On the Stability of Incompressible Elastic Cylinders in Uniaxial Extension

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Abstract

Consider a cylinder (not necessarily of circular cross-section) that is composed of a hyperelastic material and stretched parallel to its axis of symmetry. Suppose that the elastic material that constitutes the cylinder is homogeneous, transversely isotropic, and incompressible and that the deformed length of the cylinder is prescribed, the ends of the cylinder are free of shear, and the sides are left completely free. In this paper it is shown that mild additional constitutive hypotheses on the stored-energy function imply that the unique absolute minimizer of the elastic energy for this problem is a homogeneous, isoaxial deformation. This extends recent results that show the same result is valid in 2-dimensions. Prior work on this problem had been restricted to a local analysis: in particular, it was previously known that homogeneous deformations are strict (weak) relative minimizers of the elastic energy as long as the underlying linearized equations are strongly elliptic and provided that the load/displacement curve in this class of deformations does not possess a maximum.

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with centroid at the origin and consider a homogeneous, isotropic, incompressible, hyperelastic material that occupies the cylindrical region

$$\mathcal{C} := \{(x,y,z) \,:\, (x,y) \in \Omega,\ 0 < z < L\}$$

in a fixed homogeneous reference configuration. An isochoric deformation $\mathbf{u} : \overline{\mathcal{C}} \to \mathbb{R}^3$ of the body is then a continuously differentiable, one-to-one map that satisfies the constraint

$$\det \nabla\mathbf{u} \equiv 1 \quad \text{on } \overline{\mathcal{C}}. \tag{1.1}$$

The problem we herein consider is uniaxial extension. Specifically, we fix $\lambda \geq 1$ and restrict our attention to those deformations that satisfy the boundary conditions:

$$u_3(x,y,0) = 0, \quad u_3(x,y,L) = \lambda L \quad \text{for all } (x,y) \in \overline{\Omega}, \tag{1.2}$$

where we have written

$$\mathbf{u}(x,y,z) = \begin{bmatrix} u_1(x,y,z) \\ u_2(x,y,z) \\ u_3(x,y,z) \end{bmatrix}.$$

With each such deformation we associate a corresponding elastic energy

$$E(\mathbf{u}) = \int_{\mathcal{C}} W(\nabla\mathbf{u}(x,y,z))\, dV, \tag{1.3}$$

where $W : \mathrm{M}^{3\times3}_1 \to [0,\infty)$ is the stored-energy density, $\mathrm{M}^{3\times3}_1$ denotes the set of 3×3 matrices with determinant equal to 1, and $dV = dx\, dy\, dz$. If W is both isotropic and frame-indifferent, then standard representation theorems (see, e.g., [5, 12, 13, 20]) imply that there is a function $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ that satisfies

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|, |\operatorname{adj}\mathbf{F}|^2) \quad \text{for all } \mathbf{F} \in \mathrm{M}^{3\times3}_1, \tag{1.4}$$

where $|\mathbf{F}|$ denotes the square-root of the sum of the squares of the elements of $\mathbf{F}$ and $\operatorname{adj}\mathbf{F}$ denotes the inverse of the matrix $\mathbf{F} \in \mathrm{M}^{3\times3}_1$. Note that if a deformation $\mathbf{u}$ satisfies (1.1), (1.2) and minimizes (1.3), (1.4), then so does $\mathbf{g} \circ \mathbf{u}$ where $\mathbf{g}$ is any rotation or translation in the xy-plane. In order to eliminate this trivial nonuniqueness we impose the additional constraints

$$\begin{gathered} \int_{\mathcal{C}} u_1(x,y,z)\, dV = \int_{\mathcal{C}} u_2(x,y,z)\, dV = 0, \\ \int_{\mathcal{C}} \frac{\partial u_1}{\partial y}(x,y,z)\, dV = \int_{\mathcal{C}} \frac{\partial u_2}{\partial x}(x,y,z)\, dV. \end{gathered} \tag{1.5}$$

Our main result, Theorem 3.6 (see also Remarks 3.9 and 4.2 and Theorems 5.1 and 5.3), shows that if the function Φ is monotone increasing in each argument and convex, then the homogeneous deformation

$$\mathbf{u}^{\mathrm{h}}_\lambda(x,y,z) := \begin{bmatrix} \frac{1}{\sqrt{\lambda}}\,x \\ \frac{1}{\sqrt{\lambda}}\,y \\ \lambda\, z \end{bmatrix} \tag{1.6}$$

is an absolute minimizer of E. Moreover, if in addition Φ is strictly increasing, then $\mathbf{u}^{\mathrm{h}}_\lambda$ is the only absolute minimizer of the elastic energy that satisfies (1.1), (1.2), and (1.5).

The proofs of our results extend a procedure developed in [26] for energy minimization of 2-dimensional bars. The underlying approach is the following: we first take $p \geq 1$, $q \geq 1$ and consider the stored-energy functions

$$\widetilde{\sigma}(\mathbf{F},p) = \Big(\alpha^{2p} + \beta^{2p} + \gamma^{2p}\Big)^{\frac{1}{2p}}, \qquad \widehat{\sigma}(\mathbf{F},q) = \Big(\alpha^{-2q} + \beta^{-2q} + \gamma^{-2q}\Big)^{\frac{1}{q}}, \tag{1.7}$$

where α, β, and γ are the principal stretches, i.e., the eigenvalues of $\mathbf{U} = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$. (When $p = q = 1$ it follows that $\widetilde{\sigma}(\mathbf{F},1) = |\mathbf{F}|$ and $\widehat{\sigma}(\mathbf{F},1) = |\operatorname{adj}\mathbf{F}|^2$.) For each of these functions we show that the constraint of incompressibility allows us to bound the elastic energy below by an integral of a convex function of the deformed length of line segments that were initially parallel to the loading axis. Moreover, this lower bound is an equality when the image curves are straight lines that are deformed uniformly and lie parallel to the loading axis. Thus, energetically, the material *prefers* that each such straight line deform homogeneously into another parallel straight line. The result for general stored-energy functions in Theorem 3.6 then follows from Jensen's inequality applied to any convex increasing function Φ of $\widetilde{\sigma}$ and $\widetilde{\sigma}$ given in (1.7).

The general class of energy functions to which Theorem 3.6 applies (and hence for which (1.6) is a global minimizer) includes the Ogden [19] materials:

$$\sum_{i=1}^{N} \mu_i \left(\alpha^{r_i} + \beta^{r_i} + \gamma^{r_i}\right), \tag{1.8}$$

where $r_i \in (-\infty,-2] \cup [2,\infty)$ and $\mu_i \geq 0$ for each i. One of the simplest such examples is the Mooney-Rivlin material:

$$W(\mathbf{F}) = a\big[\alpha^2 + \beta^2 + \gamma^2\big] + b\big[\alpha^{-2} + \beta^{-2} + \gamma^{-2}\big],$$

$a \geq 0$, $b \geq 0$, which clearly satisfies our hypotheses.

We show in § 4 that our results can also be extended to another class of constitutive relations that are a generalization of the Ogden materials (1.8), i.e., either

$$\sigma^\dagger(\mathbf{F},\psi) = \psi\left(\alpha^2\right) + \psi\left(\beta^2\right) + \psi\left(\gamma^2\right) \quad \text{or} \quad \sigma^*(\mathbf{F},\psi) = \psi\left(\alpha^{-2}\right) + \psi\left(\beta^{-2}\right) + \psi\left(\gamma^{-2}\right), \tag{1.9}$$

where ψ is convex and strictly monotone increasing.

In the final section, § 5, we prove that (1.6) is also the unique absolute minimizer of the elastic energy for many transversely isotropic constitutive relations as well as for certain inhomogeneous stored-energy functions that depend on x and y, but not z.

In the Appendix we derive a Rayleigh-Ritz type inequality: in particular, if α^2, β^2, and γ^2 are the eigenvalues of a strictly positive-definite, symmetric matrix $\mathbf{P} \in \mathrm{M}^{3\times 3}$, then

$$\psi(\alpha^2) + \psi(\beta^2) + \psi(\gamma^2) \geq \psi(\mathbf{f}_1 \cdot \mathbf{P}\mathbf{f}_1) + \psi(\mathbf{f}_2 \cdot \mathbf{P}\mathbf{f}_2) + \psi(\mathbf{f}_3 \cdot \mathbf{P}\mathbf{f}_3)$$

for any convex function $\psi : (0, \infty) \to \mathbb{R}$ and any orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ of $\mathbb{R}^3$. The main advantage in using this inequality is that it enables us to work with stored-energy functions expressed in terms of the principal stretches, rather than the principal invariants that we used in our earlier work. It is clear from the proofs in this manuscript that the results in [23, 24, 25] remain valid for the stored-energy functions used in this paper.

