Energy Minimising Properties of the Radial Cavitation Solution in Incompressible Nonlinear Elasticity

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Abstract. Consider an incompressible hyperelastic material, occupying the unit ball $B \subset \mathbb{R}^n$ in its reference state. Suppose that the displacement is specified on the boundary, that is,

$$\mathbf{u}(\mathbf{x}) = \lambda \mathbf{x} \quad \text{for} \quad \mathbf{x} \in \partial B,$$

where $\lambda > 1$ is a given constant.

In this paper, isoperimetric arguments are used to prove that the radial deformation, producing a spherical cavity, is the energy minimiser in a general class of isochoric deformations that are discontinuous at the centre of the ball and produce a (possibly non-symmetric) cavity in the deformed body. This result has implications for the study of cavitation in certain polymers.

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1 Introduction.

Let $B \subset \mathbb{R}^n$ denote the unit ball, the physically relevant values being $n = 2$ or 3. Consider deformations $\mathbf{u} : B \to \mathbb{R}^n$ of an incompressible, nonlinearly elastic material occupying the region B in its reference state. Thus, the admissible deformations $\mathbf{u}$ satisfy the incompressibility constraint[1]

$$\det \nabla \mathbf{u} = 1 \quad \text{for a.e. } \mathbf{x} \in B. \tag{1.1}$$

Deformations satisfying the above condition are known as *isochoric* deformations. In nonlinear hyperelasticity, with each such deformation we associate a corresponding energy

$$E(\mathbf{u}) = \int_B W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.2}$$

where $W : M_1^{n\times n} \to \mathbb{R}$ is the stored-energy function and $M_1^{n\times n}$ denotes the set of $n \times n$ matrices with determinant equal to 1. In the variational approach, we seek equilibrium states by minimising (1.2) on some class of admissible deformations satisfying given boundary conditions (displacement or traction). In this paper we consider a problem which arises from the study of radial cavitation initiated by Ball in the fundamental paper [2]. The work of Ball was, in part, motivated by the work of [3] and subsequently developed by many authors (see, e.g., [6, 9, 17] and the review article [5]). In [2], Ball studies energy minimisers for compressible[2] and incompressible materials in the class of radial deformations of B. It is shown therein that if the imposed boundary tractions or displacements are sufficiently large, then the radial deformation which minimises the energy is discontinuous and corresponds to a hole forming at the centre of the deformed ball. This is the phenomenon of cavitation. To date, little is known about the minimising properties of these radial cavitation solutions in the general class of all (possibly non-symmetric) deformations.[3] The current paper addresses this problem in the incompressible case and proves that the radial incompressible minimiser is the global minimiser of the energy amongst all (possibly non-symmetric) isochoric deformations producing a hole at the centre of the deformed ball. To prove this, we will draw on and refer to a number of key ideas and results from [14].

We will consider the displacement boundary-value problem in which the deformations are required to satisfy

$$\mathbf{u}(\mathbf{x}) = \lambda \mathbf{x} \quad \text{for } \mathbf{x} \in \partial B, \tag{1.3}$$

where $\lambda > 1$ is a given constant and we study energy minimisers of (1.2) in the class of isochoric deformations. Since it is not possible for a smooth isochoric deformation to satisfy (1.3) for any $\lambda > 1$, we must enlarge the class of deformations. We therefore consider those deformations $\mathbf{u}$ that produce a single discontinuity at the centre of the deformed ball. Such discontinuous deformations can be viewed as an idealised limit, as $\epsilon \to 0$, of deformations $\mathbf{u}_\epsilon$ of punctured balls $B_\epsilon = \{\mathbf{x} \in \mathbb{R}^n : \epsilon < |\mathbf{x}| < 1\}$, where $\mathbf{u}_\epsilon(\mathbf{x}) = \lambda \mathbf{x}$ on the outer boundary of B_ϵ and the inner

[1] Vulcanized rubber is often modelled as an incompressible material.

[2] In the compressible case, the condition (1.1) is replaced by $\det \nabla \mathbf{u} > 0$.

[3] In the compressible case, partial results on minimising properties are contained in [10, 11] (see also [13]).

boundary is left free.[4] Thus, such discontinuous equilibria should approximate the behaviour of a ball containing a microvoid of radius ϵ at its centre, provided that ϵ is sufficiently small.

Radial Deformations.

