On the Global Stability of Compressible Elastic Cylinders in Tension

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16 July 2014

Abstract

Consider a three-dimensional, homogeneous, compressible, hyperelastic body that occupies a cylindrical domain in its reference configuration. We identify a variety of hypotheses on the structure of the stored-energy function under which there exists an axisymmetric, homogeneous deformation that globally minimizes the energy. For certain classes of energy functions the uniqueness of this minimizer is also established. The primary boundary condition considered is the extension of the cylinder via the prescription of its deformed axial length, but the *sliding contact* of the curved surface is also briefly considered. In particular, the results contained in this paper give conditions on the stored-energy function under which material instabilities, such as necking or the formation of shear bands, *do not occur.*

Mathematics Subject Classifications (2010): 74B20, 35J50, 49J40, 74G65.

Key words: Elastic, uniaxial tension, sliding-contact, homogeneous absolute minimizer.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected, open set with centroid at the origin and consider a homogeneous, isotropic, hyperelastic material that occupies the cylindrical region

$$\mathcal{C} := \{(x, y, z) : (x, y) \in \Omega,\ 0 < z < L\}$$

in a fixed homogeneous reference configuration.

The problem we first consider is that of *uniaxial extension.* Specifically, we fix $\lambda \geq 1$ and restrict our attention to those deformations $\mathbf{u} \in C^1(\overline{\mathcal{C}}; \mathbb{R}^3)$, with $\det \nabla\mathbf{u} > 0$ in $\overline{\mathcal{C}}$, that satisfy the boundary conditions:

$$u_3(x, y, 0) = 0, \quad u_3(x, y, L) = \lambda L \quad \text{for all } (x, y) \in \overline{\Omega}, \tag{1.1}$$

where

$$\mathbf{u}(x, y, z) = \begin{bmatrix} u_1(x, y, z) \\ u_2(x, y, z) \\ u_3(x, y, z) \end{bmatrix}. \tag{1.2}$$

With each such deformation we associate a corresponding (total) elastic energy

$$E(\mathbf{u}) = \int_{\mathcal{C}} W(\nabla\mathbf{u})\, \mathrm{d}V, \tag{1.3}$$

where $W : \mathrm{M}^{3\times3}_{+} \to [0, \infty)$ is the stored-energy density, $\mathrm{M}^{3\times3}_{+}$ denotes the set of 3×3 matrices with (strictly) positive determinant, and $\mathrm{d}V = \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$. If W is both isotropic and frame-indifferent, then standard representation theorems (see, e.g., [9, 14, 15, 25]) imply that there is a function $\Upsilon : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ that satisfies

$$W(\mathbf{F}) = \Upsilon(|\mathbf{F}|, |\operatorname{adj}\mathbf{F}|, \det\mathbf{F}) \quad \text{for all } \mathbf{F} \in \mathrm{M}^{3\times3}_{+}, \tag{1.4}$$

where $|\mathbf{F}|$ denotes the square-root of the sum of the squares of the elements of $\mathbf{F}$, $\det\mathbf{F}$ denotes the determinant of $\mathbf{F}$, and $\operatorname{adj}\mathbf{F}$ denotes the adjugate matrix of $\mathbf{F}$, i.e.,

$$(\operatorname{adj}\mathbf{F})\mathbf{F} = (\det\mathbf{F})\mathbf{I} \quad \text{for all} \quad \mathbf{F} \in \mathrm{M}^{3\times3}_{+},$$

where $\mathbf{I}$ is the identity matrix.

Note that if a deformation $\mathbf{u}$ satisfies (1.1) and minimizes (1.3)–(1.4), then so does $\mathbf{g} \circ \mathbf{u}$, where $\mathbf{g}$ is any rotation about an axis perpendicular to the xy-plane together with any translation parallel to the xy-plane. In order to eliminate this trivial nonuniqueness we impose the additional constraints

$$\begin{aligned} \int_{\mathcal{C}} u_1\, \mathrm{d}V &= \int_{\mathcal{C}} u_2\, \mathrm{d}V = 0, \\ \int_{\mathcal{C}} \frac{\partial u_1}{\partial y}\, \mathrm{d}V &= \int_{\mathcal{C}} \frac{\partial u_2}{\partial x}\, \mathrm{d}V. \end{aligned} \tag{1.5}$$

We start our analysis, in section 2, by showing that the boundary conditions (1.1) imply that any deformation that minimizes the fiber energy

$$\int_{\mathcal{C}} |\mathbf{u}_z|^p \, dV,$$

must map lines parallel to the z-axis into such lines. We then extend a result of Mizel [22] concerning the strict Baker-Ericksen inequalities (which are a consequence of the strict rank-one convexity of W) and their implications on the equality of the principal stretches perpendicular to the z-axis of any minimizer of W.

In section 3 we consider stored-energy functions of the form

$$W(\mathbf{F}) = \Psi\big(|\mathbf{F}|^q, (\det \mathbf{F})^{p/3}\big),$$

where $p \in [1,3]$, $q \geq p$, and $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is increasing in its first argument and convex. (Note that such stored energies are not necessarily polyconvex in the sense of Ball [3, 4], though it is straightforward to construct examples which are polyconvex.) For these functions we give conditions in Theorem 3.6 under which there is a unique, axisymmetric, homogeneous energy minimizer that satisfies (1.5) and the boundary condition (1.1). Moreover, if $\lambda > 1$, then this minimizer deforms the body into a state of uniaxial tension, as expected. Examples (setting $q = 2$, $p = 3/2$) of energy functions to which this result applies are *compressible neo-Hookean* materials of the form

$$W(\mathbf{F}) = \tfrac{\mu}{2}|\mathbf{F}|^2 + h\Big(\sqrt{\det \mathbf{F}}\,\Big),$$

where $\mu > 0$ and $h \in C^1((0,\infty);\mathbb{R})$ is convex.

Our method of proof is to construct, for each deformation that satisfies (1.1) with $\lambda \geq 1$, a corresponding axisymmetric homogeneous map that has no greater energy. Specifically, we use the following symmetrization procedure: to each deformation $\mathbf{u}$ that satisfies the boundary condition (1.1) we associate the homogeneous deformation

$$\mathbf{u}_\lambda^\nu(x,y,z) := \begin{bmatrix} \nu x \\ \nu y \\ \lambda z \end{bmatrix}, \tag{1.6}$$

where $\nu = \mu > 0$ is chosen so that

$$\big(\mu^2\lambda\big)^{p/3} = \fint_{\mathcal{C}} \big[\det \nabla \mathbf{u}_\lambda^\nu\big]^{p/3} \mathrm{d}V = \fint_{\mathcal{C}} \big[\det \nabla \mathbf{u}\big]^{p/3} \mathrm{d}V.$$

We then prove that:

(i) If $\mu \leq \lambda$, then $E(\mathbf{u}) \geq E(\mathbf{u}_\lambda^\mu)$;

(ii) If $\mu \geq \lambda$, then $E(\mathbf{u}) \geq E(\mathbf{u}_\lambda^\lambda)$.

Our proof of this result uses Jensen's inequality and the observation that the boundary condition (1.1) imposes a constraint on admissible deformations: material fibers originally parallel to the axis of symmetry of the cylinder in the reference configuration must have a deformed length of at least λL. Hence, it suffices to minimize E in the class of axisymmetric homogeneous deformations $\{\mathbf{u}_\lambda^\nu \mid \nu > 0\}$ to find a global energy minimizer subject to the boundary condition (1.1). A slight strengthening of the hypotheses on the energy function then yields the existence and uniqueness of this homogeneous energy minimizer.

In particular, the results contained in this paper give *conditions on the stored-energy function under which there is no bifurcation from the underlying homogeneous solution and no material instabilities such as necking or the formation of shear bands.*

If we further assume that the stored-energy function W is polyconvex in the sense of Ball [3, 4] and that W grows sufficiently fast at zero and infinity, then standard results [3, 4, 9, 23] yield the existence of an absolute minimizer of (1.3) in the set of Sobolev deformations

$$\mathcal{S}_\lambda = \{\mathbf{u} \in W^{1,1}(\mathcal{C}; \mathbb{R}^3) : \det \nabla \mathbf{u} > 0 \ a.e., \mathbf{u} \text{ satisfies (1.1) and (1.5)}\}.$$

The results in this paper are also valid for $\mathbf{u} \in \mathcal{S}_\lambda$ and show that this absolute minimizer is the homogeneous axisymmetric deformation[1] $\mathbf{u}_\lambda^\nu$ given by (1.6) for an appropriate value of ν. Moreover, this minimizer is globally one-to-one.

In section 3.4 we note ideas in [32, Section 5] can be used to extend the results in this paper to materials which are transversally isotropic (relative to the symmetry axis of the cylinder). We also note that the results in the appendix of this paper can be used to prove the same homogeneity of an energy minimizing deformation for stored-energy functions that depend on the invariant $|\operatorname{adj} \mathbf{F}|^2/(\det \mathbf{F})$, i.e., energies of the form:

$$W(\mathbf{F}) = \Psi\left(|\mathbf{F}|^q, \left[\frac{|\operatorname{adj} \mathbf{F}|^2}{\det \mathbf{F}}\right]^t, (\det \mathbf{F})^{p/3}\right),$$

where $p \in [1, 3]$, $q \geq p$, $t \geq p$ and $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is increasing in each of its first two arguments and convex.

In section 4 we extend our approach to treat the case in which the boundary condition (1.1) is replaced by a *sliding contact* (sometimes referred to as *greased sides*) boundary condition on the curved sides of a **circular** cylinder $\mathcal{C} = \Omega \times (-L, L)$ of radius $R > 0$. Mathematically, we require that there is a $\mu \geq 1$ such that the deformation (1.2) satisfies

$$[u_1(x, y, z)]^2 + [u_2(x, y, z)]^2 = \mu^2 R^2 \quad \text{for} \quad x^2 + y^2 = R^2 \quad \text{and } z \in [-L, L].$$

(The flat surfaces of the cylinder, $z = -L$ and $z = L$, are left free.)

In this case, the boundary condition imposes a constraint on the admissible deformations that each cross section $\Omega \times \{z\}$, $z \in [-L, L]$, of the cylinder in the reference configuration must map to a surface that spans the deformed boundary and hence must deform into a surface whose area is at least $\pi(\mu R)^2$. We use this observation, together with an adaptation of the

[1] If we denote the gradient of this homogeneous minimizer by $\mathbf{F}_0 \in \mathrm{M}_+^{3\times3}$, then it follows that the stored-energy function is quasiconvex at the boundary at $(\mathbf{F}_0, \mathbf{n})$, in the sense of Ball and Marsden [7], for any choice of unit vector $\mathbf{n}$ which is normal to some part of the lateral boundary $\partial\Omega \times [0, L]$ of the cylinder $\mathcal{C}$.

symmetrization procedure outlined earlier, to prove that there is an axisymmetric homogeneous deformation that globally minimizes the energy. This result applies to stored-energy functions of the form

$$W(\mathbf{F}) = \Psi\big(|\mathbf{F}|^2, |\operatorname{adj}\mathbf{F}|, (\det\mathbf{F})^{2/3}\big),$$

where $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex, with $\Psi_{,i} \geq 0$, $i = 1, 2$.

In the Appendix we extend our methods and the results in section 3 on the existence of an axisymmetric, homogeneous, global energy minimizer to stored-energy functions which are expressed in terms of elementary functions of the principal stretches. Some examples of the energy functions to which these extended results apply are given below. In all of the following examples N is a positive integer and $h \in C^1((0,\infty);[0,\infty))$ is a strictly convex function that satisfies $h(t) \to +\infty$ as $t \to 0^+$.

Compressible Ogden: Our theory includes

$$W(\mathbf{F}) = \sum_{i=1}^{N} \frac{\mu_i}{m_i}\big(\alpha^{m_i} + \beta^{m_i} + \gamma^{m_i}\big) + h\Big((\alpha\beta\gamma)^{p/3}\Big)$$

when $m_i \geq 1$, $\mu_i > 0$, and $p \in [1, p_{\max}] \cap [1,3]$ with $p_{\max} := \min\{m_i : 1 \leq i \leq N\}$. Here, and in the examples that follow, α, β, and γ denote the principal stretches of $\mathbf{F}$.

Compressible Mooney-Rivlin: Let $a > 0$ and $b > 0$. Then our theory includes

$$W(\mathbf{F}) = a|\mathbf{F}|^2 + b\left[\frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta} + \frac{\alpha\beta}{\gamma}\right] + h\Big((\alpha\beta\gamma)^{1/3}\Big).$$

In [17], Hill proposes the class of stored-energy functions[2]

$$W(\mathbf{F}) = \sum_{i=1}^{N} \frac{\mu_i}{m_i}\left[\alpha^{m_i} + \beta^{m_i} + \gamma^{m_i} + \frac{1-2\nu}{\nu}\,(\alpha\beta\gamma)^{-\nu m_i/(1-2\nu)}\right],$$

where $1 \leq m_1 < m_2 < \ldots < m_N$, $\mu_i > 0$, and $\nu \in (0, .5)$ denotes Poisson's ratio. For $m_1 > 1$ and $\lambda \geq 1$ our results imply that the absolute minimizer of the energy is given by

$$\mathbf{u}_\lambda^\kappa(x,y,z) = \begin{bmatrix} \kappa x \\ \kappa y \\ \lambda z \end{bmatrix}, \qquad \kappa = \kappa(\lambda) := \lambda^{-\nu}.$$

Note that if $m_N > 3$ then Ball's [3, 4, 9] existence theory applies to this constitutive relation. (See Murphy [24] and Horgan and Murphy [19] for interesting generalizations of this model.)

The standard technique used to establish the *stability* of an equilibrated deformation $\mathbf{u}$ is to prove that it is a strict local minimizer, in the C^1-topology, of the energy by showing that the second variation of the energy at $\mathbf{u}$ is strictly positive. Our technique can alternatively be used

[2] Hill also allows $m_i < 1$, but such values are not covered by our results. See, also, Storåkers [34].

to show that an equilibrated deformation $\mathbf{u}_\lambda^\nu$ (see (1.6)), with $\lambda \geq 1$, is a local minimizer of E. If, for example, $W(\mathbf{F}) = \Psi(|\mathbf{F}|^q, (\det \mathbf{F})^{p/3})$ where, in a neighborhood of the constant matrix $\mathbf{F}_\lambda^\nu := \nabla \mathbf{u}_\lambda^\nu$, Ψ is increasing in its first argument and convex, then $\mathbf{u}_\lambda^\nu$ is a local minimizer of E.

For the uniaxial tension problem considered in this paper, the approach of proving the positivity of the second variation was first used by Wesołowski [39, 40] for an incompressible material. The technique was subsequently used by many other authors for 2-dimensional and 3-dimensional, compressible and incompressible, elastic and elastic-plastic rods. (See, e.g., Hill and Hutchinson [18], Spector [33], Del Piero and Rizzoni [10], and Fosdick, Foti, Fraddosio, and Piccioni [12].)