As we noted in [26], the vast majority of prior results on elastic solids in uniaxial tension have analyzed the linearization stability of $\mathbf{u}^{\mathrm{h}}_{\lambda}$, that is, whether or not there exists a nontrivial solution to the system of partial differential equations that one obtains upon linearizing (about $\mathbf{u}^{\mathrm{h}}_{\lambda}$) the corresponding equilibrium equations, the boundary conditions (1.2), and equation (1.5). When $\mathbf{u}^{\mathrm{h}}_{\lambda_{\mathrm{o}}}$ is linearization stable, a second solution branch cannot bifurcate from the homogeneous branch $\lambda \mapsto \mathbf{u}^{\mathrm{h}}_{\lambda}$, at $\lambda = \lambda_{\mathrm{o}}$. Our approach yields a similar result; if Φ is convex and strictly monotone increasing in just a small neighborhood of $\nabla\mathbf{u}^{\mathrm{h}}_{\lambda_{\mathrm{o}}}$, where $\lambda_{\mathrm{o}} \geq 1$, then $\mathbf{u}^{\mathrm{h}}_{\lambda_{\mathrm{o}}}$ is a strict weak relative minimizer[1] of E and so no bifurcations can occur at $\lambda = \lambda_{\mathrm{o}}$.

For an incompressible rectangular solid, in 2-dimensions, the linearization stability of the homogeneous solution branch was investigated by Wesołowski [29] and Hill and Hutchinson [14]. In 3-dimensions, Wesołowski [30] also carried out such an analysis for a cylindrical solid with circular cross-section. Equivalently, one can derive estimates on the energy that ensure that the second variation of E is strictly positive at the deformation $\mathbf{u}^{\mathrm{h}}_{\lambda}$ and, consequently, that $\mathbf{u}^{\mathrm{h}}_{\lambda}$ is both linearization stable and a strict weak relative minimizer. For a compressible cylindrical solid in tension this technique was used by Spector [27] and, more recently, by Del Piero and Rizzoni [7] and Fosdick, Foti, Fraddosio, and Piccioni [10] (see, also, Del Piero [6]) to obtain estimates upon the values of λ where bifurcations cannot occur. Additionally, in [7] and [10] similar estimates are derived for compression ($\lambda \leq 1$); incompressible materials are also considered in [7].

In [26] we showed that, for a 2-dimensional rectangular solid, a deformation analogous to (1.6) is the absolute minimizer of the energy of a homogeneous, isotropic, incompressible, elastic bar when the stored-energy function is of the form $W(\mathbf{F}) = \Phi(|\mathbf{F}|)$ with Φ increasing and convex. Mora-Corral [17] has recently established an interesting generalization. He considers the same constitutive relations and geometry as in [26], for a total energy that is the sum of the elastic energy and a surface energy that is proportional to the length of all new cracks that form in the material. He shows that the absolute minimizer of the energy is either the homogeneous isoaxial deformation (analogous to (1.6)) that is obtained in [26] or a deformation in which the bar fractures into 2 undeformed[2] pieces along any line perpendicular to the loading axis.

[1]That is, a local minimizer in the C^1-topology (see, e.g., [11]).
[2]More precisely, rigidly deformed.

All of the constitutive relations used in this paper are polyconvex in the sense of Ball [2, 3]. If, in addition, the stored-energy function grows sufficiently fast at infinity, (e.g., for $a > 0$ and $c > 0$,

$$W(\mathbf{F}) \geq a|\mathbf{F}|^p + b|\operatorname{adj}\mathbf{F}|^q - c \quad \text{for all } \mathbf{F} \in \mathrm{M}_1^{3\times 3},$$

where either $b = 0$ and $p \geq 3$ or $b > 0$, $p \geq 2$, and $q \geq \frac{3}{2}$), then standard results [2, 3, 5, 18] yield the existence of an absolute minimizer of (1.3) in the set of Sobolev deformations

$$\mathcal{S}_\lambda^1 = \{\mathbf{u} \in W^{1,1}(\mathcal{C};\mathbb{R}^3) : \det \nabla\mathbf{u} = 1 \ a.e., \ \mathbf{u} \text{ satisfies (1.2) and (1.5)}\}.$$

The results in this paper, which are also valid $\mathbf{u} \in \mathcal{S}_\lambda^1$ (see [24, § 5]), show that this absolute minimizer[3] is unique and is given by (1.6).

Finally, we note that experiments indicate that homogeneous deformations of a cylinder composed of, for example, an elastomer can eventually become unstable under uniaxial extension due to the formation of one or more necks in the material. Thus, the constitutive assumptions we make in this paper are unsuitable for the prediction of such phenomena. It would therefore be of interest to determine if there exist isotropic, polyconvex, stored-energy functions for which $\mathbf{u}_\lambda^{\mathrm{h}}$ is not an absolute minimizer.

2 Deformations.

Definition 2.1. For $\lambda > 0$ we define the set of *admissible deformations* by

$$\mathcal{A}_\lambda := \left\{\mathbf{u} \in C^1(\overline{\mathcal{C}};\mathbb{R}^3) : \det \nabla\mathbf{u} > 0, \ \mathbf{u} \text{ satisfies (1.2) and (1.5)}\right\}.$$

We identify the subset of $\mathcal{A}_\lambda$ consisting of deformations that are a composition of plane strain in the xy-plane and a homogeneous stretch in the z-direction, i.e., deformations $\mathbf{w} \in \mathcal{A}_\lambda$ of the form

$$\mathbf{w}(x,y,z) = \begin{bmatrix} w_1(x,y) \\ w_2(x,y) \\ \lambda z \end{bmatrix}. \tag{2.1}$$

We denote by

$$\mathrm{PS}_\lambda := \{\mathbf{w} \in \mathcal{A}_\lambda : \mathbf{w} \text{ satisfies (2.1)}\}$$

the set of all such *plane-strain* deformations.

The unique curve of shortest length connecting two points is a straight line. We will make use of a slight variant of this well-known result.

[3] Additionally, since $\mathbf{u}_\lambda^{\mathrm{h}}$ is one-to-one, the integral constraint used in the existence theory of Ciarlet and Nečas [4] for mixed displacement-traction problems is not needed for this particular problem.

Lemma 2.2. *Let $\lambda > 0$, $1 \le p < \infty$, $\mathbf{u} \in \mathcal{A}_\lambda$, and $\mathbf{w} \in \mathrm{PS}_\lambda$. Then, for each $(x, y) \in \overline{\Omega}$,*

$$⨍_0^L \left|\frac{\partial \mathbf{u}}{\partial z}\right|^p dz \ge ⨍_0^L \left|\frac{\partial \mathbf{w}}{\partial z}\right|^p dz = \lambda^p, \tag{2.2}$$

where $⨍_0^L \phi\, dz$ denotes the average value of ϕ over $[0, L]$, i.e.,

$$⨍_0^L \phi(x, y, z)\, dz := \frac{1}{L}\int_0^L \phi(x, y, z)\, dz.$$

Moreover, if $p > 1$ and inequality (2.2) is an equality for every $(x, y) \in \overline{\Omega}$, then $\mathbf{u} \in \mathrm{PS}_\lambda$.

Proof. Let $p \ge 1$, $\lambda > 0$, and $\mathbf{v} \in \mathcal{A}_\lambda$. Fix $(x, y) \in \overline{\Omega}$. Then

$$⨍_0^L \left|\frac{\partial \mathbf{v}}{\partial z}\right|^p dz \ge ⨍_0^L \left|\frac{\partial v_3}{\partial z}\right|^p dz \tag{2.3}$$

with equality if and only if $\partial v_1/\partial z = \partial v_2/\partial z \equiv 0$. Next, for $p > 1$, Hölder's inequality together with the boundary conditions (1.2) yield

$$⨍_0^L \left|\frac{\partial v_3}{\partial z}\right|^p dz \ge \left|⨍_0^L \frac{\partial v_3}{\partial z}\, dz\right|^p = \left|\frac{v_3(x, y, L) - v_3(x, y, 0)}{L}\right|^p = \lambda^p \tag{2.4}$$

with equality if and only if $\partial v_3/\partial z$ is a constant. Moreover, (2.4) is clearly also valid for $p = 1$.