If $\mathbf{u} \in C^1(\overline{B}\backslash\{\mathbf{0}\};\mathbb{R}^n)$ is a radial deformation of the form

$$\mathbf{u} = \frac{r(R)}{R}\mathbf{x}, \qquad R := |\mathbf{x}|, \qquad r : [0,1] \to [0,\infty), \tag{1.4}$$

then

$$\nabla\mathbf{u}(\mathbf{x}) = r'(R)\left(\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) + \frac{r(R)}{R}\left(\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) \tag{1.5}$$

and so condition (1.1) forces

$$r'(R)\left(\frac{r(R)}{R}\right)^{n-1} = 1 \quad \text{for } R \in [0,1], \tag{1.6}$$

from which it follows that $r(R) = (R^n + A^n)^{\frac{1}{n}}$, where A is a constant. Therefore, the only kinematically admissible, isochoric, radial deformation satisfying (1.3) is

$$\mathbf{u}_\lambda^{\mathrm{rad}}(\mathbf{x}) = r_\lambda(R)\frac{\mathbf{x}}{|\mathbf{x}|}, \qquad r_\lambda(R) = \Big[R^n + (\lambda^n - 1)\Big]^{\frac{1}{n}}. \tag{1.7}$$

Note that $\mathbf{u}_\lambda^{\mathrm{rad}}$ creates a spherical hole of radius $(\lambda^n - 1)^{\frac{1}{n}} > 0$ at the centre of the ball.

Suppose further that $E(\mathbf{u}_\lambda^{\mathrm{rad}}) < \infty$. Our aim is to use isoperimetric inequalities to prove that the radial deformation (1.7) is a global minimiser in a set of deformations of the ball which can produce (possibly non-symmetric) cavities at the centre. In particular, given a deformation $\mathbf{u}$ in the *admissible* set[5]

$$\mathcal{A}_\lambda = \left\{\mathbf{u} \in C^1(\overline{B}\backslash\{\mathbf{0}\};\mathbb{R}^n) : \det\nabla\mathbf{u} \equiv 1,\ \mathbf{u} \text{ is one-to-one},\ \mathbf{u}(\mathbf{x}) = \lambda\mathbf{x} \text{ for } \mathbf{x} \in \partial B\right\},$$

we prove in Theorem 4.1 that

$$E(\mathbf{u}) \geq E(\mathbf{u}_\lambda^{\mathrm{rad}})$$

for any stored-energy function W of the form[6]

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^{n-1}, |\operatorname{adj}\mathbf{F}|) \tag{1.8}$$

where $\Phi : [0,\infty)\times[0,\infty) \to [0,\infty)$ is convex and $s \mapsto \Phi(s,t)$ and $s \mapsto \Phi(t,s)$ are non-decreasing functions for each $t > 0$.

We develop our results for deformations $\mathbf{u} \in C^1(\overline{B}\backslash\{\mathbf{0}\};\mathbb{R}^n)$, first for the case $W(\mathbf{F}) = |\mathbf{F}|^{n-1}$ (in Section 2) and then for the case $W(\mathbf{F}) = |\operatorname{adj}\mathbf{F}|$ (in Section 3). These proofs are then easily generalised to the wider class of stored-energy functions (1.8) in Section 4 and finally to general Sobolev deformations in Section 5. In order to emphasize the dimensional dependence, we present our results for general $n \geq 2$.

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[4] See, e.g., [1] for the traction problem for incompressible materials and [5, 9, 15] for the displacement problem for compressible materials.

[5] There are infinitely many such isochoric maps: simply set $\mathbf{u} = \mathbf{w}\circ\mathbf{u}^{\mathrm{rad}}$ where $\mathbf{w}$ is any non-radial isochoric deformation of the annulus $A = \{\mathbf{x} \in \mathbb{R}^n : (\lambda^n - 1)^{\frac{1}{n}} \leq |\mathbf{x}| \leq \lambda\}$ satisfying $\mathbf{w}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ that obey $|\mathbf{x}| = \lambda$.

[6] This class of stored-energy functions includes the energy functions $\int_B |\nabla\mathbf{u}|^p\,d\mathbf{x}$, $n-1 \leq p < n$. However, if $W(\mathbf{F}) \geq \alpha_1|\mathbf{F}|^n - \alpha_2$ for some $\alpha_1 > 0$ then *every* $\mathbf{u} \in \mathcal{A}_\lambda$ has infinite energy when $\lambda > 1$.

1.1 Basic Notations.

For each $\mathbf{F} \in M^{n\times n}$ (the set of $n \times n$ matrices) we use $\operatorname{adj}\mathbf{F}$ to denote the adjugate of $\mathbf{F}$ (i.e., the unique $n \times n$ matrix satisfying $\mathbf{F}(\operatorname{adj}\mathbf{F}) = (\det\mathbf{F})\mathbf{I}$, where $\mathbf{I} \in M^{n\times n}$ denotes the identity matrix). The average value of ϕ over the $(n-1)$-dimensional surface $S \subset \mathbb{R}^n$, i.e., the surface integral of ϕ over S divided by the area of S, will be denoted by[7]

$$\fint_S \phi(\mathbf{x}) := \frac{\int_S \phi(\mathbf{x})\, dA}{\int_S dA}$$

We use $|\mathbf{F}|$ to denote the Euclidean norm of the matrix $\mathbf{F}$ defined by $|\mathbf{F}|^2 = \operatorname{Trace}(\mathbf{F}^{\mathrm{T}}\mathbf{F})$ for all $\mathbf{F} \in M^{n\times n}$. For $n \geq 3$, the vector product $\mathbf{a}_1 \times \mathbf{a}_2 \ldots \times \mathbf{a}_{n-1}$ of $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1} \in \mathbb{R}^n$ is defined to be the unique vector $\mathbf{v} \in \mathbb{R}^n$ satisfying $\mathbf{x} \cdot \mathbf{v} = \det(\mathbf{x}, \mathbf{a}_1, \ldots, \mathbf{a}_{n-1})$ for all $\mathbf{x} \in \mathbb{R}^n$ (see, e.g., [16]). The notation $\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, with $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n$, denotes the determinant of the $n \times n$ matrix whose i^{th} column consists of the entries from the vector $\mathbf{a}_i$.