Finally, we note that in [31, 32] the authors used techniques related to those in the current paper to show that, under appropriate constitutive hypotheses, the energy minimizer for a 2 or 3-dimensional *incompressible* elastic rod, which is subject to uniaxial extension, is an axisymmetric homogeneous deformation.

2 Deformations; The Energy.

Definition 2.1. For $\lambda > 0$ we define the set of *admissible deformations* by

$$\mathcal{A}_\lambda := \left\{ \mathbf{u} \in C^1(\overline{\mathcal{C}}; \mathbb{R}^3) : \det \nabla \mathbf{u} > 0 \text{ on } \overline{\mathcal{C}},\ \mathbf{u} \text{ satisfies (1.1) and (1.5)} \right\}.$$

We identify the subset of $\mathcal{A}_\lambda$ consisting of deformations that are a composition of plane strain in the xy-plane and a homogeneous stretch in the z-direction, i.e., deformations $\mathbf{w} \in \mathcal{A}_\lambda$ of the form

$$\mathbf{w}(x, y, z) = \begin{bmatrix} w_1(x, y) \\ w_2(x, y) \\ \lambda z \end{bmatrix}. \tag{2.1}$$

We denote by

$$\mathrm{PS}_\lambda := \{ \mathbf{w} \in \mathcal{A}_\lambda : \mathbf{w} \text{ satisfies (2.1)} \}$$

the set of all such *plane-strain* deformations.

The unique curve of shortest length connecting two points is a straight line. We will make use of a slight variant of this well-known result. We include the proof from [32, Lemma 2.2] for the convenience of the reader.

Lemma 2.2. *Let $\lambda > 0$, $1 \leq p < \infty$, $\mathbf{u} \in \mathcal{A}_\lambda$, and $\mathbf{w} \in \mathrm{PS}_\lambda$. Then*

$$⨍_{\mathcal{C}} |\mathbf{u}_z|^p \, dV \geq ⨍_{\mathcal{C}} |\mathbf{w}_z|^p \, dV = \lambda^p, \tag{2.2}$$

where $⨍_{\mathcal{C}} \vartheta \, \mathrm{d}V$ denotes the average value of ϑ over $\mathcal{C}$, i.e.,

$$⨍_{\mathcal{C}} \vartheta \, \mathrm{d}V := \frac{1}{\mathrm{vol}(\mathcal{C})} \int_{\mathcal{C}} \vartheta \, \mathrm{d}V,$$

where $\mathrm{vol}(\mathcal{C})$ *denotes the* volume *of* $\mathcal{C}$ *and the notation* $\mathbf{v}_z$ *denotes the derivative* $\frac{\partial \mathbf{v}}{\partial z}$. *Moreover, if* $p > 1$ *and the inequality in* (2.2) *is an equality, then* $\mathbf{u} \in \mathrm{PS}_\lambda$.

Proof. Let $p \geq 1$, $\lambda > 0$, and $\mathbf{v} \in \mathcal{A}_\lambda$. Then

$$⨍_{\mathcal{C}} |\mathbf{v}_z|^p \,\mathrm{d}V \geq ⨍_{\mathcal{C}} |v_{3,z}|^p \,\mathrm{d}V \tag{2.3}$$

with equality if and only if $v_{1,z} = v_{2,z} \equiv 0$. Next, for $p > 1$, Hölder's inequality together with the boundary conditions (1.1) yield

$$⨍_{\mathcal{C}} |v_{3,z}|^p \,\mathrm{d}V \geq \left| ⨍_{\mathcal{C}} v_{3,z} \,\mathrm{d}V \right|^p = \left| ⨍_{\Omega} \frac{v_3(x,y,L) - v_3(x,y,0)}{L} \mathrm{d}x\,\mathrm{d}y \right|^p = \lambda^p \tag{2.4}$$

with equality if and only if $v_{3,z}$ is a constant. Moreover, (2.4) is clearly also valid for $p = 1$.

Finally, we note that necessary and sufficient conditions for $\mathbf{v} \in \mathrm{PS}_\lambda$ are that, for every $(x,y,z) \in \overline{\mathcal{C}}$, $v_{1,z} = v_{2,z} = 0$ and, assuming $p > 1$, $v_{3,z} \equiv \lambda$. In particular, both (2.3) and (2.4) are equalities for $\mathbf{v} \in \mathrm{PS}_\lambda$. Therefore, both the desired inequality as well as the statement concerning equality follow from (2.3) and (2.4). □

2.1 The Energy

We assume that the material in its reference configuration is homogeneous and hyperelastic; thus there exists a stored-energy density $W : \mathrm{M}_+^{3\times 3} \to [0,\infty)$. We further assume that the material is isotropic. This assumption together with invariance under a change in observer yields (see, e.g., [16, 25, 27, 37]) the existence of a symmetric function $\Phi : (0,\infty)^3 \to [0,\infty)$ that satisfies

$$W(\mathbf{F}) := \Phi(\nu_1, \nu_2, \nu_3), \text{ where } \nu_1, \nu_2, \nu_3 \text{ are the eigenvalues of } \mathbf{U} := \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}} \tag{2.5}$$

and are known as the *principal stretches.* We suppose that W (and hence Φ, see [6, 28, 29]) are C^1 (on their respective domains). The *Piola-Kirchhoff stress tensor* $\mathbf{S}$ is then the derivative of W:

$$\mathbf{S}(\mathbf{F}) := \frac{dW}{d\mathbf{F}}, \qquad \mathbf{S} : \mathrm{M}_+^{3\times 3} \to \mathrm{M}^{3\times 3}. \tag{2.6}$$

A standard assumption found in the elasticity literature (see, e.g., [3, 4, 9, 27]) is that the function W is *strictly rank-one-convex:* for every $t \in (0,1)$ and every $\mathbf{F}, \mathbf{G} \in \mathrm{M}_+^{3\times 3}$ with $\mathbf{F} - \mathbf{G} = \mathbf{a} \otimes \mathbf{b} \neq \mathbf{0}$ for some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$,

$$W\left(t\mathbf{F} + (1-t)\mathbf{G}\right) < tW(\mathbf{F}) + (1-t)W(\mathbf{G}).$$

Here $(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j$. One consequence (see [5, pp. 563–564]) of strict rank-one convexity[3] is that at every triple of principal stretches $(\nu_1,\nu_2,\nu_3)\in(0,\infty)^3$, the energy must satisfy the *strict Baker-Ericksen [2] inequalities:* (see also [37])

$$\frac{\nu_i\frac{\partial\Phi}{\partial\nu_i}(\nu_1,\nu_2,\nu_3)-\nu_j\frac{\partial\Phi}{\partial\nu_j}(\nu_1,\nu_2,\nu_3)}{\nu_i-\nu_j}>0 \tag{2.7}$$

for all $i=1,2,3$ and $j=1,2,3$ with $i\neq j$ and $\nu_i\neq\nu_j$.

We show in the next lemma that the Baker-Ericksen inequalities imply strong symmetry properties of certain (constrained and unconstrained) pointwise minima of the stored-energy function (cf. Mizel [22, Theorem A]).

Lemma 2.3 (Consequences of the Baker-Ericksen inequalities)**.** *Suppose that the stored-energy function Φ satisfies the strict Baker-Ericksen inequalities.*

(i) If $(\bar\nu_1,\bar\nu_2,\bar\nu_3)$ minimizes $(\nu_1,\nu_2,\nu_3)\mapsto\Phi(\nu_1,\nu_2,\nu_3)$ on $(0,\infty)^3$, then $\bar\nu_1=\bar\nu_2=\bar\nu_3$.

(ii) Fix $d>0$. Suppose that $(\bar\nu_1,\bar\nu_2,\bar\nu_3)$ minimizes $(\nu_1,\nu_2,\nu_3)\mapsto\Phi(\nu_1,\nu_2,\nu_3)$ on $(0,\infty)^3$ subject to the constraint $\nu_1\nu_2\nu_3=d^3$. Then $\bar\nu_1=\bar\nu_2=\bar\nu_3=d$.

(iii) Let $\lambda>0$ be fixed. Suppose that $(\bar\nu_1,\bar\nu_2)$ minimizes $(\nu_1,\nu_2)\mapsto\Phi(\nu_1,\nu_2,\lambda)$ on $(0,\infty)^2$. Then $\bar\nu_1=\bar\nu_2$.

(iv) Fix $\lambda>0$ and $d>0$. Suppose that $(\bar\nu_1,\bar\nu_2)$ minimizes $(\nu_1,\nu_2)\mapsto\Phi(\nu_1,\nu_2,\lambda)$ on $(0,\infty)^2$ subject to the constraint $\nu_1\nu_2=d^2$. Then $\bar\nu_1=\bar\nu_2=d$. Moreover, (d,d) is a strict minimizer for this constrained problem.

Proof. Parts (i) and (iii) follow directly from (2.7) since the corresponding first-order partial derivatives of Φ must vanish at a minimum. We next prove (ii) (the proof of (iv) is similar).

Consider the function $f(\nu_1,\nu_2)=\Phi\Big(\nu_1,\nu_2,\frac{d^3}{\nu_1\nu_2}\Big)$. Then

$$0=\frac{\partial f}{\partial\nu_1}(\bar\nu_1,\bar\nu_2)=\frac{1}{\bar\nu_1}\left[\bar\nu_1\Phi_{,1}\left(\bar\nu_1,\bar\nu_2,\frac{d^3}{\bar\nu_1\bar\nu_2}\right)-\frac{d^3}{\bar\nu_1\bar\nu_2}\Phi_{,3}\left(\bar\nu_1,\bar\nu_2,\frac{d^3}{\bar\nu_1\bar\nu_2}\right)\right].$$

Therefore, by (2.7), it follows that $\bar\nu_1=\frac{d^3}{\bar\nu_1\bar\nu_2}$ and (using the analogous expression for $\frac{\partial f}{\partial\nu_2}(\bar\nu_1,\bar\nu_2)$) $\bar\nu_2=\frac{d^3}{\bar\nu_1\bar\nu_2}$; thus $\bar\nu_1=\bar\nu_2$. If we interchange the roles of ν_1 and ν_3 in the above argument, we similarly conclude that $\bar\nu_3=\bar\nu_2$; therefore, all $\bar\nu_i$ are equal with product equal to d^3 and so the claim of the lemma follows. □

Remark 2.4. It is interesting to note that if $\Phi(v_1,v_2,v_3)$ satisfies the Baker-Ericksen inequalities, then so does the modified stored-energy function $\tilde\Phi(v_1,v_2,v_3)=\Phi(v_1,v_2,v_3)+h(v_1v_2v_3)$ for *any* choice of the function h (see also the related result Lemma 3.5).

[3] Strict rank-one convexity is itself a consequence of the global strong ellipticity of the linearized equations, i.e., $\mathbf{a}\otimes\mathbf{b}:\mathbb{C}(\mathbf{F})[\mathbf{a}\otimes\mathbf{b}]>0$ for every $\mathbf{F}\in\mathrm{M}^{3\times3}_+$, $\mathbf{a}\in\mathbb{R}^3$, and $\mathbf{b}\in\mathbb{R}^3$ with $\mathbf{a}\neq\mathbf{0}\neq\mathbf{b}$. Here $\mathbb{C}(\mathbf{F}):=d^2W/d\mathbf{F}^2$.

We will show in this paper that, for $\lambda \geq 1$ and for a large class of stored-energy functions W, to each deformation $\mathbf{u} \in \mathcal{A}_\lambda$ there corresponds a deformation in PS_λ that has no greater total elastic energy E. Since all deformations in PS_λ have one principal stretch that is constant and equal to λ on $\mathcal{C}$, we are motivated to consider the following result.

Proposition 2.5. *Let $W \in C^1(\mathrm{M}_+^{3\times 3}; [0,\infty))$ satisfy (2.5). Suppose that (see [1, 3, 4])*

$$\lim_{|\mathbf{F}|\to\infty} W(\mathbf{F}) = +\infty, \qquad \lim_{\det \mathbf{F}\to 0^+} W(\mathbf{F}) = +\infty. \tag{2.8}$$

Then, for any $\lambda > 0$, there exists a matrix $\mathbf{F}_\lambda \in \mathrm{M}_+^{3\times 3}$ that minimizes W among those $\mathbf{F} \in \mathrm{M}_+^{3\times 3}$ that have at least one principal stretch equal to λ. Moreover, if W is strictly rank-one convex or if W satisfies the strict Baker-Ericksen inequalities, then the two other principal stretches of $\mathbf{F}_\lambda$ must be equal.

Proof. The assumptions (2.8) imply that the continuous function W blows up as its argument approaches the boundary of its domain $\mathrm{M}_+^{3\times 3}$. The existence of a minimizer to the constrained problem is then clear. Suppose now that W satisfies the strict Baker-Ericksen inequalities and that the principal stretches of a minimizing matrix $\mathbf{F}_\lambda$ are (α, β, λ). It then follows from Lemma 2.3(iii) that $\alpha = \beta$. □

Remark 2.6. We note that the *existence* of a minimizer for problems (ii) and (iv) in Lemma 2.3 follows if the first condition in (2.8) holds. (The existence of a minimizer for problems (i) and (iii) of the lemma follows if (2.8) holds in its entirety.)

3 The Homogeneity of Energy-Minimizing Deformations.

In this section we illustrate our results by considering certain energies for which the proofs are particularly simple. These energies are also natural from the point of view of elasticity since they only depend on two of the principal invariants. Fix $\mathbf{u} \in \mathcal{A}_\lambda$. Our aim is to prove that, for a large class of stored-energy functions W, the energy functional (1.3) satisfies

$$E(\mathbf{u}) \geq E(\mathbf{u}_\lambda^\nu)$$

for an appropriately chosen $\nu > 0$, where

$$\mathbf{u}_\lambda^\nu(x,y,z) := \begin{bmatrix} \nu x \\ \nu y \\ \lambda z \end{bmatrix}, \qquad \nabla \mathbf{u}_\lambda^\nu(x,y,z) \equiv \begin{bmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{3.1}$$

3.1 The energy $\varLambda(\mathbf{F}) = |\mathbf{F}|^q$ with $q \geq 1$.

Proposition 3.1. *Fix $p \in [1,3]$, $\lambda > 0$, and $\mathbf{u} \in \mathcal{A}_\lambda$. Define $\mu = \mu(\lambda, p, \mathbf{u}) > 0$ to be the unique positive real number that satisfies*

$$\left(\mu^2\lambda\right)^{p/3} = \fint_{\mathcal{C}} (\det \nabla\mathbf{u})^{p/3}\, \mathrm{d}V. \tag{3.2}$$

If $\mu \leq \lambda$ then, for any $q \geq p$,

$$\fint_{\mathcal{C}} |\nabla\mathbf{u}|^q\, \mathrm{d}V \geq \fint_{\mathcal{C}} |\nabla\mathbf{u}_\lambda^\mu|^q\, \mathrm{d}V. \tag{3.3}$$

Moreover, if $p > 1$, then (3.3) is a strict inequality unless $\mathbf{u} \in \mathrm{PS}_\lambda$.