Finally, we note that necessary and sufficient conditions for $\mathbf{v} \in \mathrm{PS}_\lambda$ are that, for every $(x, y, z) \in \overline{\mathcal{C}}$, $\partial v_1/\partial z = \partial v_2/\partial z = 0$ and, assuming $p > 1$, $\partial v_3/\partial z = \lambda$. In particular both (2.3) and (2.4) are equalities for $\mathbf{v} \in \mathrm{PS}_\lambda$. Therefore, both the desired inequality as well as the statement concerning equality follow from (2.3) and (2.4). □

2.1 Isochoric Deformations

In this paper we are primarily interested in isochoric deformations (i.e., deformations that preserve volume) and we correspondingly define the following class of deformations in $\mathcal{A}_\lambda$.

Definition 2.3. For $\lambda > 0$ we define the set of *admissible isochoric deformations* and the set of *isochoric plane-strain deformations* by

$$\mathcal{A}^1_\lambda := \{\mathbf{u} \in \mathcal{A}_\lambda : \det \nabla\mathbf{u} \equiv 1\}, \qquad \mathrm{PS}^1_\lambda := \{\mathbf{w} \in \mathrm{PS}_\lambda : \det \nabla\mathbf{w} \equiv 1\}.$$

In particular, the *homogeneous* deformation given by (1.6) is an admissible isochoric deformation that satisfies

$$\nabla\mathbf{u}^{\mathrm{h}}_\lambda \equiv \begin{bmatrix} \frac{1}{\sqrt{\lambda}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{2.5}$$

The next Lemma shows that, in the class of isochoric deformations, the final conclusion of Lemma 2.2 is also valid when $p = 1$.

Lemma 2.4. *Let $\lambda > 0$, $\mathbf{u} \in \mathcal{A}^1_\lambda$, and $\mathbf{w} \in \mathrm{PS}^1_\lambda$. Then, for each $(x, y) \in \overline{\Omega}$,*

$$\operatorname{length}\big(\mathbf{u}(\mathcal{L}_{x,y})\big) \geq \operatorname{length}\big(\mathbf{w}(\mathcal{L}_{x,y})\big) = \lambda L, \tag{2.6}$$

where

$$\mathcal{L}_{x,y} := \{(x, y, z) : 0 \leq z \leq L\}, \qquad \operatorname{length}\big(\mathbf{u}(\mathcal{L}_{x,y})\big) := \int_0^L \left|\frac{\partial \mathbf{u}}{\partial z}\right| dz.$$

Moreover, if (2.6) *is an equality for every $(x, y) \in \overline{\Omega}$, then $\mathbf{u} \in \mathrm{PS}^1_\lambda$.*

Proof. The inequality follows from Lemma 2.2, as does the necessity of the equation $\partial u_1/\partial z = \partial u_2/\partial z \equiv 0$ for equality to hold in (2.6). To see that $\partial u_3/\partial z \equiv \lambda$ is also necessary for equality, we assume that (2.6) is an equality. We then note that the constraint $\det \nabla \mathbf{u} \equiv 1$ together with $\partial u_1/\partial z = \partial u_2/\partial z = 0$ yields

$$1 = \det \nabla \mathbf{u} = \frac{\partial u_3}{\partial z}\left[\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_2}{\partial x}\frac{\partial u_1}{\partial y}\right]. \tag{2.7}$$

However, u_1 and u_2 are independent of z. Therefore, the expression in square brackets in (2.7) is independent of z and so, for each $(x, y) \in \overline{\Omega}$, $\partial u_3/\partial z$ is independent of z. The desired result now follows from the boundary conditions (1.2). □

In our subsequent analysis we will make use of the following result (cf. Mizel [16, Theorem A]).

Lemma 2.5. *Let $\omega : (0, \infty) \to \mathbb{R}$ be convex and strictly increasing. Suppose that $\mathbf{w} \in \mathrm{PS}^1_\lambda$, $(x, y, z) \in \overline{\mathcal{C}}$, and let α, β, and λ denote the eigenvalues of $\mathbf{U} = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$, $\mathbf{F} = \nabla \mathbf{w}(x, y, z)$. Then*

$$\begin{gathered}\omega(\alpha) + \omega(\beta) + \omega(\lambda) \geq 2\omega\big(\lambda^{-1/2}\big) + \omega\big(\lambda\big),\\ \omega(\beta\lambda) + \omega(\alpha\lambda) + \omega(\alpha\beta) \geq 2\omega\big(\sqrt{\lambda}\,\big) + \omega\big(\lambda^{-1}\big).\end{gathered} \tag{2.8}$$

Moreover, each of the above inequalities is an equality at every $(x, y, z) \in \overline{\mathcal{C}}$ if and only if $\nabla \mathbf{w} \equiv \nabla \mathbf{u}^{\mathrm{h}}_\lambda$.

Remark 2.6. We note that $\nabla \mathbf{w}$ is independent of z whenever $\mathbf{w} \in \mathrm{PS}_\lambda$. In particular this Lemma therefore yields $\nabla \mathbf{w} \equiv \nabla \mathbf{u}^{\mathrm{h}}_\lambda$ whenever $(2.8)_1$ or $(2.8)_2$ is an equality for every $(x, y) \in \overline{\Omega}$.

Proof of Lemma 2.5. Let $\mathbf{w} \in \mathrm{PS}^1_\lambda$ and $(x, y, z) \in \overline{\mathcal{C}}$. Further, let α, β, and λ denote the eigenvalues of $\mathbf{U} := \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$, $\mathbf{F} := \nabla \mathbf{w}(x, y, z)$. We first note that it is clear from (2.5) that each inequality in (2.8) is an equality when $\nabla \mathbf{w}(x, y, z) = \nabla \mathbf{u}^{\mathrm{h}}_\lambda$.

Conversely, since $\det \mathbf{F} = 1$, it follows that $\alpha\beta = \lambda^{-1}$. In order to prove $(2.8)_1$, we apply Lemma A.3 with $x = \sqrt{\alpha}$ and $y = \sqrt{\beta}$ to conclude, with the aid of $\alpha\beta = \lambda^{-1}$, that

$$\omega(\alpha) + \omega(\beta) \geq 2\omega\big(\sqrt{\alpha\beta}\,\big) = 2\omega\big(\lambda^{-1/2}\big)$$

with equality if and only if $\alpha = \beta$. This establishes $(2.8)_1$. In order to prove $(2.8)_2$, we let $x = \sqrt{\beta\lambda}$ and $y = \sqrt{\alpha\lambda}$ in Lemma A.3 and again make use of the identity $\alpha\beta = \lambda^{-1}$ to arrive at

$$\omega(\beta\lambda) + \omega(\alpha\lambda) \geq 2\omega\big(\sqrt{\alpha\beta\lambda^2}\,\big) = 2\omega\big(\sqrt{\lambda}\,\big)$$

with equality if and only if $\alpha = \beta$. Equation $(2.8)_2$ now follows from a third application of the identity $\alpha\beta = \lambda^{-1}$.

Finally, we note that $\alpha = \beta$ and $\alpha\beta = \lambda^{-1}$ yield $\alpha = \beta = \lambda^{-1/2}$ and hence, by the polar decomposition theorem, $\nabla\mathbf{w}(x,y,z) = \mathbf{Q}(x,y,z)\nabla\mathbf{u}^{\mathrm{h}}_\lambda$ for some *rotation* $\mathbf{Q}(x,y,z) \in \mathrm{M}^{3\times3}$. However, since $\nabla\mathbf{u}^{\mathrm{h}}_\lambda$ is a constant matrix, a standard result in Continuum Mechanics (see, e.g., [5, pp. 44–49] or [13, p. 49]) states that the only such rotations $(x,y,z) \mapsto \mathbf{Q}(x,y,z)$ are constant maps, which together with (1.5) and (2.1) implies that $\mathbf{Q}(x,y,z) \equiv \mathbf{I}$. □

3 The Homogeneity of Isochoric Energy-Minimizing Deformations.

For the remainder of the paper we will restrict our attention to isochoric deformations. Let $\mathbf{u} \in \mathcal{A}^1_\lambda$. Our aim is to prove that the energy functional (1.3) satisfies

$$E(\mathbf{u}) \geq E(\mathbf{u}^{\mathrm{h}}_\lambda)$$

for a large class of stored-energy functions W.

