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MYRIAD RADIAL CAVITATING EQUILIBRIA IN NONLINEAR ELASTICITY*

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For Donald E. Carlson on the occasion of his 65th birthday

Abstract. It is shown that every bounded strictly increasing smooth positive function of sufficiently slow growth is the Jacobian of a radial hole creating equilibrium deformation for an appropriately constructed compressible nonlinearly elastic energy.

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1. Introduction.

Explicit solutions of model equations can be useful in gaining insight concerning the qualitative behavior of solutions to more general problems. Unfortunately, the explicit construction of radial equilibrium deformations that create new holes in a compressible nonlinearly elastic body has proven to be unexpectedly complicated. Consequently, although cavitation is a common occurrence in rubbery polymers, explicit solutions that exhibit this phenomenon are rare. The only such solutions (modulo a radial null-Lagrangian; see Horgan [3] and Steigmann [15]) that appear in the literature are for an elastic fluid (see, e.g., [3, 6]); for the Blatz–Ko constitutive relation for foam rubbers, which was obtained, in two dimensions, by Horgan and Abeyaratne [4] and, in three dimensions, by Tian-hu [18]; for a compressible neo-Hookean material, which was obtained in [11] (see also [1, section 7.6]); and for the generalized Carroll material, which was obtained by Murphy and Biwa [7] (see also Shang and Cheng [12]).

The usual method of obtaining an explicit solution is to solve the differential equation for a postulated model problem. In this paper we take a different approach; we first posit deformations that, based upon prior results, have desired properties, and then construct differential equations that have these deformations as solutions. We show, in particular, that every function in a certain class of radial cavitating deformations will satisfy an equilibrium equation that is appropriately chosen for that particular function. The radial deformations we use are those for which the Jacobian is an increasing radial function. The appropriate stored energy is then constructed as the sum of two terms. The first is a homogeneous isotropic strongly elliptic stored-energy function, while the second is a function of the Jacobian of the chosen radial deformation. This second function is constructed so that the chosen deformation will automatically satisfy the radial equilibrium equation. Further information on radial cavitation is contained in the survey article by Horgan and Polignone [5].

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2. The constitutive relation. Let $\Psi \in C^2((0,\infty)^n)$ be a symmetric function. We assume that the stored-energy function for the material is given by

$$W(\mathbf{F}) = \Phi(\nu_1, \nu_2, \ldots, \nu_n) := \Psi(\nu_1, \nu_2, \ldots, \nu_n) + h(\nu_1\nu_2 \ldots \nu_n) \tag{2.1}$$

for all $n \times n$ matrices $\mathbf{F}$ with positive determinant, where $\nu_1, \nu_2, \ldots, \nu_n$ are the principal stretches, i.e., the eigenvalues of the square root of $\mathbf{F}\mathbf{F}^T$, and $h \in C^2((0,\infty))$ is a function to be determined.

The problem of interest is to determine stationary points of the energy

$$E(\mathbf{w}) = \int_{B_\mathrm{o}} W(\nabla \mathbf{w}(\mathbf{x}))\, d\mathbf{x} \tag{2.2}$$

among orientation-preserving injective $\mathbf{w}$ that satisfy the boundary condition $\mathbf{w}(\mathbf{x}) = \lambda \mathbf{x}$ for $\mathbf{x} \in \partial B_\mathrm{o}$, where $B_\mathrm{o} := B(\mathbf{0}, R_\mathrm{o}) \subset \mathbb{R}^n$ is the ball of radius R_o centered at the origin. For a radial deformation

$$\mathbf{w}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \qquad R := |\mathbf{x}|,$$

$r : [0, R_\mathrm{o}] \to [0, \infty)$, the principal stretches at any point $\mathbf{x} \in B_\mathrm{o}$ are given by (see, e.g., [1]) $\nu_1(\mathbf{x}) = r'(R)$ and $\nu_i(\mathbf{x}) = r(R)/R$ for $i = 2, 3, \ldots, n$. Thus (2.2) reduces to[1]

$$\overline{E}(r) = \int_0^{R_\mathrm{o}} \Phi\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}, \ldots, \frac{r(R)}{R}\right) R^{n-1}\, dR \tag{2.3}$$

among those $r : [0, R_\mathrm{o}] \to [0, \infty)$ that satisfy $r' > 0$ a.e. and $r(R_\mathrm{o}) = \lambda R_\mathrm{o}$. A stationary point $\mathbf{w}$ of E corresponds to a solution r of the radial equilibrium equation

$$\frac{d}{dR}\left[R^{n-1}\Phi_{,1}\right] = (n-1)R^{n-2}\Phi_{,2}\,, \tag{2.4}$$

where

$$\Phi_{,i} = \Phi_{,i}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}, \ldots, \frac{r(R)}{R}\right)$$

and $\Phi_{,i}(v_1, v_2, \ldots, v_n)$ denotes differentiation of Φ with respect to its ith argument (see [1, Theorem 7.3]). Also, if $r(0) > 0$, then the deformed ball contains a spherical cavity, and r must satisfy the natural boundary condition

$$T(R) := \left[\frac{R}{r(R)}\right]^{n-1} \Phi_{,1}\left(r'(R), \frac{r(R)}{R}, \frac{r(R)}{R}, \ldots, \frac{r(R)}{R}\right) \to 0 \ \text{ as } R \to 0^+, \tag{2.5}$$

which corresponds to the radial component of the Cauchy stress vanishing on the cavity surface.