Proof. Fix $\lambda > 0$, $p \in [1,3]$, $\mathbf{u} \in \mathcal{A}_\lambda$, and $q \geq p$. Define μ by (3.2) and assume that $\mu \leq \lambda$. First, note that $(3.1)_2$ yields

$$|\nabla\mathbf{u}_\lambda^\mu(x,y,z)|^q \equiv \left(\lambda^2 + 2\mu^2\right)^{q/2}. \tag{3.4}$$

Next, by the arithmetic-geometric mean inequality,

$$\begin{aligned} |\nabla\mathbf{u}|^2 &= |\mathbf{u}_z|^2 + |\mathbf{u}_x|^2 + |\mathbf{u}_y|^2 \\ &\geq |\mathbf{u}_z|^2 + 2|\mathbf{u}_x||\mathbf{u}_y| \\ &\geq |\mathbf{u}_z|^2 + 2|\mathbf{u}_x \times \mathbf{u}_y|. \end{aligned} \tag{3.5}$$

However, the Cauchy-Schwarz inequality implies that the *Jacobian* $J_\mathbf{u}$ satisfies

$$J_\mathbf{u} := \det \nabla\mathbf{u} = \mathbf{u}_z \cdot (\mathbf{u}_x \times \mathbf{u}_y) \leq |\mathbf{u}_z||\mathbf{u}_x \times \mathbf{u}_y|, \tag{3.6}$$

which, when combined with (3.5), yields

$$|\nabla\mathbf{u}|^p \geq \left[|\mathbf{u}_z|^2 + 2\frac{J_\mathbf{u}}{|\mathbf{u}_z|}\right]^{p/2} = |\mathbf{u}_z|^p \left[1 + 2\left(\frac{[J_\mathbf{u}]^{p/3}}{|\mathbf{u}_z|^p}\right)^{3/p}\right]^{p/2}. \tag{3.7}$$

Define $\theta(s) := (1 + 2s^{3/p})^{p/2}$, $G(s,t) := s\theta(t/s)$, and $H(s,t) := G(s,t)^{q/p}$ so that (3.7) is equivalent to

$$|\nabla\mathbf{u}|^q \geq H\big(|\mathbf{u}_z|^p, [J_\mathbf{u}]^{p/3}\big). \tag{3.8}$$

Note that

$$\theta''(s) := 3s^{(3-2p)/p}(1 + 2s^{3/p})^{(p-4)/2}\left[\tfrac{3}{p} - 1 + s^{3/p}\right] > 0$$

for $s > 0$ and $p \in [1,3]$; thus θ is strictly convex. Therefore, by Lemma A.3, $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly convex and, consequently, so is $H = G^{q/p}$. Moreover,

$$\frac{\partial G}{\partial s} = \theta(\tau) - \tau\theta'(\tau) = \left(1 - \tau^{3/p}\right)\left(1 + 2\tau^{3/p}\right)^{(p-2)/2} > 0 \quad \text{for} \quad \tau := t/s < 1.$$

Therefore, for any $t > 0$ the mapping $s \mapsto G(s,t)$ is strictly increasing on $[t,\infty)$ and, consequently, $s \mapsto H(s,t)$ is also strictly increasing on $[t,\infty)$.

We now integrate (3.8) over the cylinder $\mathcal{C}$ and apply Jensen's inequality to the convex function H to conclude, with the aid of (3.2), that

$$\begin{aligned} \fint_{\mathcal{C}} |\nabla \mathbf{u}|^q \,\mathrm{d}V &\geq H\Big(\fint_{\mathcal{C}} |\mathbf{u}_z|^p \,\mathrm{d}V, \fint_{\mathcal{C}} [J_{\mathbf{u}}]^{p/3} \,\mathrm{d}V\Big) \\ &= H\Big(\fint_{\mathcal{C}} |\mathbf{u}_z|^p \,\mathrm{d}V, \big(\mu^2\lambda\big)^{p/3}\Big). \end{aligned} \tag{3.9}$$

Next, the assumption $\mu \leq \lambda$ implies that $\big(\mu^2\lambda\big)^{p/3} \leq \lambda^p$, which together with Lemma 2.2 and the monotonicity of $s \mapsto H(s,(\mu^2\lambda)^{p/3})$ on $[(\mu^2\lambda)^{p/3},\infty)$ yields

$$\begin{aligned} H\Big(\fint_{\mathcal{C}} |\mathbf{u}_z|^p \,\mathrm{d}V, \big(\mu^2\lambda\big)^{p/3}\Big) &\geq H\Big(\lambda^p, \big(\mu^2\lambda\big)^{p/3}\Big) \\ &= \Big[G\Big(\lambda^p, \big(\mu^2\lambda\big)^{p/3}\Big)\Big]^{q/p} \\ &= \lambda^q \left[\theta\left(\frac{\big(\mu^2\lambda\big)^{p/3}}{\lambda^p}\right)\right]^{q/p} \\ &= \big(\lambda^2 + 2\mu^2\big)^{q/2}. \end{aligned} \tag{3.10}$$

The desired inequality, (3.3), now follows from (3.9), (3.10), and (3.4).

Finally, suppose that (3.3) is an equality and that $p > 1$. Then, in particular, the inequality in (3.10) must be an equality. Since the mapping $s \mapsto H(s,t)$ is strictly increasing we conclude that

$$\fint_{\mathcal{C}} |\mathbf{u}_z|^p \,\mathrm{d}V = \lambda^p.$$

Lemma 2.2 then yields $\mathbf{u} \in \mathrm{PS}_\lambda$. □

3.2 The energy $W(\mathbf{F}) = \Psi(|\mathbf{F}|^q, (\det \mathbf{F})^{p/3})$ with $q \geq p$ and $p \in [1,3]$.

The key to including the dependence of the energy upon the Jacobian is the following simple result.

Lemma 3.2. *Let $q \geq p > 0$. Suppose that the stored energy $W \in C^1(\mathrm{M}^{3\times 3}_+;[0,\infty))$ satisfies*

$$W(\mathbf{F}) = \Psi\big(|\mathbf{F}|^q, (\det \mathbf{F})^{p/3}\big),$$

where $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to [0,\infty)$ is (strictly) convex. Then

$$t \mapsto W(\sqrt[p]{t}\,\mathbf{I}) \quad \text{is (strictly) convex on } \mathbb{R}^+.$$

Moreover, if in addition the reference configuration is stress free, then

$$\lambda \mapsto W(\lambda\mathbf{I}) \ \text{ is (strictly) increasing on } [1,\infty).$$

Proof. To obtain convexity define

$$\xi(t) := W(\sqrt[p]{t}\,\mathbf{I}) = \Psi(3^{q/2}t^{q/p}, t)$$

and note that, since $q/p \geq 1$, the assumed (strict) convexity of Ψ yields (see Proposition A.1) the (strict) convexity of ξ. Next, if we take the derivative of ξ we find that

$$\dot{\xi}(t) = \tfrac{\sqrt[p]{t}}{pt}\,\mathbf{S}(\sqrt[p]{t}\,\mathbf{I}) : \mathbf{I},$$

where the $\mathbf{S}$ is the Piola-Kirchhoff stress tensor (2.6) and $\mathbf{A}:\mathbf{B} := \mathrm{trace}(\mathbf{A}\mathbf{B}^{\mathrm{T}})$. Thus, if the reference configuration is stress free, $\dot{\xi}(1) = 0$. The (strict) convexity of ξ then implies that ξ, and hence the mapping $\lambda \mapsto W(\lambda\mathbf{I})$, is (strictly) increasing on $[1,\infty)$. □

Remark 3.3. It is interesting to note that: (1) If $p \geq 1$ then the mapping $\lambda \mapsto W(\lambda\mathbf{I}) = \xi(\lambda^p)$ is also (strictly) convex on $\mathbb{R}^+$; (2) If the reference configuration is stress free, then the mapping $\lambda \mapsto W(\lambda\mathbf{I})$ is (strictly) decreasing on $(0,1]$.

3.2.1 Energy Reduction by Symmetrization

We will next show that the result in the last subsection can be used to obtain energy reduction for a class of stored-energy functions that includes compressible neo-Hookean materials.

Theorem 3.4. *Fix $p \in [1,3]$ and $q \geq p$. Assume that the reference configuration is stress free. Suppose that the stored energy $W \in C^1(\mathrm{M}^{3\times3}_+;[0,\infty))$ satisfies*

$$W(\mathbf{F}) = \Psi\big(|\mathbf{F}|^q, (\det\mathbf{F})^{p/3}\big), \tag{3.11}$$

where $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is increasing in its first argument and convex. Let $\lambda \geq 1$ and $\mathbf{u} \in \mathcal{A}_\lambda$. Define $\mu = \mu(\lambda, p, \mathbf{u}) > 0$ to be the unique positive real number that satisfies (3.2). Then

$$\bar{E}(\mathbf{u}) := \fint_{\mathcal{C}} W(\nabla\mathbf{u})\,\mathrm{d}V \geq \fint_{\mathcal{C}} W(\nabla\mathbf{u}^\nu_\lambda)\,\mathrm{d}V = \bar{E}(\mathbf{u}^\nu_\lambda), \tag{3.12}$$

where $\mathbf{u}^\nu_\lambda$ is given by (3.1) with $\nu := \mu$ if $\mu \leq \lambda$ and $\nu := \lambda$ if $\mu \geq \lambda$. Moreover, if $\mu \leq \lambda$, $p > 1$, and Ψ is strictly increasing in its first argument, then (3.12) is a strict inequality unless $\mathbf{u} \in \mathrm{PS}_\lambda$.

Proof. Fix $p \in [1,3]$, $q \geq p$, $\lambda \geq 1$, $\mathbf{u} \in \mathcal{A}_\lambda$, and define μ by (3.2). If we now let $\mathbf{F} = \nabla\mathbf{u}$ in (3.11), integrate over the cylinder $\mathcal{C}$, and apply Jensen's inequality to the convex function Ψ we conclude, with the aid of (3.2), that

$$\begin{aligned}\fint_{\mathcal{C}} W(\nabla\mathbf{u})\,\mathrm{d}V &= \fint_{\mathcal{C}} \Psi\big(|\nabla\mathbf{u}|^q, [J_{\mathbf{u}}]^{p/3}\big)\,\mathrm{d}V\\ &\geq \Psi\bigg(\fint_{\mathcal{C}} |\nabla\mathbf{u}|^q\,\mathrm{d}V, \fint_{\mathcal{C}} [J_{\mathbf{u}}]^{p/3}\,\mathrm{d}V\bigg)\\ &= \Psi\bigg(\fint_{\mathcal{C}} |\nabla\mathbf{u}|^q\,\mathrm{d}V, \big(\mu^2\lambda\big)^{p/3}\bigg).\end{aligned} \tag{3.13}$$

Now suppose that $\mu \leq \lambda$. Then we can make use of Proposition 3.1, the monotonicity of Ψ, (3.1), and (3.11), to obtain

$$\begin{aligned}\Psi\bigg(\fint_{\mathcal{C}} |\nabla\mathbf{u}|^q\,\mathrm{d}V, \big(\mu^2\lambda\big)^{p/3}\bigg) &\geq \Psi\bigg(\fint_{\mathcal{C}} |\nabla\mathbf{u}_\lambda^\mu|^q\,\mathrm{d}V, \big(\mu^2\lambda\big)^{p/3}\bigg)\\ &= \fint_{\mathcal{C}} W\left(\nabla\mathbf{u}_\lambda^\mu\right)\mathrm{d}V,\end{aligned} \tag{3.14}$$

which when combined with (3.13) yields the desired inequality, (3.12) with $\nu = \mu$. Moreover, if $s \mapsto \Psi(s,t)$ is strictly increasing then the inequality in (3.14) is a strict inequality unless

$$\fint_{\mathcal{C}} |\nabla\mathbf{u}|^q\,\mathrm{d}V = \fint_{\mathcal{C}} |\nabla\mathbf{u}_\lambda^\mu|^q\,\mathrm{d}V.$$

Proposition 3.1 now yields $\mathbf{u} \in \mathrm{PS}_\lambda$ when $p > 1$.

Finally, suppose that $\mu \geq \lambda$ so that $\nu = \lambda$. Then if we make use of Hadamard's inequality:[4] $|\mathbf{F}|^3 \geq 3^{3/2}(\det\mathbf{F})$, we find, with the aid of Hölder's inequality and (3.2), that

$$\begin{aligned}3^{-q/2}\fint_{\mathcal{C}} |\nabla\mathbf{u}|^q\,\mathrm{d}V &\geq \fint_{\mathcal{C}} [J_{\mathbf{u}}]^{q/3}\mathrm{d}V\\ &\geq \bigg[\fint_{\mathcal{C}} [J_{\mathbf{u}}]^{p/3}\mathrm{d}V\bigg]^{q/p}\\ &= \Big[\big(\mu^2\lambda\big)^{p/3}\Big]^{q/p} = \big(\mu^2\lambda\big)^{q/3}.\end{aligned} \tag{3.15}$$

Thus, (3.15) and the monotonicity of Ψ now yield

$$\begin{aligned}\Psi\bigg(\fint_{\mathcal{C}} |\nabla\mathbf{u}|^q\,\mathrm{d}V, \big(\mu^2\lambda\big)^{p/3}\bigg) &\geq \Psi\Big(3^{q/2}\left(\mu^2\lambda\right)^{q/3}, \big(\mu^2\lambda\big)^{p/3}\Big)\\ &= W(\eta\mathbf{I}), \quad \text{where} \;\; \eta := (\mu^2\lambda)^{1/3} \geq \lambda \geq 1\end{aligned} \tag{3.16}$$

and we have made use of (3.11). However, Lemma 3.2 then implies that $W(\eta\mathbf{I}) \geq W(\lambda\mathbf{I})$, which together with (3.13) and (3.16) yields the desired result, (3.12) with $\nu = \lambda$. □

[4] See, e.g., [20, p. 220] or [30, p. 408].

3.2.2 Existence and Uniqueness of a Homogeneous Minimizer

We will now show that the result in the last subsection can be used to obtain the existence and uniqueness of a homogeneous energy minimizer. We first note that the energy under consideration satisfies the Baker-Ericksen inequalities.