We will make use of the following standard vector identities which are stated together for convenience.

Lemma 1.1. *Let $\mathbf{n}, \mathbf{t}_1, \ldots, \mathbf{t}_{n-1} \in \mathbb{R}^n$ be positively oriented and orthonormal.*

(i) If $\mathbf{A} \in M_1^{n\times n}$ then $\operatorname{adj}(\operatorname{adj}\mathbf{A}) = \mathbf{A}$;

(ii) If $\mathbf{G} \in M^{n\times n}$, $n \geq 3$, and $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1} \in \mathbb{R}^n$ then

$$\mathbf{G}\mathbf{a}_1 \times \ldots \times \mathbf{G}\mathbf{a}_{n-1} = (\operatorname{adj}\mathbf{G})^{\mathrm{T}}(\mathbf{a}_1 \times \ldots \times \mathbf{a}_{n-1})\,;$$

(iii) Since $\mathbf{n} = \mathbf{t}_1 \times \ldots \times \mathbf{t}_{n-1}$, it follows by part (ii) that

$$|(\mathbf{G}\mathbf{t}_1) \times \ldots \times (\mathbf{G}\mathbf{t}_{n-1})| = \left|(\operatorname{adj}\mathbf{G})^{\mathrm{T}}\mathbf{n}\right|,$$

for any $\mathbf{G} \in M^{n\times n}$ and any $n \geq 3$.

Remark 1.2. If $n = 2$ then part (iii) of the lemma still holds, i.e., $|\mathbf{G}\mathbf{t}_1| = |(\operatorname{adj}\mathbf{G})^{\mathrm{T}}\mathbf{n}|$.

The next lemma is a straightforward consequence of part (iii) of Lemma 1.1 together with the change of variables formula for smooth surfaces in $\mathbb{R}^n$.

Lemma 1.3. *Let $\mathbf{u} \in \mathcal{A}_\lambda$ and $n \geq 3$. Then for each $R \in (0, 1]$*

$$\int_{S_R} \left|\frac{\partial\mathbf{u}}{\partial\mathbf{t}_1} \times \ldots \times \frac{\partial\mathbf{u}}{\partial\mathbf{t}_{n-1}}\right| = \int_{S_R} \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right| = \operatorname{Area}(\mathbf{u}(S_R)), \tag{1.9}$$

where S_R denotes the sphere of radius $R > 0$ centred on the origin and $\frac{\partial\mathbf{u}}{\partial\mathbf{a}} := (\nabla\mathbf{u})\mathbf{a}$.

Remark 1.4. If $n = 2$ the above lemma still holds in the sense that

$$\int_{S_R} \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right| = \operatorname{Length}(\mathbf{u}(S_R)),$$

for each $R \in (0, 1]$, where S_R denotes the circle of radius $R > 0$ centred at the origin.

[7] In order to simplify our notation, we will often omit the surface measure dA from integrals over S.

2 The case $W(\mathbf{F}) = |\mathbf{F}|^{n-1}$.

Lemma 2.1. *Let $\mathbf{u} \in C^1(\overline{B}\backslash\{\mathbf{0}\};\mathbb{R}^n)$. At each point $\mathbf{x} \in S_R$, $R \in (0,1]$, let $\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{n-1}$ denote a positively oriented orthonormal basis with $\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}$. Then*

$$|\nabla \mathbf{u}(\mathbf{x})|^2 = \left|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|^2 + \left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_1}\right|^2 + \ldots + \left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_{n-1}}\right|^2 \quad \text{for} \quad \mathbf{x} \in S_R.$$

Proof. This is a standard consequence of the invariance of the Dirichlet integral under orthogonal changes of coordinates (see, e.g., [14, Lemma 3.3]). □

Lemma 2.2. *Let $\mathbf{u} \in C^1(\overline{B}\backslash\{\mathbf{0}\};\mathbb{R}^n)$. Then for each $\mathbf{x} \in S_R$, $R \in (0,1]$, we have*

$$|\nabla \mathbf{u}|^2 \geq \frac{1}{|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}|^2} + (n-1)\left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^{\frac{2}{n-1}}. \tag{2.1}$$