For energies of the form (2.1) the radial equilibrium equation (2.4) becomes

$$\frac{d}{dR}\left[R^{n-1}\Psi_{,1}\right] - (n-1)R^{n-2}\Psi_{,2} = -r(R)^{n-1}\frac{d}{dR}h'\left(r'(R)\left[\frac{r(R)}{R}\right]^{n-1}\right), \tag{2.6}$$

[1] The energy E is equal to $\overline{E}$ multiplied by the volume of the unit ball in $\mathbb{R}^n$.

and the natural boundary condition (2.5) reduces to

$$\lim_{R\to 0^+}\left[h'\left(r'(R)\left[\frac{r(R)}{R}\right]^{n-1}\right)+\left[\frac{R}{r(R)}\right]^{n-1}\Psi_{,1}\left(r'(R),\frac{r(R)}{R},\ldots,\frac{r(R)}{R}\right)\right]=0. \tag{2.7}$$

The main idea in this paper is that, given Ψ and r, (2.6) can be used to define the function h. In order to accomplish this, we will need the following hypotheses on the energy:

(En1) for all $q > 0$ and $t > 0$

$$\Psi_{,11}(q,t,t,\ldots,t) > 0;$$

(En2) there exists $\lambda_* > 0$ such that for every $\alpha \geq \lambda_*^n$

$$\lim_{t\to+\infty} t^{1-n}\Psi_{,1}(\alpha t^{1-n},t,t,\ldots,t) = 0;$$

(En3) for every $L > \lambda_*^n$ there are constants $\beta \in [0, n-1)$ and $K > 0$ such that

$$\left|\Psi_{,2}(\kappa t^{1-n},t,t,\ldots,t)\right| \leq K t^{\beta}$$

for all $\lambda_* < t < \infty$ and $\lambda_*^n \leq \kappa \leq L$;

(En4) for $q \neq t$ define

$$\mathcal{R}(q,t) := \frac{q\Psi_{,1}(q,t,t,\ldots,t) - t\Psi_{,2}(q,t,t,\ldots,t)}{q-t}. \tag{2.8}$$

Then for every $\mu > \lambda_*$ we assume that there exists a $B_\mu > 0$ such that

$$|\mathcal{R}(q,t)| \leq B_\mu$$

for all $\lambda_* < t < \mu$ and $0 < q < t$.

Remark 2.1. Hypothesis (En1) is a consequence of the strong-ellipticity of the energy Ψ. Hypotheses (En2)–(En4) are satisfied by many examples of stored energies (see [1, 16, 17]). In particular, Stuart [16, 17] requires a more stringent growth hypothesis than (En4): $0 \leq \mathcal{R}(q,t) \leq A + Bt^{\beta}$ for $0 < q < t$.

3. The construction. Let $\lambda_{\text{crit}} > \lambda_0 > 0$, and suppose that $J \in C^1([0,\infty))$ is a strictly monotone increasing function that satisfies

$$J(0) = \lambda_0^n, \qquad \lim_{R\to+\infty} J(R) = \lambda_{\text{crit}}^n, \qquad \int_0^\infty J'(t)t^n\,dt \leq 1. \tag{3.1}$$

Define $\rho : [0,\infty) \to [1,\infty)$ by

$$\rho(R)^n := 1 + n\int_0^R J(t)t^{n-1}\,dt \tag{3.2}$$

so that

$$\rho'(R)\left[\frac{\rho(R)}{R}\right]^{n-1} = J(R) \quad \text{for} \quad 0 < R < \infty. \tag{3.3}$$

For future reference we note the following properties of ρ.

LEMMA 3.1. *The function* ρ *given by* (3.1) *and* (3.2) *satisfies*

(i) $0 < \rho'(R)$,

(ii) $\frac{d}{dR}[\frac{\rho(R)}{R}] < 0$,

(iii) $\rho'(R) < \frac{\rho(R)}{R}$,

(iv) $0 < \rho''(R)$,

(v) $\lim_{R\to+\infty} \frac{\rho(R)}{R} = \lim_{R\to+\infty} \rho'(R) = \lambda_{\text{crit}}$.

Remark 3.2. Properties (i)–(v) are standard properties of radial minimizers and radial equilibrium deformations (see, e.g., [1, 10, 16, 17]). Since our proof shows that $(3.1)_3$ is necessary and sufficient for (iii), it is now clear that $(3.1)_3$ is also a standard property of such deformations. Note also that by (3.2), $\rho(0) = 1$.

THEOREM 3.3. *Let* Ψ *satisfy* (En1)–(En4) *and let* ρ *be given by* (3.1) *and* (3.2), *where* $\lambda_0 > \lambda_*$. *Suppose that* $h \in C^2((0,\infty);(0,\infty))$ *and satisfies*

$$
\begin{aligned}
h'(J(R)) = {} & \int_0^R \frac{(n-1)s^{n-2}}{\rho(s)^{n-1}} \hat{\Psi}_2\left(\rho'(s), \frac{\rho(s)}{s}\right) ds \\
& - R^{n-1}\hat{\Psi}_1\left(\rho'(R), \frac{\rho(R)}{R}\right)\left[\frac{1}{\rho(R)}\right]^{n-1} \\
& + \int_0^R s^{n-1}\hat{\Psi}_1\left(\rho'(s), \frac{\rho(s)}{s}\right) \frac{d}{ds}\left[\frac{1}{\rho(s)^{n-1}}\right] ds
\end{aligned} \tag{3.4}
$$

for $R \in (0,\infty)$, *where* $\hat{\Psi}_i(\rho'(s), \frac{\rho(s)}{s}) := \Psi_{,i}(\rho'(s), \frac{\rho(s)}{s}, \frac{\rho(s)}{s}, \dots, \frac{\rho(s)}{s})$. *Then* ρ *is a solution of the radial equilibrium equation* (2.6) *on* $(0,\infty)$ *and satisfies the natural boundary condition* (2.7).

Remark 3.4. It follows from the proof of the above result (see (4.4)) that $h'(\lambda_0^n) = 0$.