Lemma 3.5. *Let $q > 0$ and $p \in \mathbb{R}$. Suppose that the stored energy $W \in C^1(\mathrm{M}^{3\times 3}_+; [0,\infty))$ satisfies*

$$W(\mathbf{F}) = \Psi\big(|\mathbf{F}|^q, (\det \mathbf{F})^{p/3}\big), \tag{3.17}$$

where $\Psi \in C^1((0,\infty)^2; [0,\infty))$ with $\Psi_{,1} > 0$. Then W satisfies the strict Baker-Ericksen inequalities.

Proof. Given $\mathbf{F} \in \mathrm{M}^{3\times 3}_+$ let α, β, and γ denote the eigenvalues of $\mathbf{U} := \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$. Then, in view of (2.5) and (3.17),

$$\Phi(\alpha,\beta,\gamma) = W(\mathbf{F}) = \Psi\big((\alpha^2+\beta^2+\gamma^2)^{q/2}, (\alpha\beta\gamma)^{p/3}\big). \tag{3.18}$$

We then differentiate the last equation with respect to α and, separately, with respect to β to conclude that the strict Baker-Ericksen inequalities reduce to (cf. (2.7))

$$\frac{\alpha\Phi_{,1}(\alpha,\beta,\gamma) - \beta\Phi_{,2}(\alpha,\beta,\gamma)}{\alpha-\beta} = q(\alpha+\beta)|\mathbf{F}|^{q-2}\Psi_{,1}(|\mathbf{F}|^q, (\det \mathbf{F})^{p/3}) > 0$$

when $\alpha \neq \beta$, which establishes the desired result. □

Theorem 3.6. *Let W, Ψ, p, and q, satisfy the hypotheses of Theorem 3.4. Suppose, in addition, that $\Psi \in C^1((0,\infty)^2; \mathbb{R})$ with $\Psi_{,1} > 0$, and*

$$\lim_{|\mathbf{F}|\to\infty} W(\mathbf{F}) = +\infty, \qquad \lim_{\det \mathbf{F}\to 0^+} W(\mathbf{F}) = +\infty. \tag{3.19}$$

Then, for each $\lambda \geq 1$, there exists a $\kappa = \kappa(\lambda) > 0$, with $\kappa(\lambda) \leq \lambda$, such that the deformation

$$\mathbf{u}^\kappa_\lambda(x,y,z) := \begin{bmatrix} \kappa x \\ \kappa y \\ \lambda z \end{bmatrix} \tag{3.20}$$

is an absolute minimizer of the energy among deformations in $\mathcal{A}_\lambda$. Moreover, if in addition $p > 1$ and Ψ is strictly convex, then $\mathbf{u}^\kappa_\lambda$ is the unique energy minimizer in $\mathcal{A}_\lambda$.

Remark 3.7. At first glance the parameter q appears to be redundant since the dependence on powers greater than $q = 1$ can instead be obtained with a slight modification to the function Ψ. However, $q = 1$ forces $p = 1$, in which case we haven't obtained the uniqueness of the minimizer.

Corollary 3.8. *Suppose that all of the hypotheses of Theorem 3.6 are satisfied and that $\lambda > 1$. Then $\Phi_{,3}(\kappa,\kappa,\lambda) > 0$, where $\mathbf{u}^\kappa_\lambda$, given by (3.20), is the absolute minimizer of the energy. Thus for $\lambda > 1$ the absolute minimizer of E is a state of uniaxial tension.*

Proof of Theorem 3.6. Existence. Fix $\lambda \geq 1$. We first note that, by Lemma 3.5, W satisfies the strict Baker-Ericksen inequalities. Therefore, (3.19) and Proposition 2.5 yield a $\kappa > 0$ such that

$$\Phi(s,t,\lambda) \geq \Phi(\kappa,\kappa,\lambda) = \bar{E}(\mathbf{u}_\lambda^\kappa) \quad \text{for every } s > 0 \text{ and } t > 0, \tag{3.21}$$

where we have made use of (3.12) and (3.20). We claim that the homogeneous deformation given by (3.20) with this value of κ is an absolute minimizer of E among deformations in $\mathcal{A}_\lambda$.

Let $\mathbf{u} \in \mathcal{A}_\lambda$. Then by Theorem 3.4 there is $\nu > 0$ such that the homogeneous deformation $\mathbf{u}_\lambda^\nu$ given by (3.1) satisfies (see (3.12))

$$\bar{E}(\mathbf{u}) \geq \bar{E}(\mathbf{u}_\lambda^\nu) = \Phi(\nu,\nu,\lambda). \tag{3.22}$$

The desired minimality of $\mathbf{u}_\lambda^\kappa$ now follows from (3.22) and (3.21) with $s = t := \nu$.

Uniqueness. Fix $\lambda \geq 1$ and assume now that $p > 1$ and Ψ is strictly convex. We first claim that (3.21) is a strict inequality unless $s = t = \kappa$. To see this, let $s^* > 0$, $t^* > 0$, and define $\alpha := \sqrt{s^* t^*}$. Then, by Lemma 2.3(iv) and (3.21) (applied with $s = t = \alpha$),

$$\Phi(s^*, t^*, \lambda) \geq \Phi(\alpha,\alpha,\lambda) \geq \Phi(\kappa,\kappa,\lambda),$$

and the first inequality is strict unless $s^* = t^* = \alpha$. Hence it remains to verify that

$$\Phi(\alpha,\alpha,\lambda) > \Phi(\kappa,\kappa,\lambda) \quad \text{for all } \alpha > 0 \text{ with } \alpha \neq \kappa, \tag{3.23}$$

where κ is given in (3.21).

To this end, define $\hat{\alpha} : (0,\infty) \to (0,\infty)$ by $\hat{\alpha}(t) := t^{3/(2p)}/\sqrt{\lambda}$; thus $\hat{\alpha}$ is a bijection of $\mathbb{R}^+$ that satisfies $t = (\hat{\alpha}^2\lambda)^{p/3}$. In view of (3.18),

$$\Phi(\hat{\alpha}(t),\hat{\alpha}(t),\lambda) = \Psi\big(\sigma(t), t\big), \qquad \sigma(t) := \Big[2\lambda^{-1}t^{3/p} + \lambda^2\Big]^{q/2}. \tag{3.24}$$

Note that

$$\ddot{\sigma}(t) = \tfrac{3q}{\lambda^2 p^2}\, t^{-2+3/p}\Big[2\lambda^{-1}t^{3/p} + \lambda^2\Big]^{-2+q/2}\Big(\lambda^3[3-p] + [q + 2(q-p)]t^{3/p}\Big) > 0,$$

since $p \in [1,3]$ with $p \leq q$; thus σ is strictly convex. Since Ψ is also strictly convex with $\Psi_{,1} > 0$, Proposition A.1 together with (3.24) imply that

$$t \mapsto \Phi(\hat{\alpha}(t),\hat{\alpha}(t),\lambda) \quad \text{is strictly convex on } (0,\infty). \tag{3.25}$$

Consequently, since $\hat{\alpha}$ is a bijection of $\mathbb{R}^+$, it follows that (3.23) is satisfied.

We now suppose that $\lambda > 1$. (We will consider the case $\lambda = 1$ at the end of the proof.) We claim that $\kappa < \lambda$. To see this, assume for the sake of contradiction that $\kappa \geq \lambda$. Then apply Theorem 3.4 with $\mathbf{u} = \mathbf{u}_\lambda^\kappa$ (and hence, by (3.2), $\mu = \kappa$) to conclude, with the aid of (2.5), (3.12), and (3.21), that

$$\Phi(\kappa,\kappa,\lambda) = \bar{E}(\mathbf{u}_\lambda^\kappa) \geq \bar{E}(\mathbf{u}_\lambda^\lambda) = \Phi(\lambda,\lambda,\lambda) \geq \Phi(\kappa,\kappa,\lambda)$$

since $\nu = \lambda$ when $\mu = \kappa \geq \lambda$. Thus, by (3.23), it now follows that $\lambda = \kappa$ and so

$$\left.\frac{\partial \Phi(\alpha, \alpha, \lambda)}{\partial \alpha}\right|_{\alpha=\lambda} = 2\Phi,_1(\lambda, \lambda, \lambda) = 0. \tag{3.26}$$

Here we have made use of the symmetry[5] of Φ.

Consider

$$\varphi(t) := W(tI) = \Phi(t,t,t), \qquad \varphi'(t) = 3\Phi,_1(t,t,t).$$

Then since the reference configuration is stress free and $\lambda > 1$, Lemma 3.2 yields $\varphi(\lambda) > \varphi(1)$. Next, Remark 3.3 and the symmetry of Φ imply that φ is a strictly convex function on $[1, \infty)$. Thus, in view of (3.26),

$$\varphi(1) > \varphi(\lambda) + \varphi'(\lambda)(t - \lambda) = \varphi(\lambda),$$

which is a contradiction. This completes the proof that if $\lambda > 1$, then $\kappa < \lambda$.

Now suppose that $\mathbf{u} \in \mathcal{A}_\lambda$ satisfies $E(\mathbf{u}) = E(\mathbf{u}_\lambda^\kappa)$. We will show that $\mathbf{u} = \mathbf{u}_\lambda^\kappa$ to establish uniqueness. By Theorem 3.4 there is $\nu > 0$ such that the homogeneous deformation $\mathbf{u}_\lambda^\nu$ given by (3.1) satisfies (3.22). Thus, by the above existence argument

$$\bar{E}(\mathbf{u}) = \bar{E}(\mathbf{u}_\lambda^\nu) = \bar{E}(\mathbf{u}_\lambda^\kappa). \tag{3.27}$$

Clearly $\nu = \mu < \lambda$ since if $\nu = \lambda > 1$, then the second equality in (3.27) yields the same contradiction obtained above. We next note that (3.27) and Theorem 3.4 yield $\mathbf{u} \in \mathrm{PS}_\lambda$, i.e.,

$$\mathbf{u}(x,y,z) = \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \\ \lambda z \end{bmatrix}.$$

Moreover, by (3.23) the principal stretches of $\mathbf{u}$ must be identically equal to $(\kappa, \kappa, \lambda)$. Thus, for each $(x,y,z) \in \overline{\mathcal{C}}$, $\nabla \mathbf{u}(x,y,z)$ has polar decomposition

$$\nabla \mathbf{u}(x,y,z) = \mathbf{Q}(x,y)\mathbf{U}, \quad \text{where } \mathbf{U} = \kappa[\mathbf{I} - \mathbf{e}_z \otimes \mathbf{e}_z] + \lambda \mathbf{e}_z \otimes \mathbf{e}_z$$

is a constant matrix and, for each $(x,y) \in \overline{\Omega}$, $\mathbf{Q}(x,y)$ is a rotation about the z-axis. We note that $\mathbf{Q}$ is a gradient: $\mathbf{Q} = \nabla(\mathbf{u} \circ [\mathbf{u}_\lambda^\kappa]^{-1})$ and hence, since $\mathcal{C}$ is connected and open, a standard result (see, e.g., [9, pp. 44-45] or [15, p. 49]) implies that the only such rotations are constant maps. This together with $(1.5)_2$ shows that $\mathbf{Q} \equiv \mathbf{I}$. Thus, $\nabla \mathbf{u} = \nabla \mathbf{u}_\lambda^\kappa$ and hence, since $\mathcal{C}$ is connected and open, $\mathbf{u} = \mathbf{u}_\lambda^\kappa + \mathbf{a}$. Finally, (1.1) and $(1.5)_1$ yield $\mathbf{a} = \mathbf{0}$ and so $\mathbf{u}_\lambda^\kappa$ is the unique absolute minimizer of the energy, as claimed.

We now briefly consider the case $\lambda = 1$. The function $\hat{\alpha}$ used previously is then given by $\hat{\alpha}(t) := t^{3/(2p)}$. In view of (3.25),

$$\rho(t) := \Phi(\hat{\alpha}(t), \hat{\alpha}(t), 1) \ \text{ is strictly convex on } (0, \infty)$$

[5] $\Phi(a,b,c) = \Phi(b,a,c) = \Phi(c,b,a)$ and hence $\Phi,_1(a,b,c) = \Phi,_2(b,a,c) = \Phi,_3(c,b,a)$.

and hence ρ' is strictly monotone increasing. Since the reference configuration is stress free

$$\rho'(1) = 2\Phi_{,1}(1,1,1)\hat{\alpha}'(1) = 0$$

and hence the unique minimum of ρ occurs at 1. Thus, by (3.23) and the fact that $\hat{\alpha}$ is a bijection of $\mathbb{R}^+$, it follows that $\kappa = 1$.

Finally, suppose that $\mathbf{u} \in \mathcal{A}_\lambda$ satisfies $E(\mathbf{u}) = E(\mathbf{i})$. Then a previous argument (with $\lambda = \kappa = 1$) shows that, for each $(x,y,z) \in \overline{\mathcal{C}}$, $\nabla\mathbf{u}(x,y,z)$ has polar decomposition $\nabla\mathbf{u}(x,y,z) = \mathbf{Q}(x,y)$, where for each $(x,y) \in \overline{\Omega}$, $\mathbf{Q}(x,y)$ is a rotation about the z-axis. It then follows as before that $\mathbf{Q} \equiv \mathbf{I}$, $\nabla\mathbf{u} \equiv \mathbf{I}$, and $\mathbf{u} \equiv \mathbf{i}$, which establishes the uniqueness of the minimizing deformation $\mathbf{u} \equiv \mathbf{i}$ when $\lambda = 1$. That completes the proof. □

Proof of Corollary 3.8. Fix $\lambda > 1$ and let $\mathbf{u}^\kappa_\lambda$ be the absolute minimizer of the energy given by (3.20) in Theorem 3.6. Then, $\Phi_{,1}(\kappa,\kappa,\lambda) = 0$ and, since Φ satisfies the strict Baker-Ericksen inequalities (see (2.7) and Lemma 3.5), we obtain

$$0 < \frac{\lambda\Phi_{,3}(\kappa,\kappa,\lambda) - \kappa\Phi_{,1}(\kappa,\kappa,\lambda)}{\lambda - \kappa} = \frac{\lambda\Phi_{,3}(\kappa,\kappa,\lambda)}{\lambda - \kappa}.$$

However, in the proof of the last theorem we showed that if $\lambda > 1$, then $\kappa < \lambda$. Hence $\Phi_{,3}(\kappa,\kappa,\lambda) > 0$ as claimed. □

3.3 Sobolev Deformations and Local Results

In this subsection we generalize our results to allow for deformations given by the existence theory of Ball [3, 4] (and subsequent generalization in [8, 23, 35, 36]). We first briefly consider the possibility that some of our constitutive hypotheses are only satisfied in a neighborhood of an axisymmetric homogeneous deformation.