Proof. Suppose first that $n \geq 3$, then Lemma 2.1 together with the arithmetic-geometric mean inequality yields

$$\begin{aligned} |\nabla \mathbf{u}|^2 &\geq \left|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|^2 + (n-1)\left(\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_1}\right|\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_2}\right|\cdots\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_{n-1}}\right|\right)^{\frac{2}{n-1}} \\ &\geq \left|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|^2 + (n-1)\left(\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_1} \times \ldots \times \frac{\partial \mathbf{u}}{\partial \mathbf{t}_{n-1}}\right|\right)^{\frac{2}{n-1}}. \end{aligned} \tag{2.2}$$

Next, since $\mathbf{u}$ is isochoric, the Cauchy-Schwarz inequality implies

$$1 = (\det \nabla \mathbf{u})(\mathbf{n} \cdot \mathbf{n}) = ((\nabla \mathbf{u})\mathbf{n}) \cdot \left((\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}\right) \leq \left|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right| \left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}\right|.$$

Combining this with (2.2) and using Lemma 1.1(iii) it then follows that (2.1) holds. If $n = 2$, then (2.1) follows similarly from Lemma 2.1 on using Remark 1.2. □

Lemma 2.3. *For $t > 0$ define*

$$g(t) = \left[\frac{1}{t^2} + (n-1)t^{\frac{2}{n-1}}\right]^{\frac{n-1}{2}}.$$

Then g is convex on $(0,\infty)$ and monotone increasing for $t \geq 1$.

Proof. The monotonicity of g on $[1,\infty)$ is a straightforward calculation. The convexity follows on observing that

$$g(t) = t\phi\left(\frac{1}{t}\right), \qquad \phi(y) := \left(y^{\frac{2n}{n-1}} + n - 1\right)^{\frac{n-1}{2}}, \qquad g''(t) = \frac{1}{t^3}\phi''\left(\frac{1}{t}\right),$$

and $\phi'' \geq 0$. □

Lemma 2.4. *Let* $\mathbf{u} \in C^1(\overline{B}\backslash\{\mathbf{0}\};\mathbb{R}^n)$*. Then for each* $R \in (0,1]$

$$\fint_{S_R} |\nabla \mathbf{u}|^{n-1} \geq \fint_{S_R} g\left(\left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|\right) \geq g\left(\fint_{S_R} \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|\right).$$

Proof. This is a consequence of Lemma 2.2, the convexity of g, and Jensen's inequality. □

Lemma 2.5. *Let* $\lambda > 1$ *and* $\mathbf{u} \in \mathcal{A}_\lambda$*. Then for each* $R \in (0,1]$

$$\fint_{S_R} \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right| = \frac{\operatorname{Area}(\mathbf{u}(S_R))}{\bar{\omega}_n R^{n-1}} \geq \frac{\operatorname{Area}(\mathbf{u}_\lambda^{\mathrm{rad}}(S_R))}{\bar{\omega}_n R^{n-1}} = \fint_{S_R} \left|(\operatorname{adj}\nabla\mathbf{u}_\lambda^{\mathrm{rad}})^{\mathrm{T}}\mathbf{n}\right|$$
$$= \left(\frac{r_\lambda(R)}{R}\right)^{n-1} \geq 1,$$

where r_λ *and* $\mathbf{u}_\lambda^{\mathrm{rad}}$ *are given by (1.7) and* $\bar{\omega}_n$ *is the surface area of the unit sphere in* $\mathbb{R}^n$*.*

Proof. The proof of this result is exactly analogous to the proof of [14, Lemma 3.8] and is a consequence of the classical isoperimetric inequality on noting that the volumes enclosed by the surfaces $\mathbf{u}(S_R)$ and $\mathbf{u}_\lambda^{\mathrm{rad}}(S_R)$ are equal[8] for each $R \in (0,1]$. □

Theorem 2.6. *Let* $\lambda > 1$ *and* $\mathbf{u} \in \mathcal{A}_\lambda$*. Then for each* $R \in (0,1]$

$$\fint_{S_R} |\nabla\mathbf{u}|^{n-1} \geq \fint_{S_R} |\nabla\mathbf{u}_\lambda^{\mathrm{rad}}|^{n-1}$$

Proof. This inequality follows from Lemma 2.4, the monotonicity of g, and Lemma 2.5 on noting that for each $R \in (0,1]$,

$$g\left(\fint_{S_R} \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|\right) \geq g\left(\fint_{S_R} \left|(\operatorname{adj}\nabla\mathbf{u}_\lambda^{\mathrm{rad}})^{\mathrm{T}}\mathbf{n}\right|\right) = g\left(\left(\frac{r_\lambda(R)}{R}\right)^{n-1}\right) = |\nabla\mathbf{u}_\lambda^{\mathrm{rad}}|^{n-1},$$

by (A.1), (A.2), and the definition of g. □

3 The case $W(\mathbf{F}) = |\operatorname{adj}\mathbf{F}|$.

In this section we derive similar estimates to those obtained in the previous section but this time for the energy function $W(\mathbf{F}) = |\operatorname{adj}\mathbf{F}|$.