Finally, we use ρ to construct a family of equilibrium deformations of B_{o}. For any $R_{\text{o}} > 0$ and $\delta > 0$

$$
\mathbf{u}_\delta(\mathbf{x}) := \frac{r_\delta(|\mathbf{x}|)}{|\mathbf{x}|}\mathbf{x}, \qquad r_\delta(R) := \frac{\rho(\delta R)}{\delta} \tag{3.5}
$$

is an orientation-preserving injective radial deformation of B_{o}, and, in view of (3.3), (4.1), and [1, Lemma 4.1], $J(\delta|\mathbf{x}|)$ is the Jacobian of $\mathbf{u}_\delta$ at any point $\mathbf{x} \neq \mathbf{0}$. Clearly, each $\mathbf{u}_\delta$ is a cavitating deformation that creates a new hole of radius $1/\delta$ at the center of the ball and satisfies the boundary condition

$$
\mathbf{u}_\delta(\mathbf{x}) = \lambda\mathbf{x}, \qquad \lambda^n = \lambda(\delta, R_{\text{o}})^n := \frac{[1 + n\int_0^{\delta R_{\text{o}}} J(t)t^{n-1}\,dt]}{(\delta R_{\text{o}})^n} \tag{3.6}
$$

for $\mathbf{x} \in \partial B_{\text{o}}$. Moreover, results in [1] show that each of these deformations is contained in the Sobolev space $W^{1,p}(B_{\text{o}};\mathbb{R}^n)$ for every $p \in [1,n)$, while Theorem 3.3 shows that $\mathbf{u}_\delta$ is a stationary point for the energy.

THEOREM 3.5. *Let* W *be given by* (2.1) *and satisfy* (En1)–(En4). *Let* $\lambda > \lambda_{\text{crit}} > \lambda_0 > \lambda_*$. *Then there exists a unique* $\delta = \delta(\lambda)$ *such that* $\mathbf{u}_\delta$, *given by* (3.1), (3.2), (3.5), *and* (3.6), *is a stationary point of the energy* (2.2) *and satisfies the boundary condition* $\mathbf{u}_\delta(\mathbf{x}) = \lambda\mathbf{x}$ *for* $\mathbf{x} \in \partial B_{\text{o}}$.

Proof. Let $\lambda > \lambda_{\text{crit}}$. Then since $\rho(0) = 1$, it is clear from Lemma 3.1(ii) and (v) that there exists a unique $R_\lambda > 0$ such that $\rho(R_\lambda) = \lambda R_\lambda$. Define $\delta = R_\lambda/R_{\text{o}}$. Then by (3.5)

$$\mathbf{u}_\delta(\mathbf{x}) := \frac{\rho(\delta R_\mathrm{o})}{\delta R_\mathrm{o}}\mathbf{x} = \frac{\rho(R_\lambda)}{R_\lambda}\mathbf{x} = \lambda\mathbf{x} \quad \text{for} \quad |\mathbf{x}| = R_\mathrm{o}. \qquad \square$$

4. Proofs for the construction.

Proof of Lemma 3.1. We first note that (i) is clear from (3.2), (3.3), and the nonnegativity of J. Next, if we divide (3.2) by R^n and use the quotient rule to differentiate the result with respect to R, we find that

$$\begin{aligned}\left[\frac{\rho(R)}{R}\right]^{n-1}\frac{d}{dR}\left[\frac{\rho(R)}{R}\right] &= \frac{\left(nJ(R)R^{n-1}\right)R^n - nR^{n-1}(1+n\int_0^R J(t)t^{n-1}\,dt)}{nR^{2n}} \\ &= \frac{J(R)R^n - (1+n\int_0^R J(t)t^{n-1}\,dt)}{R^{n+1}} \\ &= \frac{-1+\int_0^R J'(t)t^n\,dt}{R^{n+1}},\end{aligned} \tag{4.1}$$

where an integration by parts has been used to deduce $(4.1)_3$ from $(4.1)_2$. It is now clear from (4.1) that $(3.1)_3$ is necessary and sufficient for (ii).

Next, if we differentiate $\rho(R)/R$ with respect to R, we see that

$$\frac{d}{dR}\left[\frac{\rho(R)}{R}\right] = \frac{1}{R}\left[\rho'(R) - \frac{\rho(R)}{R}\right], \tag{4.2}$$

and consequently (iii) is equivalent to (ii). Similarly, if we differentiate (3.3) with respect to R, we discover that

$$\rho''(R)\left[\frac{\rho(R)}{R}\right]^{n-1} = J'(R) - \rho'(R)(n-1)\left[\frac{\rho(R)}{R}\right]^{n-2}\frac{d}{dR}\left[\frac{\rho(R)}{R}\right], \tag{4.3}$$

and thus (iv) follows from (i), (ii), and the fact that J is increasing.

Finally, to obtain (v) we first note that (ii)–(iv) imply that both limits exist and are finite. Thus, if we divide (3.2) by R^n and take the limit as $R \to +\infty$, we find, using L'Hôpital's rule and $(3.1)_2$, that

$$\lim_{R\to+\infty}\left[\frac{\rho(R)}{R}\right]^n = \lim_{R\to+\infty} J(R) = \lambda_{\mathrm{crit}}^n,$$

which together with (3.3) yields (v). $\square$

Proof of Theorem 3.3. Assume for the moment that $s \mapsto s^{n-2}\hat{\Psi}_2(\rho'(s), \frac{\rho(s)}{s})$ and $s \mapsto s^{n-1}\hat{\Psi}_1(\rho'(s), \frac{\rho(s)}{s})$ are integrable on $(0, R)$, so that the right-hand side of (3.4) is well defined on $(0,\infty)$. Then, if we differentiate (3.4) with respect to R, it is clear that ρ satisfies the radial equilibrium equation (2.6) on $(0,\infty)$.