Proposition 3.9. *Let $W(\mathbf{F}) = \Phi(\nu_1,\nu_2,\nu_3)$ satisfy the strict Baker-Ericksen inequalities as well as the hypotheses of Proposition 2.5. Fix $\lambda > 1$ and suppose that $\kappa = \kappa(\lambda) \in (0,\lambda)$ is a minimizer of $t \mapsto \Phi(t,t,\lambda)$ given by Proposition 2.5. Assume, in addition, that for some $p \in [1,3]$ and $q \geq p$,*

$$W(\mathbf{F}) = \Psi\big(|\mathbf{F}|^q, (\det\mathbf{F})^{p/3}\big), \tag{3.28}$$

where $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is, in a neighborhood of $((2\kappa^2+\lambda^2)^{q/2}, (\kappa^2\lambda)^{p/3})$, increasing in its first argument and convex. Then there is a neighborhood of $\mathbf{u}^\kappa_\lambda$ (given by (3.20)) in the C^1-topology, where $\mathbf{u}^\kappa_\lambda$ is a minimizer of E.

Sketch of the proof. We first note that $\mu = \mu(\lambda,\mathbf{u})$ given by (3.2) is a continuous function of $\mathbf{u} \in \mathcal{A}_\lambda$. Thus, our hypothesis $\kappa < \lambda$ yields $\nu = \mu$ in Theorem 3.4 (rather than $\nu = \lambda$) when $\mathbf{u}$ is *sufficiently close* to $\mathbf{u}^\kappa_\lambda$; in particular, the case $\mu > \lambda$ in the proof of Theorem 3.4, which uses the global convexity of Ψ as well as the stress-free reference configuration, is not needed. Therefore, when $\mathbf{u}$ is *close* to $\mathbf{u}^\kappa_\lambda$ it will follow that the auxiliary function $\mathbf{u}^\nu_\lambda$ will be *close* to $\mathbf{u}^\kappa_\lambda$ and so the existence portion of the proof of Theorem 3.6 will yield the desired result. □

In order to consider mappings that are only weakly differentiable we will extend the stored-energy density $W \in C^1(\mathrm{M}^{3\times 3}_+;[0,\infty))$ to all matrices $\mathbf{F} \in \mathrm{M}^{3\times 3}$ by taking $W(\mathbf{F}) = +\infty$ whenever $\det \mathbf{F} \le 0$. Thus, the assumption

$$\lim_{\det \mathbf{F} \to 0^+} W(\mathbf{F}) = +\infty$$

yields $W \in C(\mathrm{M}^{3\times 3};[0,\infty])$ with $W(\mathbf{F}) = +\infty$ if and only if $\det \mathbf{F} \le 0$.

Definition 3.10. We define the set of *admissible Sobolev deformations* by

$$\mathcal{S}_\lambda = \{\mathbf{u} \in W^{1,1}(\mathcal{C};\mathbb{R}^3) : \mathbf{u} \text{ satisfies (1.1), (1.5), and } \det \nabla\mathbf{u} > 0 \ a.e.\},$$

where $W^{1,1}(\mathcal{C};\mathbb{R}^3)$ denotes the usual Sobolev space of vector-valued functions $\mathbf{u} \in \mathcal{L}^1(\mathcal{C};\mathbb{R}^3)$, whose distributional derivative $\nabla\mathbf{u}$ also lies in $\mathcal{L}^1$. We note that equation (1.1) is satisfied in the sense of trace.

Theorem 3.11 (Theorem 3.6 for Sobolev deformations)**.** *Fix $p \in [1,3]$ and $q \ge p$. Assume that the reference configuration is stress free. Suppose that the stored-energy density $W \in C(\mathrm{M}^{3\times 3};[0,\infty]) \cap C^1(\mathrm{M}^{3\times 3}_+;[0,\infty))$ satisfies $W(\mathbf{F}) = +\infty$ whenever $\det \mathbf{F} \le 0$,*

$$\lim_{|\mathbf{F}|\to\infty} W(\mathbf{F}) = +\infty, \qquad \lim_{\det \mathbf{F} \to 0^+} W(\mathbf{F}) = +\infty, \tag{3.29}$$

$$W(\mathbf{F}) = \Psi\big(|\mathbf{F}|^q, (\det \mathbf{F})^{p/3}\big), \quad \text{if } \det \mathbf{F} > 0, \tag{3.30}$$

where $\Psi \in C^1((0,\infty)^2;\mathbb{R})$ is convex and satisfies $\Psi_{,1} > 0$. Then, for each $\lambda \ge 1$, there exists a $\kappa = \kappa(\lambda) \in (0,\lambda]$ such that the (globally injective) deformation $\mathbf{u}^\kappa_\lambda$, given by (3.20), is an absolute minimizer of the energy among deformations in $\mathcal{S}_\lambda$. Moreover, if in addition $p > 1$ and Ψ is strictly convex, then $\mathbf{u}^\kappa_\lambda$ is the unique energy minimizer in $\mathcal{S}_\lambda$.

Proof. Assume the hypotheses of the Theorem. Fix $\lambda \ge 1$. Let $\kappa = \kappa(\lambda) \in (0,\lambda]$ be given by Theorem 3.6. Suppose that $\mathbf{u} \in \mathcal{S}_\lambda$. We will show that $E(\mathbf{u}) \ge E(\mathbf{u}^\kappa_\lambda)$ and that, if $p > 1$ and Ψ is strictly convex, $E(\mathbf{u}) = E(\mathbf{u}^\kappa_\lambda)$ if and only if $\mathbf{u} = \mathbf{u}^\kappa_\lambda$ a.e.

We now denote by $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$ the distributional partial derivatives of $\mathbf{u}$ and we *fix a representative of each of these $\mathcal{L}^1$ (vector-valued) functions.* We define

$$\hat{\mathcal{C}} := \{(x,y,z) \in \mathcal{C} : |\mathbf{u}_z(x,y,z)| > 0 \text{ and } \det \nabla\mathbf{u}(x,y,z) > 0\}.$$

We note that, for $\mathcal{L}^3$ almost every point $(x,y,z) \in \mathcal{C}$, $\det \nabla\mathbf{u} = \mathbf{u}_z \cdot (\mathbf{u}_x \times \mathbf{u}_y)$ and so for $a.e.$ $(x,y,z) \in \mathcal{C}$,

$$\det \nabla\mathbf{u}(x,y,z) > 0 \ \text{ implies that } \ |\mathbf{u}_z(x,y,z)| > 0.$$

Since $\det \nabla\mathbf{u} > 0$ $a.e.$ it follows that $\mathcal{L}^3(\hat{\mathcal{C}}) = \mathcal{L}^3(\mathcal{C})$ and hence

$$\int_{\mathcal{C}} f \,\mathrm{d}V = \int_{\hat{\mathcal{C}}} f \,\mathrm{d}V \ \text{ for any nonnegative, measurable function } f : \mathcal{C} \to [0,\infty]. \tag{3.31}$$

We will show that the conclusions of Lemma 2.2, Proposition 3.1, Theorem 3.4, and Theorem 3.6 remain true when the admissible deformations $\mathcal{A}_\lambda$ are enlarged to include the Sobolev deformations $\mathcal{S}_\lambda$, which will establish the desired result.

Lemma 2.2. We first observe that inequalities (2.3) and (2.4) are both valid when $\mathbf{v} \in \mathcal{S}_\lambda$ with equality in (2.3) if and only if $v_{1,z} = v_{2,z} = 0$ *a.e.* and equality in (2.4) if and only if $v_{3,z} = \varpi$ *a.e.* for some $\varpi \in \mathbb{R}$. Thus, inequality (2.2) is satisfied when $\mathbf{u} \in \mathcal{S}_\lambda$. Now suppose that $p > 1$ and (2.2) is an equality for some $\mathbf{u} \in \mathcal{S}_\lambda$. Then both (2.3) and (2.4) must be equalities and so $u_{1,z} = u_{2,z} = 0$ *a.e.* and $u_{3,z} = \lambda$ *a.e.* Consequently, $\mathbf{u}$ must be a plain strain Sobolev deformation, i.e.,

$$\mathbf{u} \in \widehat{\mathrm{PS}}_\lambda := \{\mathbf{w} \in \mathcal{S}_\lambda : \mathbf{w} \text{ satisfies (2.1) } a.e.\}\,. \tag{3.32}$$

Thus, Lemma 2.2 is valid for $\mathbf{u} \in \mathcal{S}_\lambda$ provided the set PS_λ is replaced by the set $\widehat{\mathrm{PS}}_\lambda$. Finally, we note that (3.31) then allows us to replace each of the integrals in (2.2) by an integral of the same quantity over the set $\hat{\mathcal{C}}$.

Proposition 3.1. We first consider each of the pointwise inequalities established for $\mathbf{u} \in \mathcal{A}_\lambda$ and $(x, y, z) \in \mathcal{C}$ in the proof of Proposition 3.1, i.e., (3.5)–(3.8). It is clear that each is also valid for $\mathbf{u} \in \mathcal{S}_\lambda$ *provided* we restrict (x, y, z) to the set $\hat{\mathcal{C}}$. Moreover, the function H used there is nonnegative and continuous and hence all of the integrations in this proof are integrations of nonnegative measurable functions. The proof of Proposition 3.1 therefore implies that if $\mu \leq \lambda$ and $q \geq p$, (3.3) with $\mathcal{C}$ replaced by $\hat{\mathcal{C}}$ is satisfied when $\mathbf{u} \in \mathcal{S}_\lambda$. The previous observation, (3.31), then allows us to replace $\hat{\mathcal{C}}$ by $\mathcal{C}$.

Now suppose that $p > 1$ and that (3.3) is an equality. Then in particular the inequality in (3.10), with $\mathcal{C}$ replaced by $\hat{\mathcal{C}}$, must be an equality. Since the mapping $s \mapsto H(s, t)$ is strictly increasing we conclude that

$$⨍_{\hat{\mathcal{C}}} |\mathbf{u}_z|^p \,\mathrm{d}V = \lambda^p.$$

The above extension of Lemma 2.2 then yields $\mathbf{u} \in \widehat{\mathrm{PS}}_\lambda$.

Theorem 3.4. The arguments used in the original proof are all valid for $\mathbf{u} \in \mathcal{S}_\lambda$ *provided* we replace $\mathcal{C}$ by $\hat{\mathcal{C}}$ in each integral. Inequality (3.12) is then a consequence of observation (3.31) since $W \geq 0$ is continuous and $\nabla\mathbf{u}$ is measurable. The strictness of inequality (3.12) when $p > 1$ and $\mathbf{u} \notin \widehat{\mathrm{PS}}_\lambda$ then follows as in the above modified proof of Proposition 3.1.

Theorem 3.6. The above modifications to Theorem 3.4 together with the proof of existence in Theorem 3.6 shows that $\mathbf{u}_\lambda^\kappa$ minimizes E among deformations in $\mathcal{S}_\lambda$. We now suppose that $p > 1$ and Ψ is strictly convex. We consider uniqueness when $\lambda > 1$; the proof when $\lambda = 1$ is similar.

Suppose that $\mathbf{u} \in \mathcal{S}_\lambda$ satisfies $E(\mathbf{u}) = E(\mathbf{u}_\lambda^\kappa)$. Then the arguments used in the proof of Theorem 3.6 now lead us to conclude that $\mathbf{u} \in \widehat{\mathrm{PS}}_\lambda$ with principal stretches $(\kappa, \kappa, \lambda)$ almost everywhere. Therefore, $\mathbf{v} := \mathbf{u} \circ [\mathbf{u}_\lambda^\kappa]^{-1} \in W^{1,1}$ has principal stretches that are all equal to 1 almost everywhere, i.e.,

$$(\nabla\mathbf{v})^{\mathrm{T}}(\nabla\mathbf{v}) = \mathbf{I} \ a.e. \tag{3.33}$$

We claim that (3.33) implies that $\mathbf{v} \in C^\infty$ and hence that the remainder of the argument is now the same as in the smooth case.

In order to deduce the required smoothness of $\mathbf{v}$ we first take the trace of (3.33) to get $|\nabla \mathbf{v}|^2 = 3$ *a.e.* Thus, $\nabla \mathbf{v}$ is bounded almost everywhere. In particular, $\mathbf{v} \in W^{1,2}$ and so, as first noticed by[6] Friesecke, James, and Müller [13, p. 1469], (3.33) is equivalent to $\nabla \mathbf{v} = (\operatorname{adj} \nabla \mathbf{v})^{\mathrm{T}}$ and hence the identity $\operatorname{Div}(\operatorname{adj} \nabla \mathbf{v})^{\mathrm{T}} = \mathbf{0}$ implies that $\mathbf{v} \in W^{1,2}$ is a weak solution of Laplace's equation; the standard regularity theory (see, e.g., [11, p. 339]) then yields $\mathbf{v} \in C^\infty$. □

3.4 Generalizations.

The proofs in this paper of the existence and uniqueness of axisymmetric, homogeneous global energy minimizers are can be extended to cover additional stored-energy functions.

Transversely isotropic materials. All of our proofs are for isotropic materials but can be extended to the case of materials which are transversally isotropic (relative to the symmetry axis of the cylinder) by using the results of [32, Section 5].

Other Invariants. Our proofs are for energy functions that only depend on the invariants $|\mathbf{F}|$ and $\det \mathbf{F}$ but can be extended to energies

$$W(\mathbf{F}) = \Psi\left(|\mathbf{F}|^q, \left[\frac{|\operatorname{adj} \mathbf{F}|^2}{\det \mathbf{F}}\right]^t, (\det \mathbf{F})^{p/3}\right).$$

See Appendix B and, in particular, Remark B.2.5 for such extensions.

4 Sliding-contact boundary conditions.

We now replace the boundary condition (1.1) by a *greased* or *sliding-contact* boundary condition on the curved surface of a **circular** cylinder $\mathcal{C} = \Omega \times (-L, L)$ of radius $R > 0$: Fix $\mu \geq 1$ and suppose that

$$[u_1(x,y,z)]^2 + [u_2(x,y,z)]^2 = \mu^2 R^2 \quad \text{for} \quad (x,y) \in \partial\Omega \text{ and } z \in [-L, L], \tag{4.1}$$

where[7]

$$\Omega := \left\{(x,y) \, : \, x^2 + y^2 < R^2\right\}.$$

In order to eliminate rigid deformations of the body we also assume that $(1.5)_2$ is satisfied and that

$$\int_{\mathcal{C}} u_3 \, \mathrm{d}V = 0.$$

[6] The fact that (3.33) implies $\nabla \mathbf{v}$ is constant for $\mathbf{v} \in W^{1,3}$ was first proven by Reshetnyak [26].
[7] The flat surfaces of the cylinder, $z = -L$ and $z = L$, are left free.

A corresponding homogeneous solution to this problem is given by the deformation

$$\mathbf{u}^{\mu}_{\lambda}(x,y,z) = \begin{bmatrix} \mu x \\ \mu y \\ \lambda z \end{bmatrix}, \tag{4.2}$$

where $\lambda := \delta$ is the unique positive real number that satisfies[8]

$$\Phi_{,3}(\mu,\mu,\delta) = 0 \tag{4.3}$$

and $W(\mathbf{F}) = \Phi(\nu_1,\nu_2,\nu_3)$ denotes the stored-energy density. (Note that (4.3) implies that (4.2) satisfies the natural boundary condition that the normal stress is zero on the flat surfaces of the cylinder.)