3.1 The case $W(\mathbf{F}) = (\alpha^{2p} + \beta^{2p} + \gamma^{2p})^{\frac{1}{2p}}$ with $p \geq 1$.

Proposition 3.1. *Let $\lambda \geq 1$, $\mathbf{u} \in \mathcal{A}^1_\lambda$, $(x,y) \in \overline{\Omega}$, and $p \geq 1$. Suppose that*

$$W(\mathbf{F}) = \widetilde{\sigma}(\mathbf{F},p) := \Big(\alpha^{2p} + \beta^{2p} + \gamma^{2p}\Big)^{\frac{1}{2p}}, \tag{3.1}$$

where α, β, and γ are the eigenvalues of $\mathbf{U} = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$. Then

$$⨍_0^L \widetilde{\sigma}\Big(\nabla\mathbf{u}(x,y,z),p\Big)dz \geq ⨍_0^L \widetilde{\sigma}\Big(\nabla\mathbf{u}^{\mathrm{h}}_\lambda(x,y,z),p\Big)dz. \tag{3.2}$$

Moreover, if this inequality is an equality for every $(x,y) \in \overline{\Omega}$, then $\mathbf{u} \equiv \mathbf{u}^{\mathrm{h}}_\lambda$.

Proof. Fix $p \geq 1$ and $\lambda \geq 1$. We first note that (2.5) together with (3.1) yields

$$W(\nabla\mathbf{u}^{\mathrm{h}}_\lambda(x,y,z)) \equiv \Big(\lambda^{2p} + 2\lambda^{-p}\Big)^{\frac{1}{2p}}. \tag{3.3}$$

Next, let $\mathbf{u} \in \mathcal{A}^1_\lambda$ and $(x,y) \in \overline{\Omega}$. For any $z \in [0,L]$, denote the eigenvalues of

$$\mathbf{U}(x,y,z) = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}, \qquad \mathbf{F} = \nabla\mathbf{u}(x,y,z)$$

by $\bar{\alpha} = \bar{\alpha}(x, y, z)$, $\bar{\beta} = \bar{\beta}(x, y, z)$, and $\bar{\gamma} = \bar{\gamma}(x, y, z)$. Then, in view of Proposition A.1 (with $\mathbf{P} = \mathbf{U}^2$, $\mathbf{e}_1 = \mathbf{e}_x$, $\mathbf{e}_2 = \mathbf{e}_y$, $\mathbf{e}_3 = \mathbf{e}_z$, and $\omega(t) = t^p$),

$$\Big(W(\nabla\mathbf{u})\Big)^{2p} = (\bar{\alpha})^{2p} + (\bar{\beta})^{2p} + (\bar{\gamma})^{2p} \geq \left|\frac{\partial\mathbf{u}}{\partial z}\right|^{2p} + \left|\frac{\partial\mathbf{u}}{\partial x}\right|^{2p} + \left|\frac{\partial\mathbf{u}}{\partial y}\right|^{2p}. \tag{3.4}$$

Now, $\mathbf{u}$ is isochoric and so the Cauchy-Schwarz inequality yields

$$1 = \det\nabla\mathbf{u} = \frac{\partial\mathbf{u}}{\partial z}\cdot\left(\frac{\partial\mathbf{u}}{\partial x}\times\frac{\partial\mathbf{u}}{\partial y}\right) \leq \left|\frac{\partial\mathbf{u}}{\partial z}\right|\left|\frac{\partial\mathbf{u}}{\partial x}\times\frac{\partial\mathbf{u}}{\partial y}\right|, \tag{3.5}$$

while the arithmetic-geometric mean inequality implies that

$$\left|\frac{\partial\mathbf{u}}{\partial x}\right|^{2p} + \left|\frac{\partial\mathbf{u}}{\partial y}\right|^{2p} \geq 2\left|\frac{\partial\mathbf{u}}{\partial x}\right|^{p}\left|\frac{\partial\mathbf{u}}{\partial y}\right|^{p} \geq 2\left|\frac{\partial\mathbf{u}}{\partial x}\times\frac{\partial\mathbf{u}}{\partial y}\right|^{p}. \tag{3.6}$$

If we now combine (3.4)–(3.6) we discover that

$$W(\nabla\mathbf{u}) \geq \left(\left|\frac{\partial\mathbf{u}}{\partial z}\right|^{2p} + 2\left|\frac{\partial\mathbf{u}}{\partial z}\right|^{-p}\right)^{\frac{1}{2p}}. \tag{3.7}$$

Define $g : \mathbb{R}^+ \to \mathbb{R}$ by

$$g(t) = \Big(t^{2p} + 2t^{-p}\Big)^{\frac{1}{2p}}.$$

Then

$$g'(t) = \frac{1}{t^{p+1}}\left[\frac{2 + t^{3p}}{t^p}\right]^{\frac{1-2p}{2p}}\big(t^{3p} - 1\big), \qquad g''(t) = \frac{3t^{-2}}{(2 + t^{3p})^2}\left[\frac{2 + t^{3p}}{t^p}\right]^{\frac{1}{2p}}\big((3p-1)t^{3p} + 1\big);$$

thus, g is strictly increasing on $[1, \infty)$ and strictly convex on $\mathbb{R}^+$.

We next integrate (3.7) over $[0, L]$, apply Jensen's inequality to the convex function g, and then make use of Lemma 2.2, the fact that $\lambda \geq 1$, and the monotonicity of g on $[1, \infty)$ to conclude that

$$\begin{aligned} \fint_0^L W\big(\nabla\mathbf{u}(x, y, z)\big)\,dz &\geq \fint_0^L g\left(\left|\frac{\partial\mathbf{u}}{\partial z}\right|\right)dz \\ &\geq g\left(\fint_0^L \left|\frac{\partial\mathbf{u}}{\partial z}\right| dz\right) \\ &\geq g\left(\fint_0^L \left|\frac{\partial\mathbf{u}_\lambda^{\mathrm{h}}}{\partial z}\right| dz\right) \\ &= g\left(\lambda\right) = \Big(\lambda^{2p} + 2\lambda^{-p}\Big)^{\frac{1}{2p}}, \end{aligned} \tag{3.8}$$

which, in view of (3.3), establishes the desired inequality.

Now suppose that (3.2) is an equality for every $(x,y) \in \overline{\Omega}$. Then each inequality in (3.8) must be an equality. Since g is strictly increasing we find that

$$\fint_0^L \left|\frac{\partial \mathbf{u}}{\partial z}\right| dz = \fint_0^L \left|\frac{\partial \mathbf{u}_\lambda^{\mathrm{h}}}{\partial z}\right| dz \quad \text{for every} \quad (x,y) \in \overline{\Omega}.$$

Therefore, by Lemma 2.4, $\mathbf{u} \in \mathrm{PS}_\lambda^1$. In particular, $\nabla \mathbf{u}$ is independent of z and so (3.4) together with (3.8) implies

$$(\bar{\alpha})^{2p} + (\bar{\beta})^{2p} + (\bar{\gamma})^{2p} = \lambda^{2p} + 2\lambda^{-p} \quad \text{for every} \quad (x,y,z) \in \overline{\mathcal{C}}.$$

Lemma 2.5 with $\omega(t) = t^{2p}$ then yields $\nabla \mathbf{u} \equiv \nabla \mathbf{u}_\lambda^{\mathrm{h}}$ and hence $\mathbf{u} \equiv \mathbf{u}_\lambda^{\mathrm{h}} + \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^3$. Finally, $\mathbf{a} = \mathbf{0}$ by (1.2) and $(1.5)_1$. □

When $p = 1$ Proposition 3.1 reduces to the following result.