In order to show that $s \mapsto s^{n-2}\hat{\Psi}_2(\rho'(s), \frac{\rho(s)}{s})$ is integrable on $(0, R)$ we use (3.3), (En3), $\rho' \geq 0$, and the fact that $J(s) \in [\lambda_0^n, \lambda_{\mathrm{crit}}^n]$ for each s to conclude that

$$\begin{aligned}s^{n-2}\left|\hat{\Psi}_2\left(\rho'(s), \frac{\rho(s)}{s}\right)\right| &= \rho(s)^{n-2}\left[\frac{s}{\rho(s)}\right]^{n-2}\left|\hat{\Psi}_2\left(J(s)\left[\frac{\rho(s)}{s}\right]^{1-n}, \frac{\rho(s)}{s}\right)\right| \\ &\leq K\rho(s)^{n-2}\left[\frac{\rho(s)}{s}\right]^{\beta-n+2} \leq K\rho(R)^\beta s^{n-2-\beta},\end{aligned}$$

which is clearly integrable on $(0, R)$ since $\beta < n-1$.

In order to prove that $s \mapsto s^{n-1}\hat{\Psi}_1(\rho'(s), \frac{\rho(s)}{s})$ is integrable on $(0, R)$, we will show that

$$\lim_{s\to 0^+} s^{n-1}\hat{\Psi}_1\left(\rho'(s), \frac{\rho(s)}{s}\right) = 0. \tag{4.4}$$

Now, by (3.3),

$$s^{n-1}\hat{\Psi}_1\left(\rho'(s), \frac{\rho(s)}{s}\right) = \rho(s)^{n-1}\left[\frac{s}{\rho(s)}\right]^{n-1}\hat{\Psi}_1\left(J(s)\left[\frac{\rho(s)}{s}\right]^{1-n}, \frac{\rho(s)}{s}\right). \tag{4.5}$$

Moreover, since $\lambda_0^n \leq J(s) \leq \lambda_{\text{crit}}^n$, hypothesis (En1) implies

$$\hat{\Psi}_1\left(\lambda_0^n t^{1-n}, t\right) \leq \hat{\Psi}_1\left(J(s)t^{1-n}, t\right) \leq \hat{\Psi}_1\left(\lambda_{\text{crit}}^n t^{1-n}, t\right), \quad t := \frac{\rho(s)}{s}. \tag{4.6}$$

Since $\rho \to 1$ and $t \to +\infty$ as $s \to 0^+$, (4.4) now follows from (En2), (4.5), and (4.6). In addition, the natural boundary condition (2.7) follows from (3.4), (4.4), together with the integrability of $s \mapsto s^{n-2}\hat{\Psi}_2(\rho'(s), \frac{\rho(s)}{s})$ and $s \mapsto s^{n-1}\hat{\Psi}_1(\rho'(s), \frac{\rho(s)}{s})$ on $(0, R)$.

Finally, we need to show that the right-hand side of (3.4) is bounded as $R \to \infty$ in order that h can be extended smoothly as a real-valued function on $(\lambda_{\text{crit}}^n, \infty)$. Let $R_1 > 0$. Then by (3.4)

$$\begin{aligned} h'(J(R)) - h'(J(R_1)) \quad = \quad & \int_{R_1}^{R} \frac{(n-1)s^{n-2}}{\rho(s)^{n-1}}\hat{\Psi}_2\left(\rho'(s), \frac{\rho(s)}{s}\right) ds \\ & - R^{n-1}\hat{\Psi}_1\left(\rho'(R), \frac{\rho(R)}{R}\right)\rho(R)^{1-n} \\ & + {R_1}^{n-1}\hat{\Psi}_1\left(\rho'(R_1), \frac{\rho(R_1)}{R_1}\right)\rho(R_1)^{1-n} \\ & + \int_{R_1}^{R} s^{n-1}\hat{\Psi}_1\left(\rho'(s), \frac{\rho(s)}{s}\right)\ \frac{d}{ds}\left[\rho(s)^{1-n}\right] ds \end{aligned} \tag{4.7}$$

for $R \in (R_1, \infty)$. Now, it is clear from Lemma 3.1(v) that $(4.7)_2$ is bounded as $R \to \infty$. Next, the sum of the integrals on the right-hand side of $(4.7)_1$, $(4.7)_4$ is equal to

$$(n-1)\int_{R_1}^{R} s^{-1}\left[\frac{s}{\rho(s)}\right]^{n}\left[\frac{\rho(s)}{s}\hat{\Psi}_2\left(\rho'(s), \frac{\rho(s)}{s}\right) - \rho'(s)\hat{\Psi}_1\left(\rho'(s), \frac{\rho(s)}{s}\right)\right] ds. \tag{4.8}$$

However, by Lemma 3.1, $\lambda_{\text{crit}} < \frac{\rho(s)}{s} \leq \frac{\rho(R_1)}{R_1} =: \mu$ for $R_1 \leq s < \infty$ and hence, in view of (En4), the absolute value of (4.8) is bounded by a constant times

$$\begin{aligned} \int_{R_1}^{R} \frac{1}{s}\left[\frac{\rho(s)}{s} - \rho'(s)\right] ds &= \int_{R_1}^{R} -\frac{d}{ds}\left[\frac{\rho(s)}{s}\right] ds \\ &= \frac{\rho(R_1)}{R_1} - \frac{\rho(R)}{R}, \end{aligned}$$

which is bounded as $R \to \infty$. $\square$

5. The energy: Uniqueness. In this section we note that, whenever the function h is convex, the cavitating radial equilibrium solution we have constructed is the unique global minimizer of the energy among radial deformations. The following three results can be found in Sivaloganathan [13] (see also [14]).