In contrast to the earlier boundary condition (1.1), the sliding-contact condition (4.1) imposes the geometric constraint that each cross section $\Omega \times \{z\}$, $z \in [-L,L]$, of the cylinder in the reference configuration must map to a surface that spans the deformed boundary and hence must deform into a surface whose area is at least $\pi\mu^2R^2$. Thus it follows that each deformation $\mathbf{u}$ that satisfies the boundary condition (4.1) also satisfies

$$⨍_{\Omega\times\{z\}} |\mathbf{u}_x \times \mathbf{u}_y|\,\mathrm{d}A \geq \mu^2 = ⨍_{\Omega\times\{z\}} |(\mathbf{u}^{\mu}_{\lambda})_x \times (\mathbf{u}^{\mu}_{\lambda})_y|\,\mathrm{d}A, \tag{4.4}$$

for every $z \in [-L,L]$, where $\mathrm{d}A = \,\mathrm{d}x\,\mathrm{d}y$.

The earlier arguments in this paper for the case of uniaxial extension can now be adapted to prove the following theorem.

Theorem 4.1 (Energy reduction by symmetrization)**.** *Assume that the reference configuration is stress free. Suppose that the stored energy* $W \in C^1(\mathrm{M}^{3\times3}_+;[0,\infty))$ *is of the form*

$$W(\mathbf{F}) = \Psi\big(|\mathbf{F}|^2, |\operatorname{adj}\mathbf{F}|, (\det\mathbf{F})^{2/3}\big), \tag{4.5}$$

where $\Psi : \mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}$ *is convex with* $\Psi_{,i} \geq 0$, $i = 1,2$*. Fix* $\mu \geq 1$*. For each deformation* $\mathbf{u}$ *that satisfies the boundary condition* (4.1)*, we define the homogeneous deformation*

$$\mathbf{u}^{\mu}_{\bar\lambda}(x,y,z) := \begin{bmatrix} \mu x \\ \mu y \\ \bar\lambda z \end{bmatrix}, \tag{4.6}$$

where $\bar\lambda$ *is defined by*

$$(\mu^2\bar\lambda)^{2/3} = ⨍_{\mathcal{C}} (\det\nabla\mathbf{u})^{2/3}\mathrm{d}V. \tag{4.7}$$

(i) If $\bar\lambda \leq \mu$ *then the energy functional* (1.3) *satisfies* $E(\mathbf{u}) \geq E(\mathbf{u}^{\mu}_{\bar\lambda})$*;*

(ii) If $\bar\lambda > \mu$ *then the energy functional* (1.3) *satisfies* $E(\mathbf{u}) \geq E(\mathbf{u}^{\mu}_{\mu})$*.*

---

[8]The existence of such a δ follows from the continuity of W together with the growth conditions (2.8). The uniqueness follows either from the strict rank-one convexity of W or, more simply, from the strict tension-extension inequality $\Phi_{,33} > 0$.

Proof. Our proof follows the proof of Theorem 3.4.

Case (i): $\bar{\lambda} < \mu$. We first consider the stored-energy function $\tilde{W}(\mathbf{F}) = |\mathbf{F}|^2$. Here we replace the earlier calculation (3.5)–(3.7) by

$$\begin{aligned} |\nabla \mathbf{u}|^2 &= |\mathbf{u}_z|^2 + |\mathbf{u}_x|^2 + |\mathbf{u}_y|^2 \\ &\geq |\mathbf{u}_z|^2 + 2|\mathbf{u}_x||\mathbf{u}_y| \\ &\geq \frac{[J_\mathbf{u}]^2}{|\mathbf{u}_x \times \mathbf{u}_y|^2} + 2|\mathbf{u}_x \times \mathbf{u}_y| \\ &= |\mathbf{u}_x \times \mathbf{u}_y| \left[\left(\frac{[J_\mathbf{u}]^{2/3}}{|\mathbf{u}_x \times \mathbf{u}_y|} \right)^3 + 2 \right] \\ &=: \tilde{H}\Big(|\mathbf{u}_x \times \mathbf{u}_y|,\, [J_\mathbf{u}]^{2/3}\Big), \end{aligned} \tag{4.8}$$

where $a \mapsto \tilde{H}(a, d)$ is monotone increasing for $a \geq d$ and, by Lemma A.3, $\tilde{H}$ is a convex function. Thus, if we integrate (4.8) over the cylinder and make use of Jensen's inequality we find that

$$⨍_\mathcal{C} |\nabla \mathbf{u}|^2 \mathrm{d}V \geq \tilde{H}\Big(⨍_\mathcal{C} |\mathbf{u}_x \times \mathbf{u}_y| \, \mathrm{d}V,\, ⨍_\mathcal{C} (\det \nabla \mathbf{u})^{2/3} \mathrm{d}V \Big)$$

and hence, in view of (4.4), (4.6), and (4.7),

$$\begin{aligned} ⨍_\mathcal{C} |\nabla \mathbf{u}|^2 \mathrm{d}V &\geq \tilde{H}\left(\mu^2, (\mu^2 \bar{\lambda})^{2/3}\right) \\ &= \tilde{H}\left(⨍_\mathcal{C} |(\mathbf{u}^\mu_{\bar{\lambda}})_x \times (\mathbf{u}^\mu_{\bar{\lambda}})_y| \, \mathrm{d}V,\, ⨍_\mathcal{C} (\det \nabla \mathbf{u}^\mu_{\bar{\lambda}})^{2/3} \mathrm{d}V \right) = ⨍_\mathcal{C} |\nabla \mathbf{u}^\mu_{\bar{\lambda}}|^2 \, \mathrm{d}V. \end{aligned} \tag{4.9}$$

We next consider the case $\hat{W}(\mathbf{F}) = |\operatorname{adj} \mathbf{F}|$ (cf. (B.12) and (4.8)).

$$\begin{aligned} |\operatorname{adj} \nabla \mathbf{u}|^2 &= |(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_x|^2 + |(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_y|^2 + |(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_z|^2 \\ &\geq |(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_z|^2 + 2 J_\mathbf{u} |(\nabla \mathbf{u}) \mathbf{e}_z| \\ &\geq |(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_z|^2 + \frac{2\,[J_\mathbf{u}]^2}{|(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_z|} \\ &= |(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_z|^2 \left[1 + 2 \frac{[J_\mathbf{u}]^2}{|(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_z|^3} \right] \\ &= |(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_z|^2 \left[1 + 2 \left(\frac{[J_\mathbf{u}]^{2/3}}{|(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_z|} \right)^3 \right]. \end{aligned}$$

Thus

$$|\operatorname{adj} \nabla \mathbf{u}| \geq \hat{H}\big(|(\operatorname{adj} \nabla \mathbf{u})^\mathrm{T} \mathbf{e}_z|, [J_\mathbf{u}]^{2/3}\big),$$

where

$$\hat{H}(a,d) = a\left[1 + 2\left(\frac{d}{a}\right)^3\right]^{1/2} =: a\hat{\theta}\left(\frac{d}{a}\right)$$

satisfies $\hat{H}(\cdot,\cdot)$ is convex (by Lemma A.3) and monotone increasing in its first argument for $a \geq d$.

If we integrate over the cylinder and make use of Jensen's inequality we now find that

$$\begin{aligned}
\fint_{\mathcal{C}} |\operatorname{adj}\nabla\mathbf{u}|\,\mathrm{d}V &\geq \hat{H}\left(\fint_{\mathcal{C}} |(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{e}_z|\,\mathrm{d}V,\ \fint_{\mathcal{C}} (\det\nabla\mathbf{u})^{2/3}\mathrm{d}V\right)\\
&= \hat{H}\left(\fint_{\mathcal{C}} |(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{e}_z|\,\mathrm{d}V,\ (\mu^2\bar{\lambda})^{2/3}\right)\\
&\geq \hat{H}\left(\mu^2, (\mu^2\bar{\lambda})^{2/3}\right) \qquad (4.10)\\
&= \hat{H}\left(\fint_{\mathcal{C}} |(\operatorname{adj}\nabla\mathbf{u}^{\mu}_{\bar{\lambda}})^{\mathrm{T}}\mathbf{e}_z|\,\mathrm{d}V,\ \fint_{\mathcal{C}} \left(\det\nabla\mathbf{u}^{\mu}_{\bar{\lambda}}\right)^{2/3}\mathrm{d}V\right)\\
&= \fint_{\mathcal{C}} |\operatorname{adj}\nabla\mathbf{u}^{\mu}_{\bar{\lambda}}|\,\mathrm{d}V,
\end{aligned}$$

where we have made use of (4.4), (4.6), (4.7) and the fact that $|\mathbf{u}_x \times \mathbf{u}_y| = |(\operatorname{adj}\nabla\mathbf{u})^{\mathrm{T}}\mathbf{e}_z|$.

If we now combine (4.6), (4.9), and (4.10) and make use of Jensen's inequality, we obtain the general claim of the theorem for stored-energy functions of the form (4.5):

$$\begin{aligned}
\fint_{\mathcal{C}} W(\nabla\mathbf{u})\mathrm{d}V &\geq \Psi\left(\fint_{\mathcal{C}} |\nabla\mathbf{u}|^2\mathrm{d}V, \fint_{\mathcal{C}} |\operatorname{adj}\nabla\mathbf{u}|\,\mathrm{d}V, \fint_{\mathcal{C}} (\det\nabla\mathbf{u})^{2/3}\mathrm{d}V\right)\\
&\geq \Psi\left(\fint_{\mathcal{C}} |\nabla\mathbf{u}^{\mu}_{\bar{\lambda}}|^2\mathrm{d}V, \fint_{\mathcal{C}} |\operatorname{adj}\nabla\mathbf{u}^{\mu}_{\bar{\lambda}}|\,\mathrm{d}V, \fint_{\mathcal{C}} (\det\nabla\mathbf{u}^{\mu}_{\bar{\lambda}})^{2/3}\mathrm{d}V\right)\\
&= \fint_{\mathcal{C}} W(\nabla\mathbf{u}^{\mu}_{\bar{\lambda}})\mathrm{d}V.
\end{aligned}$$

Case (ii): $\bar{\lambda} > \mu$. In this case, the result that $E(\mathbf{u}) \geq E(\mathbf{u}^{\mu}_{\mu})$ follows from a straightforward adaptation of the proof of the second part of Theorem 3.4. (See also the last part of the proof of Theorem C.2.) □

Remark 4.2. The existence of an axisymmetric, homogeneous, global energy minimizer follows on a slight strengthening of the above hypotheses on the stored-energy function. (The details are again analogous to those in the proof of Theorem 3.6 using Theorem 3.4.)

Remark 4.3. We note that, analogously to section 3, the results of Theorem 4.1 should extend to stored-energy functions of the form

$$W(\mathbf{F}) = \Psi\left(|\mathbf{F}|^{2q}, |\operatorname{adj}\mathbf{F}|^r, (\det\mathbf{F})^{2p/3}\right),$$

where $p \in [1, 3/2]$, $q \geq p$, and $r \geq p$.

5 Concluding Remarks.

As noted earlier, the boundary condition of uniaxial extension (1.1), imposes a constraint on admissible deformations that material fibers parallel to the axis of symmetry of the cylinder in the reference configuration must have a deformed length of at least λL. In contrast, the sliding contact boundary condition (4.1) imposes a constraint on admissible deformations that each cross section of the cylinder, $z = c$, in the reference configuration must map to a surface whose area is at least $\pi\mu^2 R^2$. It is interesting to note the difference in the corresponding structure of the allowed stored-energy functions in our proofs of the homogeneity of an energy minimizer in the case of these two boundary conditions. In particular, the stored-energy functions (3.17) do not contain an argument of the form $|\operatorname{adj}\mathbf{F}|$ (in contrast to (4.5)). The technical reason for this is that the arguments of section 3.1, when applied to $\Lambda(\mathbf{F}) = |\operatorname{adj}\mathbf{F}|^q,\ q \geq 1$, do not yield a jointly convex functional of $|\mathbf{u}_z|^p$ and $[J_{\mathbf{u}}]^{p/3}$.

However, an intuitive reason for the failure of the proof appears to be that the term $|\operatorname{adj}\mathbf{F}|$ relates to deformed areas of surfaces within the body. A related problem is that of minimizing, for a given enclosed volume, the surface area of a droplet bridging two parallel planes. Here it is known that the minimum is attained for a cylinder when the distance between the planes is small but not when the distance between the planes is sufficiently large. This instability appears related to the Rayleigh instability observed in capillary jets (see Vogel [38]).

As noted earlier, the results in this paper are independent of the geometry of the reference configuration (in particular, the aspect ratio when the reference state is a circular cylinder), hence we would not expect our results/methods to extend to the case of compression (i.e., $\lambda < 1$ in (1.1)) as a consequence of instabilities such as buckling.

In summary, in the problems considered in this paper, there appears to be a subtle interplay between the geometric constraints imposed on admissible deformations by boundary conditions and the structure of the stored-energy functions for which no instability occurs in the corresponding homogeneous solution. This connection seems worthy of further investigation.

A Appendix

We here gather a couple of the technical results used in this manuscript.

Proposition A.1. *Fix $N \in \mathbb{N}$. Let $\Psi : (0,\infty)^{N+1} \to \mathbb{R}$ be (strictly) convex with*

$$t_i \mapsto \Psi\big(t_1, t_2, \dots, t_i, \dots, t_N, t_{N+1}\big)$$

increasing for $i = 1, 2, \dots, N$. Suppose that $\mathbf{v} : (0,\infty) \to (0,\infty)^N$ with v_i convex for $i = 1, 2, \dots, N$. Then

$$\psi(t) := \Psi\big(\mathbf{v}(t), t\big) \text{ is (strictly) convex.}$$

Proof. Let $s, t \in (0, \infty)$ and $\lambda \in (0, 1)$. Then the convexity of each component of $\mathbf{v}$ implies that

$$v_i(\lambda t + (1-\lambda)s) \leq \lambda v_i(t) + (1-\lambda)v_i(s) \quad \text{for } i = 1, 2, \ldots, N. \tag{A.1}$$

Next, (A.1) and the monotonicity of Ψ in its first N-arguments yields

$$\begin{aligned} \psi\left(\lambda t + (1-\lambda)s\right) &= \Psi\big(\mathbf{v}(\lambda t + (1-\lambda)s), \lambda t + (1-\lambda)s\big) \\ &\leq \Psi\big(\lambda \mathbf{v}(t) + (1-\lambda)\mathbf{v}(s), \lambda t + (1-\lambda)s\big). \end{aligned} \tag{A.2}$$

Finally, the strict convexity of Ψ gives us

$$\begin{aligned} \Psi\big(\lambda \mathbf{v}(t) + (1-\lambda)\mathbf{v}(s), \lambda t + (1-\lambda)s\big) &< \lambda \Psi\big(\mathbf{v}(t), t\big) + (1-\lambda)\Psi\big(\mathbf{v}(s), s\big) \\ &= \lambda \psi\left(t\right) + (1-\lambda)\psi\left(s\right), \end{aligned}$$

which together with (A.2) yields the strict convexity of ψ. □

Remark A.2. If $N = 2$ and Ψ is independent of its second argument then Proposition A.1 reduces to the classical result that the composition of 2 convex functions is convex whenever the outer function is also increasing.

