mathbf{H}^T \in \mathrm{Sym}^{n\times n} \quad \text{for all} \quad \mathbf{H} \in M^{n\times n}, \\ &\mathbb{C}[\mathbf{W}] = p\mathbf{W} \quad \text{for all skew-symmetric} \quad \mathbf{W} \in M^{n\times n}, \end{aligned} \tag{9.10}$$

respectively.

Now define $\mathbb{B} : \mathrm{Sym}^{n\times n} \to M^{n\times n}$ by

$$\mathbb{B}[\mathbf{E}] = \mathbb{C}[\mathbf{E}] + \tfrac{p}{\lambda}\mathbf{E}.$$

Then, by $(9.10)_1$, the range of $\mathbb{B}$ is contained in $\mathrm{Sym}^{n\times n}$; equation (9.4) can be used to show that $\mathbb{B}$ isotropic; and, the fact that $\mathbb{C}$ is the second derivative of W implies that it and hence $\mathbb{B}$ is symmetric. Therefore, by Proposition 9.2, the are real constants μ and ν such that

$$\mathbb{C}[\mathbf{E}] + \tfrac{p}{\lambda}\mathbf{E} = 2\mu\mathbf{E} + \nu(\mathrm{tr}\,\mathbf{E})\mathbf{I} \quad \text{for every} \quad \mathbf{E} \in \mathrm{Sym}^{n\times n}. \tag{9.11}$$

Thus, given any $\mathbf{H} \in M^{n\times n}$, $\mathbf{H} = \frac{1}{2}(\mathbf{H} - \mathbf{H}^T) + \frac{1}{2}(\mathbf{H} + \mathbf{H}^T)$, where the first term is skew-symmetric and the second term is symmetric, and consequently, by $(9.10)_2$, (9.11), and the linearity of $\mathbb{C}$

$$\begin{aligned} \mathbb{C}[\mathbf{H}] &= \tfrac{1}{2}\mathbb{C}[\mathbf{H} - \mathbf{H}^T] + \tfrac{1}{2}\mathbb{C}[\mathbf{H} + \mathbf{H}^T] \\ &= \tfrac{p}{2}(\mathbf{H} - \mathbf{H}^T) + (\mu - \tfrac{p}{2\lambda})(\mathbf{H} + \mathbf{H}^T) + \nu(\mathrm{tr}\,\mathbf{H})\mathbf{I}, \end{aligned}$$

which implies the claimed representation (9.1) for the elasticity tensor $\mathbb{C}$.

Finally, we establish the restrictions on the coefficients a, b, c for strong ellipticity. Suppose that (9.1) holds, then on setting $\mathbf{H} = \mathbf{m} \otimes \mathbf{n} \neq \mathbf{0}$ in (9.1) we obtain

$$\mathbf{m} \otimes \mathbf{n} : \mathbb{C}[\mathbf{m} \otimes \mathbf{n}] = a|\mathbf{m}|^2|\mathbf{n}|^2 + (b + c)(\mathbf{m} \cdot \mathbf{n})^2. \tag{9.12}$$

If we set $\mathbf{m} \cdot \mathbf{n} = 0$ in (9.12) we obtain

$$a|\mathbf{m}|^2|\mathbf{n}|^2 > 0$$

and hence $a > 0$. Next, setting $\mathbf{m} = \mathbf{n}$ in (9.12) we obtain

$$(a + b + c)|\mathbf{m}|^4 > 0$$

and hence $a+b+c > 0$. The converse implication follows easily from (9.12) and the Cauchy-Schwarz inequality. □

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