Lemma 3.1. *Let* $\mathbf{u} \in C^1(\overline{B}\backslash\{\mathbf{0}\};\mathbb{R}^n)$*. Then for each* $\mathbf{x} \in S_R$*,* $R \in (0,1]$*, we have*

$$|\operatorname{adj}\nabla\mathbf{u}|^2 \geq \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^2 + \frac{n-1}{\left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{n}\right|^{\frac{2}{n-1}}}. \tag{3.1}$$

[8] The change of variables formula implies that every smooth, isochoric, one-to-one map preserves volumes.

Proof. Let $n \geq 3$. Writing $\mathbf{G} := \operatorname{adj} \nabla \mathbf{u}$ we note that, using the arithmetic-geometric mean inequality,

$$|\mathbf{G}|^2 = \left|\mathbf{G}^{\mathrm{T}}\right|^2 = \left|\mathbf{G}^{\mathrm{T}}\mathbf{n}\right|^2 + \left|\mathbf{G}^{\mathrm{T}}\mathbf{t}_1\right|^2 + \ldots + \left|\mathbf{G}^{\mathrm{T}}\mathbf{t}_{n-1}\right|^2$$
$$\geq \left|\mathbf{G}^{\mathrm{T}}\mathbf{n}\right|^2 + (n-1)\left|\left(\mathbf{G}^{\mathrm{T}}\mathbf{t}_1\right) \times \ldots \times \left(\mathbf{G}^{\mathrm{T}}\mathbf{t}_{n-1}\right)\right|^{\frac{2}{n-1}}. \tag{3.2}$$

Next, observe that, by (i) and (iii) of Lemma 1.1,

$$\left((\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{t}_1\right) \times \ldots \times \left((\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{t}_{n-1}\right) = (\nabla \mathbf{u})\mathbf{n}. \tag{3.3}$$

Finally, by the Cauchy-Schwarz inequality and the fact that $\det \nabla \mathbf{u} = 1$,

$$1 = |\mathbf{n}|^2 = \left|(\nabla \mathbf{u})\mathbf{n} \cdot (\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}\right| \leq \left|(\nabla \mathbf{u})\mathbf{n}\right| \left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}\right|. \tag{3.4}$$

Equations (3.2)–(3.4) then yield (3.1). The case $n = 2$ follows similarly from $(3.2)_1$ on using (i) of Lemma 2.1 and Remark 1.2 (with $\mathbf{G} = (\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}$). □

Lemma 3.2. *For $t > 0$ define*

$$h(t) = \left(t^2 + \frac{n-1}{t^{\frac{2}{n-1}}}\right)^{\frac{1}{2}}.$$

Then h is convex on $(0, \infty)$ and monotone increasing for $t \geq 1$.

Proof. The monotonicity of h is a straightforward calculation. The convexity follows on observing that

$$h(t) = t\psi\left(\frac{1}{t}\right), \qquad \psi(y) := \left(1 + (n-1)y^{\frac{2n}{n-1}}\right)^{\frac{1}{2}}, \qquad h''(t) = \frac{1}{t^3}\psi''\left(\frac{1}{t}\right),$$

and $\psi'' \geq 0$. □

Theorem 3.3. *Let $\lambda > 1$ and $\mathbf{u} \in \mathcal{A}_\lambda$. Then for each $R \in (0, 1]$*

$$\fint_{S_R} |\operatorname{adj} \nabla \mathbf{u}| \geq \fint_{S_R} |\operatorname{adj} \nabla \mathbf{u}_\lambda^{\mathrm{rad}}|.$$

Proof. First note that by Lemma 1.3, $\int_{S_R} |(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}|$ is the area of $\mathbf{u}(S_R)$. Hence, by Lemma 3.1, the convexity of h, and Jensen's inequality

$$\fint_{S_R} |\operatorname{adj} \nabla \mathbf{u}| \geq h\left(\fint_{S_R} \left|(\operatorname{adj} \nabla \mathbf{u})^{\mathrm{T}}\mathbf{n}\right|\right) \geq h\left(\fint_{S_R} \left|(\operatorname{adj} \nabla \mathbf{u}_\lambda^{\mathrm{rad}})^{\mathrm{T}}\mathbf{n}\right|\right),$$

where the last inequality follows from the monotonicity of h and Lemma 2.5. Finally, note that by (A.2) and (A.3),

$$h\left(\fint_{S_R} \left|(\operatorname{adj} \nabla \mathbf{u}_\lambda^{\mathrm{rad}})^{\mathrm{T}}\mathbf{n}\right|\right) = h\left(\left(\frac{r_\lambda(R)}{R}\right)^{(n-1)}\right) = |\operatorname{adj} \nabla \mathbf{u}_\lambda^{\mathrm{rad}}|,$$

where r_λ and $\mathbf{u}_\lambda^{\mathrm{rad}}$ are related by (1.7). □

4 The general case: $W(\mathbf{F}) = \Phi(|\mathbf{F}|^{n-1}, |\operatorname{adj}\mathbf{F}|)$.

Now suppose that $W(\mathbf{F}) = \Phi(|\mathbf{F}|^{n-1}, |\operatorname{adj}\mathbf{F}|)$. If we now combine the arguments of Sections 2 and 3 we obtain the following result.