PROPOSITION 5.1. *Assume that*

$$\hat{\Phi}_{11}(q,t) > 0 \tag{5.1}$$

for all $q > 0$ *and* $t > 0$. *Let* $r_c \in C^1([0,\infty)) \cap C^2((0,\infty))$ *be a* cavitating equilibrium solution*; i.e.,* r_c *satisfies* (2.4) *on* $(0,\infty)$, (2.5), $r_c(0) > 0$, *and* $r_c' > 0$ *a.e. Suppose that* $R_o > 0$, *and define* $\lambda = r_c(R_o)/R_o$. *Let* $r \in \mathcal{A}_\lambda$,

$$\mathcal{A}_\lambda := \{r \in W^{1,1}((0,R_o)) : r(R_o) = \lambda R_o,\ r(0) \geq 0,\ r' > 0 \text{ a.e.}\}, \tag{5.2}$$

satisfy

$$0 < \limsup_{R\to 0^+} \left[\frac{r(R)}{R}\right]. \tag{5.3}$$

Then

$$\overline{E}(r_c) < \overline{E}(r)$$

unless $r \equiv r_c$, *where* $\overline{E}$ *is given by* (2.3).

COROLLARY 5.2. *Let* $\lambda > 0$ *and* $R_o > 0$. *Then, under the hypotheses of the previous proposition, there exists at most one cavitating equilibrium solution* $r_c \in C^1([0,\infty)) \cap C^2((0,\infty))$ *that satisfies* $r_c(R_o) = \lambda R_o$.

Remarks. 1. The statement of Theorem 6.8 in [13] actually requires that (5.3) be satisfied with lim sup replaced by lim inf. However, the remark after the proof of [13, Theorem 6.9] notes that the result remains valid under this weaker hypothesis.

2. Theorems 6.8 and 6.9 in [13] also appear to require the weakened Baker–Ericksen inequality $\mathcal{R}(q,t) \geq 0$, where $\mathcal{R}$ is given by (2.8). However, an examination of the proofs in [13, 14] shows that this inequality is used only to extend a solution of the radial equilibrium equation from $(0,R_o)$ to $(0,\infty)$, a step not needed in our presentation.

Proof of Corollary 5.2. Let $\lambda > 0$ and $R_o > 0$. If r_c is any cavitating equilibrium solution that satisfies $r_c(R_o) = \lambda R_o$, then $r_c \in \mathcal{A}_\lambda$ and r_c satisfies (5.3). Thus, by the previous proposition, two distinct cavitating equilibrium solutions r_{c_1} and r_{c_2} would satisfy $\overline{E}(r_{c_2}) < \overline{E}(r_{c_1})$ and $\overline{E}(r_{c_1}) < \overline{E}(r_{c_2})$, which is a contradiction. ∎

COROLLARY 5.3. *Let* $\lambda > 0$ *and* $R_o > 0$. *Suppose that* $r_c \in C^1([0,\infty)) \cap C^2((0,\infty))$ *is a cavitating equilibrium solution that satisfies* $r_c(R_o) = \lambda R_o$. *Then, under the hypotheses of the previous proposition,* $\overline{E}(r_c) < \overline{E}(r_h)$, *where* $r_h(R) := \lambda R$.

Proof. For any $\lambda > 0$ and $R_o > 0$ the homogeneous deformation $r_h(R) := \lambda R$ satisfies $r_h \in \mathcal{A}_\lambda$ and (5.3). The result then follows from Proposition 5.1. ∎

In order to make use of Proposition 5.1 we will need the following additional hypothesis on the energy:

(En5) there exist $\phi, \psi : (0,\infty) \to \mathbb{R}$ and $\Psi^* \in C((0,\infty)^n;\mathbb{R})$, with $\phi > 0, \psi \geq 0$, and $\Psi \geq 0$, that satisfy

$$\Psi(\nu_1,\nu_2,\ldots,\nu_n) = \sum_{i=1}^{n} \phi(\nu_i) + \sum_{i\neq j} \psi(\nu_i\nu_j) + \Psi^*(\nu_1,\nu_2,\ldots,\nu_n),$$

where $t \mapsto \Psi^*(t, t, \ldots, t)$ is bounded on $(0, \lambda^*]$ and $\psi \equiv 0$ if $n = 2$.

We now use Proposition 5.1 to show that if our equilibrium solutions are supersolutions for the energy Ψ, then they are energy minimizers among the radial deformations.

THEOREM 5.4. *Let Ψ satisfy* (En1)–(En5) *and let ρ be given by* (3.1) *and* (3.2), *where $\lambda_0 > \lambda_*$ and $J' > 0$ on $[0, \infty)$. Suppose that*

$$\frac{d}{dR}\left[R^{n-1}\hat{\Psi}_1\left(\rho'(R), \frac{\rho(R)}{R}\right)\right] < (n-1)R^{n-2}\hat{\Psi}_2\left(\rho'(R), \frac{\rho(R)}{R}\right). \tag{5.4}$$

Finally, suppose in addition that $h \in C^2((0,\infty);(0,\infty))$ satisfies (3.4) *and $h''(s) \geq 0$ for $s \in (0, \lambda_0^n) \cup [\lambda_{\text{crit}}^n, \infty)$. Then the radial cavitating deformation $r_{\delta(\lambda)}$ given by Theorem* 3.5 *satisfies*

$$\overline{E}(r_{\delta(\lambda)}) < \overline{E}(r)$$

for every $r \in \mathcal{A}_\lambda$ (see (5.2)*).*

Proof. We first show that h'' is nonnegative on its domain of definition. Let $s \in (0, \infty)$. If $s \notin [\lambda_0^n, \lambda_{\text{crit}}^n)$, then $h''(s) \geq 0$, by hypothesis. If $s \in [\lambda_0^n, \lambda_{\text{crit}}^n)$, then by (3.1) there exists an $R \in [0, \infty)$ such that $J(R) = s$. Therefore, by (3.3) and the radial equilibrium equation (2.6),

$$\frac{d}{dR}\left[R^{n-1}\Psi_{,1}\right] - (n-1)R^{n-2}\Psi_{,2} = -\rho(R)^{n-1}h''\left(J(R)\right)J'(R). \tag{5.5}$$

The desired result now follows from (5.4), (5.5), and the assumed positivity of ρ and J'.