A proof of the following well-known result on convex functions can be found in, for example, [21, Lemma 2.1] or [30, Lemma A.1].

Lemma A.3. *Let $\vartheta : \mathbb{R}^+ \to \mathbb{R}$ be (strictly) convex. Then*

$$Q(s, t) = t\vartheta\Big(\frac{s}{t}\Big)$$

is (strictly) convex on $\mathbb{R}^+ \times \mathbb{R}^+$.

B The Principal Stretches; Other Invariants.

In this section we show that slight variants of our method will allow for a variety of constitutive relations that depend on the principal stretches. The key to these results is the following classical lemma[9], a proof of which can be found in the appendix of [32].

Lemma B.1. *Let $\mathbf{P} \in \mathrm{M}^{3\times 3}$ be symmetric and strictly positive definite with eigenvalues λ_1, λ_2, and λ_3. Suppose that $\omega : (0, \infty) \to \mathbb{R}$ is convex. Then*

$$\omega(\lambda_1) + \omega(\lambda_2) + \omega(\lambda_3) \geq \omega(\mathbf{f}_1 \cdot \mathbf{P}\mathbf{f}_1) + \omega(\mathbf{f}_2 \cdot \mathbf{P}\mathbf{f}_2) + \omega(\mathbf{f}_3 \cdot \mathbf{P}\mathbf{f}_3)$$

for any orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ of $\mathbb{R}^3$.

We also made use of a slight generalization of the arithmetic-geometric mean inequality. Again see, e.g., the appendix of [32] for a proof.

[9] See S. Kaniel, *J. Math. Mech.* **19** (1970), 681–707 for a proof of Lemma B.1 in the case $\omega(t) = t^p$.

Lemma B.2. *Let $\omega : (0,\infty) \to \mathbb{R}$ be monotone increasing and convex. Then, for every $\mathbf{a} \in \mathbb{R}^n$ with $a_i > 0$, $i = 1, 2, \ldots, n$,*

$$\sum_{i=1}^{n} \omega\big((a_i)^n\big) \geq n\omega\left(\prod_{i=1}^{n} a_i\right).$$

Moreover, this inequality is strict if either ω is strictly monotone or strictly convex unless all of the a_i are equal.

Let $q \geq 1$ and $r \geq 1$. Suppose that $\phi, \psi : (0,\infty) \to \mathbb{R}$ are monotone increasing with ψ convex and $s \mapsto \phi(\sqrt{s^q}\,)$ convex. In this section we will consider the energy functions

$$\varLambda(\mathbf{F}) := \phi\big(\alpha^q\big) + \phi\big(\beta^q\big) + \phi\big(\gamma^q\big) \ \text{ with } p_{\max} := q \leq 2, \tag{B.1}$$

$$\varLambda(\mathbf{F}) := \psi\Big(\Big[\tfrac{\alpha\beta}{\gamma}\Big]^q\Big) + \psi\Big(\Big[\tfrac{\beta\gamma}{\alpha}\Big]^q\Big) + \psi\Big(\Big[\tfrac{\gamma\alpha}{\beta}\Big]^q\Big) \ \text{ with } p_{\max} := q, \tag{B.2}$$

$$\varLambda(\mathbf{F}) := \Big(\alpha^{2r} + \beta^{2r} + \gamma^{2r}\Big)^{q/(2r)} \ \text{ with } p_{\max} := \min\{q, 3r\}, \tag{B.3}$$

$$\varLambda(\mathbf{F}) := \Big[\Big(\tfrac{\alpha\beta}{\gamma}\Big)^r + \Big(\tfrac{\beta\gamma}{\alpha}\Big)^r + \Big(\tfrac{\gamma\alpha}{\beta}\Big)^r\Big]^{q/r} \ \text{ with } p_{\max} := \min\{q, 3r\}. \tag{B.4}$$

where α, β, and γ are the eigenvalues of $\mathbf{U} = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$ and $\mathbf{F} \in \mathrm{M}^{3\times 3}_{+}$.

The main result of this section is the following. The proof is contained in the subsections that follow.

Proposition B.3. *Suppose that $\varLambda$ is given by* (B.1)*,* (B.2)*,* (B.3)*, or* (B.4)*. Let $\lambda > 0$, $\mathbf{u} \in \mathcal{A}_\lambda$, and $p \in [1, p_{\max}]$. Define $\mu = \mu(\lambda, p, \mathbf{u}) > 0$ to be the unique positive real number that satisfies* (3.2)*. Then*

$$\fint_{\mathcal{C}} \varLambda(\nabla\mathbf{u})\mathrm{d}V \geq \varLambda\big([\mu^2\lambda]^{1/3}\mathbf{I}\big). \tag{B.5}$$

Moreover, if $\mu \leq \lambda$ then

$$\fint_{\mathcal{C}} \varLambda(\nabla\mathbf{u})\mathrm{d}V \geq \fint_{\mathcal{C}} \varLambda(\nabla\mathbf{u}^{\mu}_{\lambda})\mathrm{d}V, \tag{B.6}$$

where $\mathbf{u}^{\mu}_{\lambda}$ is given by (3.1) *with $\nu := \mu$. If, in addition, ϕ is strictly increasing and $p > 1$, then* (B.6) *will be a strict inequality unless $\mathbf{u} \in \mathrm{PS}_\lambda$.*

Remark B.4.

1. Since $\omega(s) := \phi(\sqrt{s^q})$ in (B.1) is convex and increasing with $q \in [1,2]$, so is $\phi(t) = \omega(t^{2/q})$ (see Remark A.2).
2. For $\phi \in C^2((0,\infty);\mathbb{R})$, the hypothesis $s \mapsto \phi(\sqrt{s^q})$ is convex is equivalent to
$$t\phi''(t) \geq \big(1 - \tfrac{q}{2}\big)\phi'(t) \ \text{ for all } t > 0.$$
3. Suppose that the energy is given by (B.1) with $q > 2$. Then one can rewrite (B.1) as
$$\varLambda(\mathbf{F}) = \hat{\phi}\big(\alpha^2\big) + \hat{\phi}\big(\beta^2\big) + \hat{\phi}\big(\gamma^2\big), \qquad \hat{\phi}(t) := \phi\big(t^{q/2}\big),$$
where $\omega(s) := \hat{\phi}(s^{2/q}) = \phi(s)$ is increasing and convex. Thus the restriction $q \leq 2$ is not essential.

4. If $r = 1$ the energy (B.3) is the same as the energy in section 3.1.

5. If $r = 1$ the energy (B.4) can also be written:

$$\Lambda(\mathbf{F}) = \left[\frac{|\operatorname{adj}\mathbf{F}|^2}{\det\mathbf{F}}\right]^q.$$

In order to establish the existence of a homogeneous minimizer we will also need the following result.

Proposition B.5. *Suppose that Λ is given by* (B.1), (B.2), (B.3), *or* (B.4). *Let $p \in [1, p_{\max}]$. Then $\sigma(t) := \Lambda(\sqrt[p]{t}\,\mathbf{I})$ is increasing and convex.*

Proof. If Λ is given by (B.1), then $\sigma(t) := 3\phi(t^{q/p})$, which is monotone increasing and convex since $q \geq p > 0$ and ϕ is increasing and convex (see Remark A.2). If Λ is given by (B.2), then $\sigma(t) := 3\psi(t^{q/p})$, which is increasing and convex since $q \geq p > 0$ and ψ is increasing and convex. If Λ is given by (B.3), then $\sigma(t) := 3^{q/(2r)}t^{q/p}$, which is increasing and convex since $q \geq p > 0$. Finally, if Λ is given by (B.4), then $\sigma(t) := 3^{q/r}t^{q/p}$, which is increasing and convex since $q \geq p > 0$. □

B.1 The energy $\Lambda(\mathbf{F}) = \phi(\alpha^q) + \phi(\beta^q) + \phi(\gamma^q)$ for $q \in [1, 2]$.

In this subsection we prove Proposition B.3 when Λ is given by (B.1).

Proof of Proposition B.3: Case 1. Let Λ be given by (B.1) with $q \in [1, 2]$, where $\phi \in C^1(\mathbb{R}^+; \mathbb{R})$ is increasing and $s \mapsto \phi(\sqrt{s^q})$ is convex. Fix $\lambda > 0$, $\mathbf{u} \in \mathcal{A}_\lambda$, and $p \in [1, q]$. Let μ be given by (3.2). Define $\mathbf{F} = \mathbf{F}(x, y, z) := \nabla\mathbf{u}(x, y, z)$, $J_\mathbf{u} := \det\nabla\mathbf{u}(x, y, z)$, and $\mathbf{U} := \sqrt{\mathbf{F}^\mathrm{T}\mathbf{F}}$ with eigenvalues α, β, and γ. Then, since $q \geq p > 0$, Hölder's inequality and (3.2) yield

$$\fint_\mathcal{C} [J_\mathbf{u}]^{q/3}\mathrm{d}V \geq \left[\fint_\mathcal{C} [J_\mathbf{u}]^{p/3}\mathrm{d}V\right]^{q/p} = \left[\left(\mu^2\lambda\right)^{p/3}\right]^{q/p} = \left[\mu^2\lambda\right]^{q/3}. \tag{B.7}$$

Next, since ϕ is increasing and convex, Lemma B.2, Jensen's inequality, (B.1), (B.7), and the identity $J_\mathbf{u} = \alpha\beta\gamma$ imply that

$$\begin{aligned}
\fint_\mathcal{C} \Lambda(\nabla\mathbf{u})\,\mathrm{d}V &= \fint_\mathcal{C} \left[\phi(\alpha^q) + \phi(\beta^q) + \phi(\gamma^q)\right]\mathrm{d}V \\
&\geq \fint_\mathcal{C} 3\phi\big([J_\mathbf{u}]^{q/3}\big)\mathrm{d}V \\
&\geq 3\phi\Big(\fint_\mathcal{C} [J_\mathbf{u}]^{q/3}\mathrm{d}V\Big) \\
&\geq 3\phi\big(\left[\mu^2\lambda\right]^{q/3}\big) = \Lambda\big(\left[\mu^2\lambda\right]^{1/3}\mathbf{I}\big),
\end{aligned}$$

which establishes (B.5).

Now assume that $\mu \le \lambda$. Define $\omega : \mathbb{R}^+ \to \mathbb{R}$ by $\omega(s) := \phi(\sqrt{s^q})$ so that ω is convex. If we then apply Lemma B.1 with $\mathbf{P} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ (with eigenvalues α^2, β^2, and γ^2) we conclude that, for any $(x, y, z) \in \mathcal{C}$,

$$\begin{aligned}\omega(\alpha^2) + \omega(\beta^2) + \omega(\gamma^2) &\ge \omega(\mathbf{e}_z \cdot \mathbf{P}\mathbf{e}_z) + \omega(\mathbf{e}_x \cdot \mathbf{P}\mathbf{e}_x) + \omega(\mathbf{e}_y \cdot \mathbf{P}\mathbf{e}_y)\\ &= \omega\left(|\mathbf{F}\mathbf{e}_z|^2\right) + \omega\left(|\mathbf{F}\mathbf{e}_x|^2\right) + \omega\left(|\mathbf{F}\mathbf{e}_y|^2\right)\end{aligned}$$

or, equivalently,

$$\mathit{\Lambda}(\nabla\mathbf{u}) = \phi\big(\alpha^q\big) + \phi\big(\beta^q\big) + \phi\big(\gamma^q\big) \ge \phi\left(|\mathbf{u}_z|^q\right) + \phi\left(|\mathbf{u}_x|^q\right) + \phi\left(|\mathbf{u}_y|^q\right). \tag{B.8}$$

Next, (B.8) together with (3.6), Lemma B.2, and the monotonicity of ϕ yield

$$\begin{aligned}\mathit{\Lambda}(\nabla\mathbf{u}) &\ge \phi(|\mathbf{u}_z|^q) + 2\phi\left(\sqrt{|\mathbf{u}_x|^q|\mathbf{u}_y|^q}\right)\\ &\ge \phi(|\mathbf{u}_z|^q) + 2\phi\left(\sqrt{|\mathbf{u}_x \times \mathbf{u}_y|^q}\right)\\ &\ge \phi(|\mathbf{u}_z|^q) + 2\phi\left(\sqrt{\frac{[J_{\mathbf{u}}]^q}{|\mathbf{u}_z|^q}}\right).\end{aligned} \tag{B.9}$$

Define

$$H(s,t) := \phi(s^{q/p}) + 2\phi\left(\frac{t^{3q/(2p)}}{s^{q/(2p)}}\right) = \phi(s^{q/p}) + 2\phi\left(\left[s\left(\frac{t}{s}\right)^{3/2}\right]^{q/p}\right).$$

Then, in view of (B.9),

$$\mathit{\Lambda}(\nabla\mathbf{u}) \ge H\Big(|\mathbf{u}_z|^p, [J_{\mathbf{u}}]^{p/3}\Big).$$

Since ϕ is convex (see Remark B.4.1) and increasing it follows that $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex (see Lemma A.3 and Remark A.2). Moreover, since ϕ' is positive and increasing, it follows that if $t < s$, then

$$\begin{aligned}\tfrac{p}{q}s^{[1-q/p]}\frac{\partial H}{\partial s} &= \phi'\Big(s^{q/p}\Big) - \left[\frac{t}{s}\right]^{3q/(2p)} \phi'\left(\frac{t^{3q/(2p)}}{s^{q/(2p)}}\right)\\ &\ge \phi'\Big(s^{q/p}\Big) - \phi'\left(\left[s\left(\frac{t}{s}\right)^{3/2}\right]^{q/p}\right)\\ &\ge \phi'\Big(s^{q/p}\Big) - \phi'\Big(s^{q/p}\Big) \ge 0;\end{aligned}$$

thus, for any $t > 0$, the mapping $s \mapsto H(s,t)$ is increasing on $[t, \infty)$. The remainder of the proof of (B.6), when $\mu \le \lambda$, is then similar to the proof of (3.3) in Proposition 3.1. □

B.2 The energy $\Lambda(\mathbf{F}) = \psi(\alpha^q\beta^q/\gamma^q) + \psi(\beta^q\gamma^q/\alpha^q) + \psi(\gamma^q\alpha^q/\beta^q)$ with $q \geq 1$.