Theorem 4.1. *Let* $\lambda > 1$ *and*

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^{n-1}, |\operatorname{adj}\mathbf{F}|),$$

where $\Phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is convex and* $s \mapsto \Phi(s,t)$ *and* $s \mapsto \Phi(t,s)$ *are non-decreasing functions for each* $t > 0$. *Then for every* $\mathbf{u} \in \mathcal{A}_\lambda$

$$E(\mathbf{u}) = \int_B W(\nabla\mathbf{u})\, d\mathbf{x} \geq \int_B W(\nabla\mathbf{u}_\lambda^{\mathrm{rad}})\, d\mathbf{x} = E(\mathbf{u}_\lambda^{\mathrm{rad}}).$$

Proof. We first note that $\Phi \geq 0$ and so if $\mathbf{u} \in \mathcal{A}_\lambda$ has infinite energy we are done. Thus we assume that $E(\mathbf{u}) < \infty$. Using Jensen's inequality, the monotonicity of Φ, and Theorems 2.6 and 3.3 we obtain

$$\begin{aligned}
\int_B W(\nabla\mathbf{u})\, d\mathbf{x} &= \int_0^1 \left(\int_{S_R} \Phi(|\nabla\mathbf{u}|^{n-1}, |\operatorname{adj}\nabla\mathbf{u}|) \right) dR \\
&\geq \int_0^1 \bar{\omega}_n R^{n-1} \Phi\left(\fint_{S_R} |\nabla\mathbf{u}|^{n-1}, \fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}| \right) dR \\
&\geq \int_0^1 \bar{\omega}_n R^{n-1}\ \Phi\left(\fint_{S_R} |\nabla\mathbf{u}_\lambda^{\mathrm{rad}}|^{n-1}, \fint_{S_R} |\operatorname{adj}\nabla\mathbf{u}_\lambda^{\mathrm{rad}}| \right) dR \\
&= \int_0^1 \bar{\omega}_n R^{n-1} \Phi\left(|\nabla\mathbf{u}_\lambda^{\mathrm{rad}}|^{n-1}\ , |\operatorname{adj}\nabla\mathbf{u}_\lambda^{\mathrm{rad}}| \right) dR = \int_B W(\nabla\mathbf{u}_\lambda^{\mathrm{rad}})\, d\mathbf{x}.
\end{aligned}$$

□

5 Extension to Sobolev Deformations.

The results in [12, Theorem 4.1] guarantee the existence of an absolute minimiser for any continuous polyconvex energy, satisfying appropriate growth conditions, amongst deformations that create a single (possibly non-symmetric) hole at the centre of the ball[9]. A straightforward modification of the proof of this theorem yields a corresponding existence result in the class of isochoric deformations. In this section we outline the main steps in an argument that demonstrates that the radial function, $\mathbf{u}_\lambda^{\mathrm{rad}}$ given by (1.7), is in fact the absolute minimiser of the energy whose existence is given by such a modified theorem. We start with the necessary technical preliminaries.

[9]The existence results in [12] follow by applying the direct method of the calculus of variations as used by Ball [1] together with the extensions in [8, 12, 18] to include cavitation.

For $p > n-1$ the *distributional Jacobian* of a mapping $\mathbf{u} \in W^{1,p}(B;\mathbb{R}^n) \cap L^\infty(B;\mathbb{R}^n)$ is the linear functional, $(\mathrm{Det}\,\nabla\mathbf{u}) : C_0^\infty(B) \to \mathbb{R}$, given by

$$(\mathrm{Det}\,\nabla\mathbf{u})(\phi) = -\frac{1}{n}\int_B \phi_{,\alpha} u^i (\mathrm{adj}\,\nabla\mathbf{u})_i^\alpha \, d\mathbf{x} \quad \text{for all } \phi \in C_0^\infty(B). \tag{5.1}$$

If $\mathbf{u} \in C^2(\overline{B};\mathbb{R}^n)$ then integrating (5.1) by parts yields

$$(\mathrm{Det}\,\nabla\mathbf{u})(\phi) = \int_B \phi(\det \nabla\mathbf{u})\, d\mathbf{x} \quad \text{for all } \phi \in C_0^\infty(B).$$

Thus, the distributional Jacobian is equal to the ordinary Jacobian, $\det\nabla\mathbf{u}$, and we write

$$\mathrm{Det}\,\nabla\mathbf{u} = (\det\nabla\mathbf{u})\mathcal{L}^n$$

in the sense of distributions, where $\mathcal{L}^n$ denotes n-dimensional Lebesgue measure. If, however, the map $\mathbf{u}$ has a discontinuity at the centre of the ball, then roughly speaking, we pick up an extra contribution from the point of discontinuity in the integration by parts process and find that

$$\mathrm{Det}\,\nabla\mathbf{u} = (\det\nabla\mathbf{u})\mathcal{L}^n + \alpha\delta_{\mathbf{0}}, \tag{5.2}$$

where $\delta_{\mathbf{0}}$ denotes the Dirac measure supported at the centre $\mathbf{0}$ of the ball. The coefficient α is the volume of the hole created at the origin. Thus specifying that a deformation satisfy (5.2), for some $\alpha \geq 0$, in the sense of distributions is a precise mathematical formulation of the requirement that a deformation opens at most one hole at the centre of B.