Corollary 3.2. *Let $\lambda \geq 1$, $\mathbf{u} \in \mathcal{A}_\lambda^1$, and $(x,y) \in \overline{\Omega}$. Then*

$$\fint_0^L \left|\nabla \mathbf{u}(x,y,z)\right| dz \geq \fint_0^L \left|\nabla \mathbf{u}_\lambda^{\mathrm{h}}(x,y,z)\right| dz.$$

Moreover, if this inequality is an equality for every $(x,y) \in \overline{\Omega}$, then $\mathbf{u} \equiv \mathbf{u}_\lambda^{\mathrm{h}}$.

3.2 The case $W(\mathbf{F}) = (\alpha^{-2q} + \beta^{-2q} + \gamma^{-2q})^{1/q}$ with $q \geq 1$.

Proposition 3.3. *Let $\lambda \geq 1$, $\mathbf{u} \in \mathcal{A}_\lambda^1$, $(x,y) \in \overline{\Omega}$, and $q \geq 1$. Suppose that*

$$W(\mathbf{F}) = \widehat{\sigma}(\mathbf{F}, q) := \Big((\beta\gamma)^{2q} + (\alpha\gamma)^{2q} + (\alpha\beta)^{2q}\Big)^{1/q}, \tag{3.9}$$

where α, β, and γ are the eigenvalues of $\mathbf{U} = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$. Then

$$\fint_0^L \widehat{\sigma}\Big(\nabla \mathbf{u}(x,y,z), q\Big) dz \geq \fint_0^L \widehat{\sigma}\Big(\nabla \mathbf{u}_\lambda^{\mathrm{h}}(x,y,z), q\Big) dz.$$

Moreover, if this inequality is an equality for every $(x,y) \in \overline{\Omega}$, then $\mathbf{u} \equiv \mathbf{u}_\lambda^{\mathrm{h}}$.

Remark 3.4. Since all of our deformations are isochoric, $1 = \det \mathbf{F} = \alpha\beta\gamma$ and hence

$$(\beta\gamma)^{2q} + (\alpha\gamma)^{2q} + (\alpha\beta)^{2q} = \alpha^{-2q} + \beta^{-2q} + \gamma^{-2q}.$$

Proof of Proposition 3.3. Fix $q \geq 1$ and $\lambda \geq 1$. We first note that (2.5) together with (3.9) implies

$$W\big(\nabla \mathbf{u}_\lambda^{\mathrm{h}}(x,y,z)\big) \equiv \big(2\lambda^q + \lambda^{-2q}\big)^{1/q}.$$

Next, let $\mathbf{u} \in \mathcal{A}^1_\lambda$ and $(x,y) \in \overline{\Omega}$. For any $z \in [0,L]$, denote the eigenvalues of

$$\mathbf{U}(x,y,z) = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}, \qquad \mathbf{F} := \nabla\mathbf{u}(x,y,z)$$

by $\bar{\alpha} = \bar{\alpha}(x,y,z)$, $\bar{\beta} = \bar{\beta}(x,y,z)$, and $\bar{\gamma} = \bar{\gamma}(x,y,z)$. Then the eigenvalues of

$$\mathbf{P} = (\operatorname{adj}\mathbf{F})(\operatorname{adj}\mathbf{F})^{\mathrm{T}} = \operatorname{adj}(\mathbf{F}^{\mathrm{T}}\mathbf{F}) = \operatorname{adj}(\mathbf{U}^2)$$

are $(\bar{\beta}\bar{\gamma})^2$, $(\bar{\alpha}\bar{\gamma})^2$, and $(\bar{\alpha}\bar{\beta})^2$. Thus, in view of Proposition A.1 (with $\mathbf{P}$ as given above, $\mathbf{e}_1 = \mathbf{e}_x$, $\mathbf{e}_2 = \mathbf{e}_y$, $\mathbf{e}_3 = \mathbf{e}_z$, and $\omega(t) = t^q$),

$$\Big(W(\nabla\mathbf{u})\Big)^q = (\bar{\beta}\bar{\gamma})^{2q} + (\bar{\alpha}\bar{\gamma})^{2q} + (\bar{\alpha}\bar{\beta})^{2q} \geq \left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_x\right|^{2q} + \left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_y\right|^{2q} + \left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_z\right|^{2q}, \tag{3.10}$$

where $\mathbf{G} := \operatorname{adj}\nabla\mathbf{u}$.

Once again $\mathbf{u}$ is isochoric and so the Cauchy-Schwarz inequality yields

$$1 = |\mathbf{e}_z|^2|\det\nabla\mathbf{u}| = \left|(\nabla\mathbf{u})\mathbf{e}_z \cdot (\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{e}_z\right| \leq \left|(\nabla\mathbf{u})\mathbf{e}_z\right| \left|(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{e}_z\right|, \tag{3.11}$$

while the arithmetic-geometric mean inequality implies that

$$\left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_x\right|^{2q} + \left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_y\right|^{2q} \geq 2\left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_x\right|^{q}\left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_y\right|^{q} \geq 2\left|\left(\mathbf{G}^{\mathrm{T}}\mathbf{e}_x\right) \times \left(\mathbf{G}^{\mathrm{T}}\mathbf{e}_y\right)\right|^{q}. \tag{3.12}$$

Consequently, if we combine (3.10)–(3.12) we discover, with the aid of the identity

$$\left[(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{e}_x\right] \times \left[(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{e}_y\right] = (\nabla\mathbf{u})\mathbf{e}_z = \frac{\partial\mathbf{u}}{\partial z}, \tag{3.13}$$

that

$$W(\nabla\mathbf{u}) \geq \left(2\left|\frac{\partial\mathbf{u}}{\partial z}\right|^q + \left|\frac{\partial\mathbf{u}}{\partial z}\right|^{-2q}\right)^{1/q}.$$

Define $h : \mathbb{R}^+ \to \mathbb{R}$ by

$$h(t) = \left(2t^q + t^{-2q}\right)^{1/q}.$$

Then

$$h'(t) = 2\,\frac{\sqrt[q]{2t^q + t^{-2q}}}{t\,(1+2t^{3q})}\left(t^{3q} - 1\right), \qquad h''(t) = 6\,\frac{\sqrt[q]{2t^q + t^{-2q}}}{t^2\,(1+2t^{3q})}\left[(3q-1)t^{3q} + 1\right];$$

thus, h is strictly increasing on $[1,\infty)$ and strictly convex on $\mathbb{R}^+$. The remainder of the proof is now analogous to the proof of Proposition 3.1. □

When $q = 1$ Proposition 3.3 reduces to the following result.

Corollary 3.5. *Let $\lambda \geq 1$, $\mathbf{u} \in \mathcal{A}^1_\lambda$, and $(x,y) \in \overline{\Omega}$. Then*

$$⨍_0^L \left|\operatorname{adj}\nabla\mathbf{u}(x,y,z)\right|^2 dz \geq ⨍_0^L \left|\operatorname{adj}\nabla\mathbf{u}^{\mathrm{h}}_\lambda(x,y,z)\right|^2 dz.$$

Moreover, if this inequality is an equality for every $(x,y) \in \overline{\Omega}$, then $\mathbf{u} \equiv \mathbf{u}^{\mathrm{h}}_\lambda$.

3.3 The general case: $W(\mathbf{F}) = \varPhi(\widetilde{\sigma}(\mathbf{F}, p_i), \widehat{\sigma}(\mathbf{F}, q_j))$.

We now suppose that

$$W(\mathbf{F}) = \varPhi\big(\widetilde{\sigma}(\mathbf{F}, p_1), \ldots, \widetilde{\sigma}(\mathbf{F}, p_N), \widehat{\sigma}(\mathbf{F}, q_1), \ldots, \widehat{\sigma}(\mathbf{F}, q_M)\big) \tag{3.14}$$

to obtain the following result.