Now let $\lambda > \lambda_{\text{crit}}$, $r \in \mathcal{A}_\lambda$, and suppose that

$$0 < \limsup_{R \to 0^+}\left[\frac{r(R)}{R}\right]. \tag{5.6}$$

Then, by Proposition 5.1, all we need show is that (5.1) is satisfied. If we differentiate (2.1) twice with respect to ν_1 and set $\nu_1 = q$ and $\nu_2 = \nu_3 = \cdots = \nu_n = t$, we find that

$$\hat{\Phi}_{11}(q,t) = \hat{\Psi}_{11}(q,t) + t^{2n-2}h''(qt^{n-1}).$$

Consequently, in view of (En1), a sufficient condition for (5.1) is that h'' be nonnegative on its domain of definition, which has previously been shown.

Before proceeding further we note that the convexity of h, together with $h'(\lambda_0^n) = 0$ (see the remark following Theorem 3.3), implies that h is bounded below by $h(\lambda_0^n)$ on $(0, \infty)$.

Next, suppose alternatively that (5.6) is not satisfied and therefore that

$$0 = \lim_{R \to 0^+}\left[\frac{r(R)}{R}\right].$$

Then, in particular, $r(0) = 0$, and hence

$$\int_0^{R_o} nr'r^{n-1}dR = \int_0^{R_o}\frac{d}{dR}\left[r^n\right]dR = r(R_o)^n = \lambda^n R_o^n = n\int_0^{R_o}\lambda^n R^{n-1}dR. \tag{5.7}$$

Now, in view of the convexity of h,

$$h\left(r'(R)\left[\frac{r(R)}{R}\right]^n\right) \geq h(\lambda^n) + \left(r'(R)\left[\frac{r(R)}{R}\right]^n - \lambda^n\right)h'(\lambda^n),$$

and consequently, by (5.7),

$$\int_0^{R_o} h\left(r'(R)\left[\frac{r(R)}{R}\right]^{n-1}\right)R^{n-1}dR \geq \int_0^{R_o} h(\lambda^n)R^{n-1}dR. \tag{5.8}$$

If $\overline{E}(r) = +\infty$, we are done. If instead $\overline{E}(r) < \infty$, then by (En5) and since h is bounded below,

$$R \mapsto R^{n-1}\phi\left(\frac{r(R)}{R}\right) \in L^1((0, R_o)), \tag{5.9}$$

$$R \mapsto R^{n-1}\psi\left(\left[\frac{r(R)}{R}\right]^2\right) \in L^1((0, R_o)). \tag{5.10}$$

We claim that

$$0 = \liminf_{R\to 0^+} R^n\hat{\Psi}\left(\frac{r(R)}{R}, \frac{r(R)}{R}\right). \tag{5.11}$$

Otherwise, $R^n\hat{\Psi}\big(\frac{r(R)}{R}, \frac{r(R)}{R}\big) \geq K > 0$ for small R, and thus

$$R^{n-1}\hat{\Psi}\left(\frac{r(R)}{R}, \frac{r(R)}{R}\right) \geq \frac{K}{R}.$$

This inequality, together with (En5), implies

$$nR^{n-1}\phi\left(\frac{r(R)}{R}\right) + \frac{1}{2}n(n-1)R^{n-1}\psi\left(\left[\frac{r(R)}{R}\right]^2\right) \geq \frac{K}{R} - \hat{\Psi}^*\left(\frac{r(R)}{R}, \frac{r(R)}{R}\right),$$

which, since $r(R)/R$ and hence Ψ^* are bounded, contradicts (5.9) or (5.10).

Next, $\Psi_{,11} > 0$. Therefore $\hat{\Psi}(q, t) \geq \hat{\Psi}(t, t) + (q - t)\hat{\Psi}_{,1}(t, t)$, and hence

$$\begin{aligned}\hat{\Psi}\left(r'(R), \frac{r(R)}{R}\right) &\geq \hat{\Psi}\left(\frac{r(R)}{R}, \frac{r(R)}{R}\right) + \left(r'(R) - \frac{r(R)}{R}\right)\hat{\Psi}_{,1}\left(\frac{r(R)}{R}, \frac{r(R)}{R}\right)\\ &= \frac{1}{n}R^{1-n}\frac{d}{dR}\left[R^n\hat{\Psi}\left(\frac{r(R)}{R}, \frac{r(R)}{R}\right)\right],\end{aligned}$$

which, when multiplied by nR^{n-1} and integrated over (R_k, R_o), yields

$$\int_{R_k}^{R_o} n\hat{\Psi}\left(r'(R), \frac{r(R)}{R}\right)R^{n-1}dR \geq \left[R_o^n\hat{\Psi}(\lambda, \lambda)\right] - \left[R_k^n\hat{\Psi}\left(\frac{r(R_k)}{R_k}, \frac{r(R_k)}{R_k}\right)\right]. \tag{5.12}$$