There are two other mathematical difficulties that must be addressed to obtain the existence of an energy minimizer. Firstly, the distributional Jacobian as defined above does not detect cavities that form at the boundary of the ball[10] and, secondly, the minimizer obtained using the direct method of the calculus of variations must not interpenetrate matter (see [8, §11]).

A possible solution to the first problem (and the one adopted in [7, 12]) is to extend the deformation using the boundary-values, i.e., for some $\epsilon > 0$ define

$$\mathbf{u}^e(\mathbf{x}) := \begin{cases} \mathbf{u}(\mathbf{x}), & \mathbf{x} \in B \\ \lambda\mathbf{x}, & \mathbf{x} \in B_{1+\epsilon}\backslash B \end{cases} \tag{5.3}$$

and then require that (5.2) be satisfied by $\mathbf{u}^e$ on the extended domain $B_{1+\epsilon}$.

The specific mathematical difficulty encountered in the second problem is that the weak limit of a sequence of one-to-one mappings contained in the Sobolev space $W^{1,p}(B;\mathbb{R}^n)$, with $p < n$, need not be one-to-one. A possible solution to this problem (and the one which is adopted in [7, 8, 12]) is to require that deformations satisfy condition (INV). Condition (INV) is preserved under weak convergence ([8, Lemma 3.3]) and (INV) together with the positivity of the Jacobian implies that a mapping is one-to-one almost everywhere ([8, Lemma 3.4]). Condition (INV) is essentially[11] the requirement that a mapping be monotone in the sense

[10] See [7, Remark 3.3] and [8, Figures 5 and 6].

[11] See [8, §3] for a precise statement of condition (INV).

of Lebesgue and that a hole created in the body does not contain material originating from another part of the body.

We now proceed with the extension of the results in §2–5 to Sobolev deformations.

Definition 5.1. Fix $p \in (n-1, n)$, $\lambda > 1$, and $\epsilon > 0$. We define the set of *admissible Sobolev deformations* by

$$\mathcal{A}_\lambda^p = \left\{ \mathbf{u} \in W^{1,p}(B;\mathbb{R}^n) : \begin{array}{c} \mathbf{u}^e \text{ satisfies condition (INV) on } B_{1+\epsilon}, \\ \mathbf{u}(\mathbf{x}) = \lambda \mathbf{x} \text{ for } \mathbf{x} \in \partial B, \quad \det \nabla \mathbf{u} = 1 \text{ } a.e. \text{ in } B, \\ \operatorname{Det} \nabla \mathbf{u}^e = (\det \nabla \mathbf{u}^e)\mathcal{L}^n + \frac{\bar{\omega}_n}{n}\,(\lambda^n - 1)\,\delta_{\mathbf{0}} \end{array} \right\},$$

where $\bar{\omega}_n$ denotes the area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ ($\bar{\omega}_2 = 2\pi, \bar{\omega}_3 = 4\pi$) and $\mathbf{u}^e$ is given by (5.3).

The next result then extends Lemma 2.5 and follows from the proof of [14, Lemma 5.7].

Lemma 5.2. *Let $\lambda > 1$, $p \in (n-1, n)$, and $\mathbf{u} \in \mathcal{A}_\lambda^p$. Suppose that $\mathbf{u}_\lambda^{\mathrm{rad}}$ is given by (1.7). Then for almost every $R \in (0, 1+\epsilon)$*

$$\mathcal{H}^{n-1}(\mathbf{u}(S_R)) \geq \mathcal{H}^{n-1}(\mathbf{u}_\lambda^{\mathrm{rad}}(S_R)),$$

where $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional surface (Hausdorff) measure.

Finally, we arrive at our main result.

Theorem 5.3. *Let $\lambda > 1$ and suppose that W satisfies*

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|^{n-1}, |\operatorname{adj} \mathbf{F}|),$$

where $\Phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ is convex and $s \mapsto \Phi(s,t)$ and $s \mapsto \Phi(t,s)$ are non-decreasing functions for each $t > 0$. Assume that there are constants $p \in (n-1, n)$, $c_0 > 0$, and c_2 such that, for all $s > 0$ and $t > 0$,

$$\Phi(s,t) \geq c_0 s^{\frac{p}{n-1}} - c_2.$$

Suppose further that $E(\mathbf{u}) < \infty$ for some $\mathbf{u} \in \mathcal{A}_\lambda^p$. Then $\mathbf{u}_\lambda^{\mathrm{rad}}$, given by (1.7), is a global minimiser of the energy on $\mathcal{A}_\lambda^p$.