Theorem 3.6. *Let $\lambda \geq 1$, $M, N \in \mathbb{Z}^+$, $p_i \geq 1$, $i = 1, \ldots, N$, and $q_j \geq 1$, $j = 1, \ldots, M$. Suppose that $\widetilde{\sigma}$, $\widehat{\sigma}$, and W are given by* (3.1), (3.9), *and* (3.14), *where $\varPhi : (0, \infty)^{N+M} \to \mathbb{R}$ is monotone increasing in each argument and convex. Then, for any $\mathbf{u} \in \mathcal{A}^1_\lambda$,*

$$E(\mathbf{u}) = \int_{\mathcal{C}} W(\nabla \mathbf{u})\, dV \geq \int_{\mathcal{C}} W(\nabla \mathbf{u}^{\mathrm{h}}_\lambda)\, dV = E(\mathbf{u}^{\mathrm{h}}_\lambda). \tag{3.15}$$

Moreover, if in addition $\varPhi$ is a strictly increasing function of any of its arguments (for all values of the remaining arguments), then inequality (3.15) *is strict when $\mathbf{u} \not\equiv \mathbf{u}^{\mathrm{h}}_\lambda$.*

Proof. In order to simplify the notation we present the proof when $N = M = 1$ and note that the extension to other N and M is straightforward. By Jensen's inequality, the monotonicity of $\varPhi$, Proposition 3.1, and Proposition 3.3

$$\begin{aligned} \fint_0^L \varPhi\Big(\widetilde{\sigma}\big(\nabla \mathbf{u}, p_1\big), \widehat{\sigma}\big(\nabla \mathbf{u}, q_1\big)\Big) dz &\geq \varPhi\Big(\fint_0^L \widetilde{\sigma}\big(\nabla \mathbf{u}, p_1\big)\, dz, \fint_0^L \widehat{\sigma}\big(\nabla \mathbf{u}, q_1\big)\, dz\Big) \\ &\geq \varPhi\Big(\fint_0^L \widetilde{\sigma}\big(\nabla \mathbf{u}^{\mathrm{h}}_\lambda, p_1\big)\, dz, \fint_0^L \widehat{\sigma}\big(\nabla \mathbf{u}^{\mathrm{h}}_\lambda, q_1\big)\, dz\Big) \\ &= \varPhi\Big(\widetilde{\sigma}\big(\nabla \mathbf{u}^{\mathrm{h}}_\lambda, p_1\big), \widehat{\sigma}\big(\nabla \mathbf{u}^{\mathrm{h}}_\lambda, q_1\big)\Big), \\ &= \fint_0^L \varPhi\Big(\widetilde{\sigma}\big(\nabla \mathbf{u}^{\mathrm{h}}_\lambda, p_1\big), \widehat{\sigma}\big(\nabla \mathbf{u}^{\mathrm{h}}_\lambda, q_1\big)\Big) dz, \end{aligned} \tag{3.16}$$

where the last two equalities follow from the fact that $\nabla \mathbf{u}^{\mathrm{h}}_\lambda$ is constant on $\overline{\mathcal{C}}$ (see (2.5)). The desired energy inequality, (3.15), now follows upon integrating (3.16) over Ω and then multiplying by L.

In order to see that the inequality is strict when $\mathbf{u} \not\equiv \mathbf{u}^{\mathrm{h}}_\lambda$, we observe that Proposition 3.1, Proposition 3.3, and the strict monotonicity of $\varPhi$, imply that the second of the above inequalities is a strict inequality when $\mathbf{u} \not\equiv \mathbf{u}^{\mathrm{h}}_\lambda$. □

Remark 3.7. If one replaces the assumption that $\varPhi$ is globally convex and monotone increasing by the assumption that it is convex and strictly monotone increasing in a neighborhood of $\nabla \mathbf{u}^{\mathrm{h}}_{\lambda_{\mathrm{o}}}$, where $\lambda_{\mathrm{o}} \geq 1$, it then follows that $\mathbf{u}^{\mathrm{h}}_{\lambda_{\mathrm{o}}}$ is a strict weak relative minimizer of the energy.

Example 3.8. The choice $\varPhi(s, t) = as^{2p} + bt^q$ shows that our results are valid for the Ogden [19] materials:

$$W(\mathbf{F}) = a(\alpha^{2p} + \beta^{2p} + \gamma^{2p}) + b(\alpha^{-2q} + \beta^{-2q} + \gamma^{-2q}),$$

$a \geq 0$, $b \geq 0$, $p \geq 1$, and $q \geq 1$.

Remark 3.9. If $\mathcal{C} \subset \mathbb{R}^n$, $n \geq 3$, i.e., $\mathcal{C} = \Omega \times (0, L)$ with $\Omega \subset \mathbb{R}^{n-1}$ bounded and open, then a slight modification of our proof, with $p = q = 1$ (see Corollaries 3.2 and 3.5 in this manuscript and the proof of lemma 3.1 in [23]), shows that if $\lambda \geq 1$ and

$$W(\mathbf{F}) = \Phi(|\mathbf{F}|, |\operatorname{adj}\mathbf{F}|^{n-1}),$$

where $\Phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is monotone increasing in each argument and convex, then the energy is minimized by $\mathbf{u}^{\mathrm{h}}_\lambda(x_1, x_2, \ldots, x_n) = (\lambda^{-1/(n-1)}x_1, \ldots, \lambda^{-1/(n-1)}x_{n-1}, \lambda x_n)$. Note that the first argument of Φ is homogeneous of degree 1, while the last argument is homogeneous of degree $(n-1)^2$. This differs from the similar results for thick spherical shells in tension[4] (see [23, 24, 25]) where all the arguments of the corresponding functions are homogeneous of the same degree, degree $n - 1$ for incompressible shells and degree n for compressible shells.

Remark 3.10. When the stored-energy density W is isotropic, the representation theorem for isotropic functions (see, e.g., [5, 12, 13, 20]) implies that for any $p, q \in \mathbb{R}$ with $pq \neq 0$ there is a function $\Upsilon : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ that satisfies

$$W(\mathbf{F}) = \Upsilon(|\mathbf{F}|^p, |\operatorname{adj}\mathbf{F}|^q) \quad \text{for all } \mathbf{F} \in \mathrm{M}^{3\times 3}_1.$$

It may then be of interest to determine necessary and sufficient conditions on p, q, and Υ for this representation to be polyconvex. Similarly, it may be of interest to determine such conditions on p_i ($i = 1, \ldots, N$), q_j ($j = 1, \ldots, M$), and the function Φ so that the representation given by (3.14) is polyconvex. It is clear that the monotonicity and convexity conditions we impose are sufficient, but not necessary for polyconvexity. The results of Mielke [15], Rosakis [21], Ambrosio, Fonseca, Marcellini, and Tartar [1], Šilhavý [22], and Steigmann [28] might be useful in addressing these problems.

4 Generalized Ogden Materials.

In this section we show that a slight variant of our method will allow for constitutive relations given by (1.9).

Proposition 4.1. *Let* $\lambda \geq 1$, $\mathbf{u} \in \mathcal{A}^1_\lambda$, *and* $(x, y) \in \overline{\Omega}$. *Suppose that*

$$W(\mathbf{F}) = \sigma\big(\mathbf{F}, \psi\big) := \psi\left(\alpha^2\right) + \psi\left(\beta^2\right) + \psi\left(\gamma^2\right) \tag{4.1}$$

or

$$W(\mathbf{F}) = \sigma\big(\mathbf{F}, \psi\big) := \psi\left(\alpha^{-2}\right) + \psi\left(\beta^{-2}\right) + \psi\left(\gamma^{-2}\right), \tag{4.2}$$

where α, β, *and* γ *are the eigenvalues of* $\mathbf{U} = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$, *and* $\psi \in C^1((0, \infty); \mathbb{R})$ *is convex and strictly increasing. Then*

$$\int_0^L \sigma\big(\nabla\mathbf{u}(x, y, z), \psi\big)\, dz \geq \int_0^L \sigma\big(\nabla\mathbf{u}^{\mathrm{h}}_\lambda(x, y, z), \psi\big)\, dz.$$

Moreover, if this inequality is an equality for every $(x, y) \in \overline{\Omega}$, *then* $\mathbf{u} \equiv \mathbf{u}^{\mathrm{h}}_\lambda$.

[4] However, see Theorem 4.4 in [24] for an incompressible shell under compression in which this hypothesis does arise.

Remark 4.2. For any convex increasing function ψ, the functions in (4.1) and (4.2) can be used in place of (or in addition to) $\widetilde{\sigma}(\mathbf{F},p)$ and $\widehat{\sigma}(\mathbf{F},q)$ in Theorem 3.6.

