On the Symmetry of Energy Minimising Deformations in Nonlinear Elasticity I: Incompressible Materials

Jeyabal Sivaloganathan
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY, U.K.

Scott J. Spector
Department of Mathematics
Southern Illinois University
Carbondale, IL 62901, U.S.A.

Dedicated to Robin Knops

31st December 2007

Abstract. Consider an incompressible, nonlinear, hyperelastic material which occupies the region $A \subset \mathbb{R}^n$, $n \geq 2$, in its reference configuration, where A denotes the annular region

$$A = \{\mathbf{x} \in \mathbb{R}^n : a < |\mathbf{x}| < b\},$$

$0 < a < b$. Deformations of A are therefore isochoric maps $\mathbf{u} : A \to \mathbb{R}^n$ and so satisfy the incompressibility constraint

$$\det \nabla \mathbf{u} = 1.$$

The boundary of the annulus ∂A is separated into two disjoint pieces $\partial A = \partial A_o \cup \partial A_I$, where $\partial A_I = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = a\}$ and $\partial A_o = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = b\}$ denote the inner and outer boundary components respectively. We study displacement and mixed displacement/zero-traction boundary-value problems in which we impose a displacement boundary condition of the form

$$\mathbf{u}(\mathbf{x}) = \sigma \mathbf{x}$$

on one of the boundary components (where $\sigma > 0$ is a given constant) and the displacement on the remaining boundary component is either prescribed (in the case of the pure displacement boundary-value problem) or left unspecified (in the case of the mixed boundary-value problem).

In this paper we use isoperimetric arguments to prove that the radially symmetric solutions to these problems are global energy minimisers in various classes of (possibly non-symmetric) isochoric deformations of the annulus.

Mathematics Subject Classifications (2000): 74B20, 49K20, 35J50, 74G65

Key words: Symmetry, incompressible elasticity, radial minimiser.