Proof. The existence of a minimizer, $\mathbf{u}_m \in \mathcal{A}_\lambda^p$, under the above hypotheses follows from a slight modification of the proof of [12, Theorem 4.1]. The fact that $E(\mathbf{u}_m) \geq E(\mathbf{u}_\lambda^{\mathrm{rad}})$ is then a consequence of the arguments used to prove Theorem 4.1. See the proof of [14, Theorem 5.8] for details. □

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A Appendix.

Lemma A.1. *Let* $\mathbf{u}_\lambda^{\mathrm{rad}}(\mathbf{x})$ *be given by (1.7). Then*

$$\nabla\mathbf{u}_\lambda^{\mathrm{rad}}(\mathbf{x}) = \left(\frac{R}{r_\lambda}\right)^{(n-1)}\left(\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) + \left(\frac{r_\lambda}{R}\right)\left(\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right)$$

and

$$|\nabla\mathbf{u}_\lambda^{\mathrm{rad}}|^2 = \mathrm{Tr}\left[(\nabla\mathbf{u}_\lambda^{\mathrm{rad}})^{\mathrm{T}}\nabla\mathbf{u}_\lambda^{\mathrm{rad}}\right] = \left(\frac{R}{r_\lambda}\right)^{2(n-1)} + (n-1)\left(\frac{r_\lambda}{R}\right)^2. \tag{A.1}$$

Moreover,

$$\mathrm{adj}(\nabla\mathbf{u}_\lambda^{\mathrm{rad}})(\mathbf{x}) = \left(\frac{r_\lambda}{R}\right)^{(n-1)}\left(\frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) + \left(\frac{R}{r_\lambda}\right)\left(\mathbf{I} - \frac{\mathbf{x}\otimes\mathbf{x}}{|\mathbf{x}|^2}\right) \tag{A.2}$$

and

$$|\,\mathrm{adj}(\nabla\mathbf{u}_\lambda^{\mathrm{rad}})|^2 = \left(\frac{r_\lambda}{R}\right)^{2(n-1)} + (n-1)\left(\frac{R}{r_\lambda}\right)^2. \tag{A.3}$$

Proof. These identities follow from (1.4), (1.5), and the incompressibility constraint (1.6). □

References

[1] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337–403.

[2] J. M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Phil. Trans. R. Soc. Lond.* A **306** (1982), 557–611.

[3] A. N. Gent and P. B. Lindley, Internal rupture of bonded rubber cylinders in tension, *Proc. R. Soc. Lond. A* **249** (1958), 195–205.

[4] C.O. Horgan and R. Abeyaratne, A bifurcation problem for a compressible nonlinearly elastic medium: growth of a micro–void, *J. Elasticity* **16** (1986), 189–200.

[5] C. O. Horgan and D. A. Polignone, Cavitation in nonlinearly elastic solids: A review, *Appl. Mech. Rev.* **48** (1995), 471–485.

[6] F. Meynard, Existence and nonexistence results on the radially symmetric cavitation problem, *Quart. Appl. Math.* **50** (1992), 201–226.

[7] S. Müller, J. Sivaloganathan, and S. J. Spector, An isoperimetric estimate and $\mathrm{W}^{1,p}$-quasiconvexity in nonlinear elasticity, *Calc. Var. Partial Differential Equations* **8** (1999), 159–176.

[8] S. Müller and S. J. Spector, An existence theory for nonlinear elasticity that allows for cavitation, *Arch. Rational Mech. Anal.* **131** (1995), 1–66.

[9] J. Sivaloganathan, Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity, *Arch. Rational Mech. Anal.* **96** (1986), 97–136.

[10] J. Sivaloganathan, Singular minimizers in the calculus of variations: a degenerate form of cavitation, *Anal. non linéaire* **9** (1992), 657–681.

[11] J. Sivaloganathan, On the stability of cavitating equilibria, *Q. Appl. Math.* **53** (1995), 301–313.

[12] J. Sivaloganathan and S. J. Spector, On the existence of minimizers with prescribed singular points in nonlinear elasticity, *J. Elasticity* **59** (2000), 83–113.

[13] J. Sivaloganathan and S. J. Spector, Necessary conditions for a minimum at a radial cavitating singularity in nonlinear elasticity, *Anal. non linéaire* **25** (2008), 201–213.

[14] J. Sivaloganathan and S. J. Spector, On the Symmetry of Energy Minimising Deformations in Nonlinear Elasticity I: Incompressible Materials. Preprint, (2008): (http://www.math.siu.edu/spector/incompressible.pdf).

[15] J. Sivaloganathan, S. J. Spector, and V. Tilakraj, The convergence of regularized minimizers for cavitation problems in nonlinear elasticity, *SIAM J. Appl. Math.* **66**, (2006) 736–757.

[16] M. Spivak, *Calculus on Manifolds*, Benjamin, 1965.

[17] C. A. Stuart, Radially symmetric cavitation for hyperelastic materials, *Anal. non linéaire* **2** (1985), 33–66.

[18] V. Šverák, Regularity properties of deformations with finite energy, *Arch. Rational Mech. Anal.* **100** (1988), 105–127.