Proof of Proposition 4.1. Assume firstly that W and σ are given by (4.1). Fix $\lambda \geq 1$, $\mathbf{u} \in \mathcal{A}^1_\lambda$ and $(x,y) \in \overline{\Omega}$. For any $z \in [0,L]$, denote the eigenvalues of

$$\mathbf{U}(x,y,z) = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}, \qquad \mathbf{F} = \nabla\mathbf{u}(x,y,z)$$

by $\bar{\alpha} = \bar{\alpha}(x,y,z)$, $\bar{\beta} = \bar{\beta}(x,y,z)$, and $\bar{\gamma} = \bar{\gamma}(x,y,z)$. Then, in view of Proposition A.1 (with $\mathbf{P} = \mathbf{U}^2$, $\mathbf{e}_1 = \mathbf{e}_x$, $\mathbf{e}_2 = \mathbf{e}_y$, $\mathbf{e}_3 = \mathbf{e}_z$, and $\omega = \psi$),

$$\begin{aligned} W(\nabla\mathbf{u}) &= \psi\left((\bar{\alpha})^2\right) + \psi\left((\bar{\beta})^2\right) + \psi\left((\bar{\gamma})^2\right) \\ &\geq \psi\bigg(\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2\bigg) + \psi\bigg(\left|\frac{\partial\mathbf{u}}{\partial x}\right|^2\bigg) + \psi\bigg(\left|\frac{\partial\mathbf{u}}{\partial y}\right|^2\bigg). \end{aligned} \tag{4.3}$$

Next,

$$\left|\frac{\partial\mathbf{u}}{\partial x}\right| \left|\frac{\partial\mathbf{u}}{\partial y}\right| \geq \left|\frac{\partial\mathbf{u}}{\partial x} \times \frac{\partial\mathbf{u}}{\partial y}\right|$$

and hence Lemma A.3 together with the monotonicity of ψ yields

$$\psi\left(\left|\frac{\partial\mathbf{u}}{\partial x}\right|^2\right) + \psi\left(\left|\frac{\partial\mathbf{u}}{\partial y}\right|^2\right) \geq 2\psi\left(\left|\frac{\partial\mathbf{u}}{\partial x} \times \frac{\partial\mathbf{u}}{\partial y}\right|\right). \tag{4.4}$$

If we now combine (4.3) with (4.4) we find, with the aid of (3.5) and the monotonicity of ψ, that

$$W(\nabla\mathbf{u}) \geq \psi\left(\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2\right) + 2\psi\left(\left|\frac{\partial\mathbf{u}}{\partial z}\right|^{-1}\right).$$

Define $g : \mathbb{R}^+ \to \mathbb{R}$ by

$$g(t) = \psi\big(t^2\big) + 2\psi\big(t^{-1}\big).$$

Then, since ψ is increasing and convex, g is convex. Also,

$$g'(t) = 2t^{-2}\left[t^3\psi'\left(t^2\right) - \psi'\left(t^{-1}\right)\right]$$

and hence, since ψ' is increasing, g is strictly increasing on $[1,\infty)$. The remainder of the proof in this case is then similar to the corresponding parts of the proof of Proposition 3.1.

In the case when W and σ are given by (4.2), we first note that (4.2) together with the constraint $1 = \det\mathbf{F} = \alpha\beta\gamma$ implies that

$$W(\mathbf{F}) = \sigma\big(\mathbf{F},\psi\big) = \psi\left(\beta^2\gamma^2\right) + \psi\left(\alpha^2\gamma^2\right) + \psi\big(\alpha^2\beta^2\big).$$

Next, as in the proof of Proposition 3.3, we apply Proposition A.1 (with $\mathbf{P} = (\operatorname{adj}\mathbf{F})(\operatorname{adj}\mathbf{F})^{\mathrm{T}}$, $\mathbf{e}_1 = \mathbf{e}_x$, $\mathbf{e}_2 = \mathbf{e}_y$, $\mathbf{e}_3 = \mathbf{e}_z$, and $\omega(t) = \psi(t^2)$), to arrive at

$$\begin{aligned} W(\nabla\mathbf{u}) &= \psi\left((\bar\beta\bar\gamma)^2\right) + \psi\left((\bar\alpha\bar\gamma)^2\right) + \psi\left((\bar\alpha\bar\beta)^2\right) \\ &\geq \psi\big(\left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_x\right|^2\big) + \psi\big(\left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_y\right|^2\big) + \psi\big(\left|\mathbf{G}^{\mathrm{T}}\mathbf{e}_z\right|^2\big), \end{aligned} \tag{4.5}$$

where $\mathbf{G} := \operatorname{adj}\nabla\mathbf{u}$. Lemma A.3 (with $\omega = \psi$), (3.11)–(3.13) (with $q = 1$), and (4.5) (cf. the proof of Proposition 3.3) then yield

$$W(\nabla\mathbf{u}) \geq 2\psi\left(\left|\frac{\partial\mathbf{u}}{\partial z}\right|\right) + \psi\left(\left|\frac{\partial\mathbf{u}}{\partial z}\right|^{-2}\right).$$

The function

$$h(t) = 2\psi(t) + \psi\big(t^{-2}\big)\,.$$

is convex with

$$h'(t) = 2t^{-3}\Big[t^3\psi'(t) - \psi'\left(t^{-2}\right)\Big];$$

and so h is strictly increasing on $[1,\infty)$. The remainder of the proof then follows the similar part of the proof of Proposition 3.1. □

5 Transversely Isotropic and Inhomogeneous Materials.

In this section we show that our results are valid for certain inhomogeneous stored-energy functions as well as certain ones that are transversely isotropic rather than isotropic.

We first recall that a homogeneous, incompressible, hyperelastic body is frame-indifferent and transversely isotropic, with respect to the z-axis, provided that, for any $\mathbf{F} \in \mathrm{M}_1^{3\times3}$,

$$W(\mathbf{QF}) = W(\mathbf{F}), \qquad W(\mathbf{FR}) = W(\mathbf{F})$$

for every rotation $\mathbf{R} \in \mathrm{M}_1^{3\times3}$ about the z-axis and every rotation $\mathbf{Q} \in \mathrm{M}_1^{3\times3}$. A standard representation theorem (see, e.g., [8, 9] or [12, p. 26]) then yields a function $\varPsi : (\mathbb{R}^+)^4 \to \mathbb{R}$ that satisfies

$$W(\mathbf{F}) = \varPsi\big(|\mathbf{F}|, |\operatorname{adj}\mathbf{F}|^2, |\mathbf{F}\mathbf{e}_z|, |\mathbf{F}\mathbf{e}_x\cdot\mathbf{F}\mathbf{e}_z|^2 + |\mathbf{F}\mathbf{e}_y\cdot\mathbf{F}\mathbf{e}_z|^2\big), \tag{5.1}$$

for every $\mathbf{F} \in \mathrm{M}_1^{3\times3}$. We note that $|\mathbf{F}\mathbf{e}_x\cdot\mathbf{F}\mathbf{e}_z| = |\mathbf{F}\mathbf{e}_y\cdot\mathbf{F}\mathbf{e}_z| = 0$ when $\mathbf{F} = \nabla\mathbf{u}_\lambda^{\mathrm{h}}$ and that, by Lemma 2.4,

$$\fint_0^L |(\nabla\mathbf{u})\mathbf{e}_z|\,dz \geq \fint_0^L \big|\big(\nabla\mathbf{u}_\lambda^{\mathrm{h}}\big)\mathbf{e}_z\big|\,dz = \lambda,$$

for every $\mathbf{u} \in \mathcal{A}_\lambda^1$. The next result therefore follows from these two observations, Corollaries 3.2 and 3.5, and the proof of Theorem 3.6.