In particular, choose a sequence $R_k \to 0^+$ as $k \to \infty$ so that $R_k^n\hat{\Psi}\big(\frac{r(R_k)}{R_k}, \frac{r(R_k)}{R_k}\big)$ converges to its lim inf, which is zero by (5.11). Then, if we let $k \to \infty$ in (5.12) and apply the dominated convergence theorem, we find that

$$\int_0^{R_o} \hat{\Psi}\left(r'(R), \frac{r(R)}{R}\right)R^{n-1}dR \geq \int_0^{R_o} \hat{\Psi}(\lambda, \lambda)\,R^{n-1}dR,$$

which, together with (2.1) and (5.8), yields $\overline{E}(r) \geq \overline{E}(r_h)$. The desired result now follows from Corollary 5.3. $\square$

6. Examples.

6.1. Ogden materials. In order to illustrate the form that hypotheses (En1)–(En5) take for a well-analyzed class of materials, we now restrict our attention to three dimensions and consider materials whose constitutive relation is of the form

$$\Psi(\lambda_1, \lambda_2, \lambda_3) = \phi(\lambda_1) + \phi(\lambda_2) + \phi(\lambda_3) + \psi(\lambda_1\lambda_2) + \psi(\lambda_2\lambda_3) + \psi(\lambda_1\lambda_3), \tag{6.1}$$

where $\phi, \psi \in C^2((0, \infty))$. (Such constitutive relations were used by Ogden [8] to match theory with experiments.)

For such materials we make the following assumptions (cf. [1, 16, 13, 14, 9] and especially [17, 10]):

(Og1) for all $s > 0$

$$\phi''(s) > 0, \qquad \psi''(s) \geq 0;$$

(Og2) there exist $\beta \in [1, 2)$, $\gamma \in [0, 1)$, and $B > 0$ such that for every $s > 0$

$$|\phi'(s)| \leq B\left[s^{-\gamma} + s^{\beta}\right], \qquad |\psi'(s)| \leq B[s^{-\gamma} + s^{(\beta-1)/2}].$$

We now show that (Og1) and (Og2) imply (En1)–(En5). First, it is clear that (En5) is satisfied with $\Psi^* \equiv 0$. Next, we differentiate (6.1) with respect to λ_1 and let $\lambda_1 = q$ and $\lambda_2 = \lambda_3 = t$ to get

$$\begin{aligned} \hat{\Psi}_1(q, t) &= \phi'(q) + 2t\psi'(qt), \\ \hat{\Psi}_{11}(q, t) &= \phi''(q) + 2t^2\psi''(qt). \end{aligned} \tag{6.2}$$

Then (Og1) and $(6.2)_2$ yield (En1). If we differentiate (6.1) with respect to λ_2 and let $\lambda_1 = q$ and $\lambda_2 = \lambda_3 = t$, we get

$$\hat{\Psi}_2(q, t) = \phi'(t) + q\psi'(qt) + t\psi'(t^2), \tag{6.3}$$

and hence, when $q = \kappa t^{-2}$, we find that

$$\hat{\Psi}_2(\kappa t^{-2}, t) = \phi'(t) + \kappa t^{-2}\psi'(\kappa t^{-1}) + t\psi'(t^2). \tag{6.4}$$

In order to obtain (En3) we take the absolute value of (6.4) and use the triangle inequality and (Og2) to conclude that

$$|\hat{\Psi}_2(\kappa t^{-2}, t)| \leq B[(t^{-\gamma} + t^{\beta}) + (\kappa^{1-\gamma}t^{\gamma-2} + \kappa^{(\beta+1)/2}t^{-(\beta+3)/2}) + (t^{1-2\gamma} + t^{\beta})],$$

which implies (En3). Similarly, if we take $q = \alpha t^{-2}$ in $(6.2)_1$ and use the triangle inequality and (Og2), we obtain

$$|t^{-2}\hat{\Psi}_1(\alpha t^{-2}, t)| \leq B[\alpha^{-\gamma}t^{2(\gamma-1)} + \alpha^{\beta}t^{-2(\beta+1)} + 2\alpha^{-\gamma}t^{\gamma-1} + 2\alpha^{(\beta-1)/2}t^{-(\beta+1)/2}],$$

which approaches zero as $t \to \infty$ since $\gamma < 1$. This implies (En2).

In order to obtain (En4) we first use $(6.2)_1$ and (6.3) to get

$$\mathcal{R}(q, t) = \frac{t\phi'(t) - q\phi'(q)}{t - q} + \frac{t^2\psi'(t^2) - qt\psi'(qt)}{t - q}. \tag{6.5}$$

We then fix $\mu > \lambda_*$ and consider two cases: $\frac{1}{2}\lambda_* \leq q < t < \mu$ and $0 < q < \frac{1}{2}\lambda_* < \lambda_* < t < \mu$.

Case I. $0 < q < \frac{1}{2}\lambda_* < \lambda_* < t < \mu$. Then $|t-q|^{-1} \leq 2/\lambda_*$, and hence (6.5) together with (Og2) and the triangle inequality yield

$$|\mathcal{R}(q,t)| \leq \frac{2B}{\lambda_*}[t^{1-\gamma} + t^{\beta+1} + q^{1-\gamma} + q^{\beta+1} + t^{2(1-\gamma)} + t^{\beta+1} + (qt)^{1-\gamma} + (qt)^{(\beta+1)/2}],$$

which is bounded for $0 < q < t < \mu$ since $\gamma < 1$. This implies (En4) for small q.