Theorem 5.1. *Let $\lambda \geq 1$. Suppose that W is given by* (5.1)*, where $\varPsi$ is monotone increasing in each argument and convex. Then, for any $\mathbf{u} \in \mathcal{A}^1_\lambda$,*

$$E(\mathbf{u}) = \int_{\mathcal{C}} W(\nabla\mathbf{u})\, dV \geq \int_{\mathcal{C}} W(\nabla\mathbf{u}^{\mathrm{h}}_\lambda)\, dV = E(\mathbf{u}^{\mathrm{h}}_\lambda). \tag{5.2}$$

Moreover, if in addition $\varPsi$ is a strictly increasing function of any of its arguments (for all values of the remaining arguments), then inequality (5.2) *is strict when $\mathbf{u} \not\equiv \mathbf{u}^{\mathrm{h}}_\lambda$.*

Remark 5.2. More generally, it is clear that $\mathbf{u}^{\mathrm{h}}_\lambda$ is a (strict) minimizer of the energy if the first two arguments of $\varPsi$ in (5.1) are replaced by the corresponding $N + M$ arguments used in Theorem 3.6. Also, $\varPsi$ need not be increasing or convex in its last argument. One just needs to require that $\varPsi$ is continuous (and hence integrable) on $(\mathbb{R}^+)^4$, that $(r, s, t) \mapsto \varPsi(r, s, t, 0)$ is increasing and convex on $(\mathbb{R}^+)^3$, and that $\varPsi(r, s, t, \tau) \geq \varPsi(r, s, t, 0)$ for all $(r, s, t, \tau) \in (\mathbb{R}^+)^4$. (In the proof of this result one first makes use of this inequality before applying Jensen's inequality.)

Finally, we note that our results also apply to stored-energy functions that depend explicitly on x and y.

Theorem 5.3. *Let $\lambda \geq 1$. Suppose that W is given by*

$$W(\mathbf{F}, x, y) = \varPsi\big(|\mathbf{F}|, |\operatorname{adj}\mathbf{F}|^2, |\mathbf{F}\mathbf{e}_z|, |\mathbf{F}\mathbf{e}_x \cdot \mathbf{F}\mathbf{e}_z|^2 + |\mathbf{F}\mathbf{e}_y \cdot \mathbf{F}\mathbf{e}_z|^2, (x, y)\big), \tag{5.3}$$

where $\varPsi$ is continuous and, for each $(x, y) \in \overline{\Omega}$, $(r, s, t, \tau) \mapsto \varPsi(r, s, t, \tau, (x, y))$ is monotone increasing in each argument and convex. Then, for every $\mathbf{u} \in \mathcal{A}^1_\lambda$,

$$E(\mathbf{u}) = \int_{\mathcal{C}} W\big(\nabla\mathbf{u}(x, y, z), x, y\big)\, dV \geq \int_{\mathcal{C}} W\big(\nabla\mathbf{u}^{\mathrm{h}}_\lambda(x, y, z), x, y\big)\, dV = E(\mathbf{u}^{\mathrm{h}}_\lambda). \tag{5.4}$$

Moreover, if in addition $(r, s) \mapsto \varPsi(r, s, t, \tau, (x, y))$ is a strictly increasing function of either of its arguments, for all $(x, y) \in \overline{\Omega}$ and all values of the remaining arguments, then inequality (5.4) *is strict when $\mathbf{u} \not\equiv \mathbf{u}^{\mathrm{h}}_\lambda$.*

Proof. Let W be given by (5.3). Fix $(x, y) \in \overline{\Omega}$. Then it follows from (3.1), (3.9), and (3.16) (with $p = q = 1$) that

$$\int_0^L W\big(\nabla\mathbf{u}(x, y, z), x, y\big)\, dz \geq \int_0^L W\big(\nabla\mathbf{u}^{\mathrm{h}}_\lambda(x, y, z), x, y\big)\, dz. \tag{5.5}$$

The desired inequality now follows upon integration of (5.5) over Ω and multiplication of the result by L. The proof that (5.4) is a strict inequality when $\mathbf{u} \not\equiv \mathbf{u}^{\mathrm{h}}_\lambda$ is once again comparable to the proof of Theorem 3.6. □

A Appendix.

Proposition A.1 (A Rayleigh-Ritz Inequality). *Let $\mathbf{P} \in \mathrm{M}^{n\times n}$ be symmetric and strictly positive definite with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Suppose that $\omega : (0,\infty) \to \mathbb{R}$ is convex. Then*

$$\sum_{i=1}^{n} \omega(\lambda_i) \geq \sum_{i=1}^{n} \omega(\mathbf{f}_i \cdot \mathbf{P}\mathbf{f}_i)$$

for every orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ of $\mathbb{R}^n$.

Remark A.2. It is clear from the proof that if $\omega : \mathbb{R} \to \mathbb{R}$ is convex, then the result is valid for all symmetric matrices $\mathbf{P} \in \mathrm{M}^{n\times n}$.

Proof of Proposition A.1. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ denote an orthonormal basis of eigenvectors for $\mathbf{P}$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Suppose that $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ is any orthonormal basis for $\mathbb{R}^n$. Then there exist scalars Q_{ij} such that, for each $j = 1, 2, \ldots, n$,

$$\mathbf{f}_j = \sum_{i=1}^{n} Q_{ij}\mathbf{e}_i \qquad \text{and hence} \qquad \mathbf{P}\mathbf{f}_j = \sum_{i=1}^{n} Q_{ij}\lambda_i\mathbf{e}_i. \tag{A.1}$$

Define $\mathbf{Q} \in \mathrm{M}^{n\times n}$ to be the matrix that satisfies $\mathbf{e}_i \cdot \mathbf{Q}\mathbf{e}_j := Q_{ij}$.

Next, since $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_i\}$, $i = 1, 2, \ldots, n$, are each orthonormal bases, it follows from $(\mathrm{A}.1)_1$ that the rows of $\mathbf{Q}$ form an orthonormal basis. Therefore $\mathbf{Q}$ is an orthogonal matrix and so the columns of $\mathbf{Q}$ also form an orthonormal basis. Thus, in particular,

$$1 = \sum_{i=1}^{n} Q_{ij}^2, \qquad 1 = \sum_{j=1}^{n} Q_{ij}^2. \tag{A.2}$$

We take the inner product of $(\mathrm{A}.1)_1$ with $(\mathrm{A}.1)_2$ and make use of the fact that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ to conclude that, for each $j = 1, 2, \ldots, n$,

$$\mathbf{f}_j \cdot \mathbf{P}\mathbf{f}_j = \sum_{i=1}^{n} \lambda_i Q_{ij}^2.$$

The convexity of ω together with $(\mathrm{A}.2)_1$ then yields

$$\omega\big(\mathbf{f}_j \cdot \mathbf{P}\mathbf{f}_j\big) = \omega\left(\sum_{i=1}^{n} \lambda_i Q_{ij}^2\right) \leq \sum_{i=1}^{n} \omega(\lambda_i) Q_{ij}^2,$$

which when summed over j implies

$$\sum_{j=1}^{n} \omega\big(\mathbf{f}_j \cdot \mathbf{P}\mathbf{f}_j\big) \leq \sum_{i=1}^{n} \left(\omega(\lambda_i) \sum_{j=1}^{n} Q_{ij}^2\right).$$

The desired result now follows from $(\mathrm{A}.2)_2$. $\square$

Lemma A.3 (A Generalized Arithmetic-Geometric Mean Inequality)**.** *Let $\omega : (0,\infty) \to \mathbb{R}$ be monotone increasing and convex. Then, for every $x > 0$ and $y > 0$,*

$$\omega\big(x^2\big) + \omega\big(y^2\big) \geq 2\omega(xy).$$

Moreover, this inequality is strict if $x \neq y$ and either ω is strictly monotone or strictly convex.

Proof. The arithmetic-geometric mean inequality together with the monotonicity of ω yields

$$\tfrac{1}{2}\big(x^2 + y^2\big) \geq xy \qquad \text{and hence} \qquad \omega\big(\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2\big) \geq \omega(xy)$$

with strict inequality if $x \neq y$ and ω is strictly increasing. The desired result now follows from the last inequality together with the convexity of ω, i.e.,

$$\tfrac{1}{2}\omega\big(x^2\big) + \tfrac{1}{2}\omega\big(y^2\big) \geq \omega\big(\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2\big)$$

with strict inequality if $x \neq y$ and ω is strictly convex. □

Remark A.4. More generally, it is clear from the above proof that if $p_1 + p_2 + \cdots + p_n = 1$ and $p_i > 0$, then

$$\sum_{i=1}^{n} \tfrac{1}{p_i}\omega\Big((x_i)^{p_i}\Big) \geq \omega\bigg(\prod_{i=1}^{n} x_i\bigg).$$

In particular the choice $p_i = 1/n$ yields

$$\sum_{i=1}^{n} \omega\Big((x_i)^n\Big) \geq n\,\omega\bigg(\prod_{i=1}^{n} x_i\bigg).$$

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