Case II. $\frac{1}{2}\lambda_* \leq q < t < \mu$. Then (6.5) together with the mean-value theorem applied to the functions $\tilde{\phi}(s) := s\phi'(s)$ and $\tilde{\psi}(s) := s\psi'(s)$ yield

$$\mathcal{R}(q,t) = \tilde{\phi}'(c^*) + t\tilde{\psi}'(\hat{c}) \tag{6.6}$$

for some $c^* \in (q,t)$ and $\hat{c} \in (qt, t^2)$. Thus, since ϕ and ψ are C^2, the function $|\mathcal{R}|$ is bounded when $\frac{1}{2}\lambda_* \leq q < t < \mu$. Therefore (En4) is also valid for larger q.

6.2. Deformations. It is easy to construct radial cavitating *deformations*: the specification of a strictly monotone increasing radial Jacobian $J(R)$ that satisfies (3.1) suffices. The content of Theorem 3.3 is that such a deformation will satisfy the radial equilibrium equation for *every* stored energy function of the form (6.1) that satisfies (Og1) and (Og2), *provided* h is defined by (3.4). The difficulty is then ensuring that this equilibrium deformation is the unique radial minimizer of the energy, i.e., that the combination of deformation and stored energy satisfies (5.4).

One method of obtaining a family of such deformations is to perturb from a known solution. We illustrate this idea when the initial solution is isochoric. The resulting deformations will be nearly incompressible, as is expected in many elastomers (see, e.g., [2] or [8]), since the resulting energy will heavily penalize even small changes in volume. First, let's restrict our attention to the energy

$$\Psi(\lambda_1, \lambda_2, \lambda_3) := \phi(\lambda_1) + \phi(\lambda_2) + \phi(\lambda_3), \tag{6.7}$$

where ϕ satisfies (Og1), (Og2) and ϕ' is a convex function. (For example, $\phi(t) = t^{\beta+1}$ with $\beta \in [1,2)$.) Then (5.4) reduces to

$$\frac{d}{dR}\left[R^2\phi'(\rho'(R))\right] < 2R\phi'\left(\frac{\rho(R)}{R}\right)$$

or, equivalently,

$$R\phi''(\rho'(R))\rho''(R) < 2\left[\phi'\left(\frac{\rho(R)}{R}\right) - \phi'(\rho'(R))\right]. \tag{6.8}$$

However, the mapping $t \mapsto \phi'(t)$ is convex:

$$\phi'\left(\frac{\rho(R)}{R}\right) \geq \phi'(\rho'(R)) + \phi''(\rho'(R))\left[\frac{\rho(R)}{R} - \rho'(R)\right],$$

and so, in view of (6.8), a sufficient condition for (5.4) is that the function ρ satisfies

$$R\rho''(R) < 2\left[\frac{\rho(R)}{R} - \rho'(R)\right] \tag{6.9}$$

(or, equivalently, $\frac{d}{dR}\mathrm{trace}(\mathbf{F}) > 0$).

Now, let $\lambda_0 > 0$, and suppose that $\theta : [0,1] \to [0,1]$ is continuous with $\theta > 0$ on $(0,1]$ and that

$$\theta(s) = o(s^6) \quad \text{as} \quad s \to 0^+. \tag{6.10}$$

Then for each $s \in [0,1]$ define

$$\rho(R,s)^3 := 1 + 3\int_0^R J(t,s)t^2\,dt, \tag{6.11}$$

$$J(R,s) := \lambda_0^3 + \theta(s)(1 - e^{-sR}) \tag{6.12}$$

and note that

$$J_R(R,s) = s\theta(s)e^{-sR}, \qquad s\theta(s)\int_0^\infty t^3 e^{-st}\,dt = \frac{6\theta(s)}{s^3}, \tag{6.13}$$

where the subscript R denotes the partial derivative with respect to R. (If $\lambda_0 = 1$, the deformation at $s = 0$ is isochoric.) By (6.13), for each $s > 0$, $R \mapsto J(R,s)$ is strictly monotone increasing and satisfies $(3.1)_3$, provided $0 < \theta(s) \le s^3/6$, which is a consequence of (6.10) for all sufficiently small s.

Next, by (4.2), (4.3), and $(6.13)_1$,

$$R\rho''(R)\left[\frac{\rho(R)}{R}\right]^2 = s\theta(s)Re^{-sR} - 2\rho'(R)\left[\frac{\rho(R)}{R}\right]\left[\rho'(R) - \frac{\rho(R)}{R}\right],$$

which shows that (6.9) is equivalent to

$$s\theta(s)Re^{-sR} < 2\left[\frac{\rho(R)}{R}\right]\left[\frac{\rho(R)}{R} - \rho'(R)\right]^2. \tag{6.14}$$

However, in view of (4.1), (4.2), and $(6.13)_1$,

$$R^6\left[\frac{\rho(R)}{R}\right]^4\left[\frac{\rho(R)}{R} - \rho'(R)\right]^2 = \left[1 - s\theta(s)\int_0^R t^3 e^{-st}\,dt\right]^2,$$

so that (6.14) is the same as

$$s\theta(s)R^7\left[\frac{\rho(R)}{R}\right]^3 < 2e^{sR}\left[1 - s\theta(s)\int_0^R t^3 e^{-st}\,dt\right]^2. \tag{6.15}$$

Finally, we note that in view of (6.10), $\theta(s) \le s^3/12$ for s sufficiently small, and consequently, by $(6.13)_2$,

$$s\theta(s)\int_0^R t^3 e^{-st}\,dt \le \frac{1}{2} \tag{6.16}$$

for small s. In addition, $e^t \ge (1+t^7)/7!$ and hence, upon multiplying (6.15) by $(sR)^{-7}$ and making use of (6.16), it suffices to show

$$2\frac{\theta(s)}{s^6}\left[\frac{\rho(R)}{R}\right]^3 < \frac{1}{7!}\left[(sR)^{-7} + 1\right] \tag{6.17}$$

in order to obtain (6.15). However, for small s, (6.17) is a consequence of (6.10)–(6.12), which completes the example.

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