In this subsection we prove Proposition B.3 when Λ is given by (B.2).

Proof of Proposition B.3: Case 2. Let Λ be given by (B.2) with $q \geq 1$, where $\psi \in C^1(\mathbb{R}^+;\mathbb{R})$ is increasing and convex. Fix $\lambda > 0$, $\mathbf{u} \in \mathcal{A}_\lambda$, and $p \in [1, q]$. Let μ be given by (3.2). Define $\mathbf{F} = \mathbf{F}(x, y, z) := \nabla\mathbf{u}(x, y, z)$, $J_\mathbf{u} := \det \nabla\mathbf{u}(x, y, z)$, and $\mathbf{U} := \sqrt{\mathbf{F}^\mathrm{T}\mathbf{F}}$, with eigenvalues α, β, and γ.

We first note that, since ψ is increasing and convex, Lemma B.2, Jensen's inequality, (B.7), and (B.2) imply that

$$\begin{aligned}
\fint_{\mathcal{C}} \Lambda(\nabla\mathbf{u})\,\mathrm{d}V &= \fint_{\mathcal{C}} \left[\psi\left(\left[\frac{\alpha\beta}{\gamma}\right]^q\right) + \psi\left(\left[\frac{\beta\gamma}{\alpha}\right]^q\right) + \psi\left(\left[\frac{\gamma\alpha}{\beta}\right]^q\right)\right]\mathrm{d}V \\
&\geq \fint_{\mathcal{C}} 3\psi\Big([J_\mathbf{u}]^{q/3}\Big)\mathrm{d}V \\
&\geq 3\psi\left(\fint_{\mathcal{C}} [J_\mathbf{u}]^{q/3}\mathrm{d}V\right) \\
&\geq 3\psi\Big(\left[\mu^2\lambda\right]^{q/3}\Big) = \Lambda\Big(\left[\mu^2\lambda\right]^{1/3}\mathbf{I}\Big),
\end{aligned}$$

which establishes (B.5).

Now suppose that $\mu \leq \lambda$. Note that $\mathbf{P} := (\operatorname{adj}\mathbf{F})(\operatorname{adj}\mathbf{F})^\mathrm{T}/(\det\mathbf{F})$ is symmetric and strictly positive definite with eigenvalues $\alpha\beta/\gamma$, $\beta\gamma/\alpha$, and $\gamma\alpha/\beta$. Then, in view of (B.2), Lemma B.1 (with $\omega(t) = \psi(t^q)$) yields, for any $(x, y, z) \in \mathcal{C}$,

$$\begin{aligned}
\Lambda(\nabla\mathbf{u}) &= \psi\left(\left[\frac{\alpha\beta}{\gamma}\right]^q\right) + \psi\left(\left[\frac{\beta\gamma}{\alpha}\right]^q\right) + \psi\left(\left[\frac{\gamma\alpha}{\beta}\right]^q\right) \\
&\geq \psi\big([\mathbf{e}_z\cdot\mathbf{P}\mathbf{e}_z]^q\big) + \psi\big([\mathbf{e}_x\cdot\mathbf{P}\mathbf{e}_x]^q\big) + \psi\big([\mathbf{e}_y\cdot\mathbf{P}\mathbf{e}_y]^q\big) \qquad \text{(B.10)} \\
&= \psi\left(\frac{|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_z|^{2q}}{[J_\mathbf{u}]^q}\right) + \psi\left(\frac{|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_x|^{2q}}{[J_\mathbf{u}]^q}\right) + \psi\left(\frac{|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_y|^{2q}}{[J_\mathbf{u}]^q}\right),
\end{aligned}$$

where $J_\mathbf{u} := \det\nabla\mathbf{u} = \alpha\beta\gamma$. As in the proof in section B.1, Lemma B.2 now implies that

$$\psi\left(\frac{|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_x|^{2q}}{[J_\mathbf{u}]^q}\right) + \psi\left(\frac{|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_y|^{2q}}{[J_\mathbf{u}]^q}\right) \geq 2\psi\left(\frac{|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_x \times (\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_y|^q}{[J_\mathbf{u}]^q}\right) \qquad \text{(B.11)}$$

However,

$$\begin{aligned}
(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_x \times (\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_y &= (\det\mathbf{F})\mathbf{F}(\mathbf{e}_x \times \mathbf{e}_y) = [J_\mathbf{u}]\mathbf{F}\mathbf{e}_z, \\
J_\mathbf{u} = [(\operatorname{adj}\mathbf{F})\mathbf{F}]^\mathrm{T}\mathbf{e}_z\cdot\mathbf{e}_z &= (\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_z\cdot\mathbf{F}\mathbf{e}_z \leq |(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_z||\mathbf{F}\mathbf{e}_z|,
\end{aligned} \qquad \text{(B.12)}$$

and $\mathbf{F}\mathbf{e}_z = \mathbf{u}_z$.

If we now combine (B.10), (B.11), and (B.12) we find, with the aid of the monotonicity of ψ, that

$$\mathit{\Lambda}(\nabla\mathbf{u}) \geq \psi\left(\frac{[J_{\mathbf{u}}]^q}{|\mathbf{u}_z|^{2q}}\right) + 2\psi\big(|\mathbf{u}_z|^q\big). \tag{B.13}$$

Define

$$H(s,t) := \psi\left(\frac{t^{3q/p}}{s^{2q/p}}\right) + 2\psi\Big(s^{q/p}\Big) = \psi\left(\left[s\left(\frac{t}{s}\right)^3\right]^{q/p}\right) + 2\psi\Big(s^{q/p}\Big).$$

Then, in view of (B.13),

$$\mathit{\Lambda}(\nabla\mathbf{u}) \geq H\Big(|\mathbf{u}_z|^p, [J_{\mathbf{u}}]^{p/3}\Big).$$

Since ψ is convex and increasing and $q \geq p > 0$, it follows that $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is convex (see Lemma A.3 and Remark A.2). Moreover, since ψ' is positive and increasing, it follows that if $t < s$, then

$$\begin{aligned}
\tfrac{p}{2q} s^{[1-p/q]} \frac{\partial H}{\partial s} &= \psi'\Big(s^{q/p}\Big) - \left[\frac{t}{s}\right]^{3q/p} \psi'\left(\frac{t^{3q/p}}{s^{2q/p}}\right) \\
&\geq \psi'\Big(s^{q/p}\Big) - \psi'\left(\left[s\left(\frac{t}{s}\right)^3\right]^{q/p}\right) \\
&\geq \psi'\Big(s^{q/p}\Big) - \psi'\Big(s^{q/p}\Big) \geq 0;
\end{aligned}$$

thus, for any $t > 0$, the mapping $s \mapsto H(s,t)$ is increasing on $[t, \infty)$. The remainder of the proof of (B.6) is then similar to the proof of (3.3) in Proposition 3.1. □

B.3 The energy $\mathit{\Lambda}(\mathbf{F}) = (\alpha^{2r} + \beta^{2r} + \gamma^{2r})^{q/(2r)}$ with $q \geq 1$ and $r \geq 1$.

In this subsection we prove Proposition B.3 when $\mathit{\Lambda}$ is given by (B.3).

Proof of Proposition B.3: Case 3. Let $\mathit{\Lambda}$ be given by (B.3) with $r \geq 1$ and $q \geq 1$. Fix $\lambda > 0$, $\mathbf{u} \in \mathcal{A}_\lambda$, and $p \geq 1$ with $p \leq q$ and $p \leq 3r$. Let μ be given by (3.2). Define $\mathbf{F} = \mathbf{F}(x,y,z) := \nabla\mathbf{u}(x,y,z)$, $J_{\mathbf{u}} := \det \nabla\mathbf{u}(x,y,z)$, and $\mathbf{U} := \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$, with eigenvalues α, β, and γ. Then Lemma B.2, (B.7), and (B.3) yield

$$\begin{aligned}
\fint_{\mathcal{C}} \mathit{\Lambda}(\nabla\mathbf{u})\,\mathrm{d}V &= \fint_{\mathcal{C}} \Big(\alpha^{2r} + \beta^{2r} + \gamma^{2r}\Big)^{q/(2r)} \mathrm{d}V \\
&\geq \fint_{\mathcal{C}} \Big(3[\alpha\beta\gamma]^{2r/3}\Big)^{q/(2r)} \mathrm{d}V \\
&= 3^{q/(2r)} \fint_{\mathcal{C}} [J_{\mathbf{u}}]^{q/3} \mathrm{d}V \\
&\geq 3^{q/(2r)} \left[\mu^2\lambda\right]^{q/3} = \mathit{\Lambda}\Big(\left[\mu^2\lambda\right]^{1/3} \mathbf{I}\Big),
\end{aligned}$$

which establishes (B.5).

Now suppose that $\mu \le \lambda$. If we then apply Lemma B.1 with $\omega(t) = t^r$ and $\mathbf{P} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ (with eigenvalues α^2, β^2, and γ^2) we conclude that, for any $(x, y, z) \in \mathcal{C}$,

$$\begin{aligned}\alpha^{2r} + \beta^{2r} + \gamma^{2r} &\ge (\mathbf{e}_z \cdot \mathbf{P}\mathbf{e}_z)^r + (\mathbf{e}_x \cdot \mathbf{P}\mathbf{e}_x)^r + (\mathbf{e}_y \cdot \mathbf{P}\mathbf{e}_y)^r \\ &= |\mathbf{F}\mathbf{e}_z|^{2r} + |\mathbf{F}\mathbf{e}_x|^{2r} + |\mathbf{F}\mathbf{e}_y|^{2r}\end{aligned}$$

or, equivalently,

$$\big[\Lambda(\nabla\mathbf{u})\big]^{2r/q} = \alpha^{2r} + \beta^{2r} + \gamma^{2r} \ge |\mathbf{u}_z|^{2r} + |\mathbf{u}_x|^{2r} + |\mathbf{u}_y|^{2r}. \tag{B.14}$$

Next, (B.14) together with (3.6), the arithmetic-geometric mean inequality, and the monotonicity of $t \mapsto t^r$ yield

$$\begin{aligned}\big[\Lambda(\nabla\mathbf{u})\big]^{2r/q} &\ge |\mathbf{u}_z|^{2r} + 2\big(|\mathbf{u}_x||\mathbf{u}_y|\big)^r \\ &\ge |\mathbf{u}_z|^{2r} + 2|\mathbf{u}_x \times \mathbf{u}_y|^r \\ &\ge |\mathbf{u}_z|^{2r} + 2\left(\frac{J_{\mathbf{u}}}{|\mathbf{u}_z|}\right)^r \\ &= |\mathbf{u}_z|^{2r}\left(1 + 2\frac{|J_{\mathbf{u}}|^r}{|\mathbf{u}_z|^{3r}}\right)\end{aligned}$$

and hence

$$\big[\Lambda(\nabla\mathbf{u})\big]^{p/q} \ge |\mathbf{u}_z|^p \left(1 + 2\left[\frac{|J_{\mathbf{u}}|^{p/3}}{|\mathbf{u}_z|^p}\right]^{3r/p}\right)^{p/(2r)}. \tag{B.15}$$

Define

$$\theta(s) := \Big(1 + 2s^{3r/p}\Big)^{p/(2r)}, \qquad G(s,t) := s\theta\left(\frac{t}{s}\right), \qquad H(s,t) := G(s,t)^{q/p}$$

and note that (B.15) implies

$$\Lambda(\nabla\mathbf{u}) \ge H\Big(|\mathbf{u}_z|^p, |J_{\mathbf{u}}|^{p/3}\Big).$$

Now

$$\theta''(s) = \tfrac{3}{p} s^{[-2+3r/p]}\Big(1 + 2s^{3r/p}\Big)^{[-2+p/(2r)]}\Big[ps^{3r/p} + (3r - p)\Big] > 0,$$

for $s > 0$ and $3r \ge p$. Consequently, (see, e.g., [30, Lemma A.1]) G and hence $H = G^{q/p}$ is strictly convex. Moreover,

$$\frac{\partial G}{\partial s} = \theta(\tau) - \tau\theta'(r) = \Big(1 - \tau^{3r/p}\Big)\Big(1 + 2\tau^{3r/p}\Big)^{[-1+p/(2r)]} > 0$$

for $\tau := t/s < 1$; thus for any $t > 0$ the mappings $s \mapsto G(s,t)$ and $s \mapsto H(s,t)$ are strictly increasing on $[t, \infty)$. The remainder of the proof of (B.6) is then similar to the proof of (3.3) in Proposition 3.1. □

B.4 The energy $\varLambda(\mathbf{F}) = [(\alpha\beta/\gamma)^r + (\beta\gamma/\alpha)^r + (\gamma\alpha/\beta)^r]^{q/r}$ with $q \geq 1$ and $r \geq 1$.

In this subsection we prove Proposition B.3 when $\varLambda$ is given by (B.4).

Proof of Proposition B.3: Case 4. Let $\varLambda$ be given by (B.4) with $r \geq 1$ and $q \geq 1$. Fix $\lambda > 0$, $\mathbf{u} \in \mathcal{A}_\lambda$, and $p \geq 1$ with $p \leq q$ and $p \leq 3r$. Let μ be given by (3.2). Define $\mathbf{F} = \mathbf{F}(x, y, z) := \nabla\mathbf{u}(x, y, z)$, $J_\mathbf{u} := \det\nabla\mathbf{u}(x, y, z)$, and $\mathbf{U} := \sqrt{\mathbf{F}^\mathrm{T}\mathbf{F}}$, with eigenvalues α, β, and γ. Then Lemma B.2, (B.7), and (B.4) yield

$$\begin{aligned}
\fint_{\mathcal{C}} \varLambda(\nabla\mathbf{u})\,\mathrm{d}V &= \fint_{\mathcal{C}} \left[\left(\frac{\alpha\beta}{\gamma}\right)^r + \left(\frac{\beta\gamma}{\alpha}\right)^r + \left(\frac{\gamma\alpha}{\beta}\right)^r\right]^{q/r} \mathrm{d}V \\
&\geq \fint_{\mathcal{C}} \Big(3[\alpha\beta\gamma]^{r/3}\Big)^{q/r} \mathrm{d}V \\
&= 3^{q/r} \fint_{\mathcal{C}} [J_\mathbf{u}]^{q/3} \mathrm{d}V \\
&\geq 3^{q/r}\left[\mu^2\lambda\right]^{q/3} = \varLambda\Big(\left[\mu^2\lambda\right]^{1/3}\mathbf{I}\Big),
\end{aligned}$$

which establishes (B.5).

Now suppose that $\mu \leq \lambda$. Then, if we apply Lemma B.1 with $\omega(t) = t^r$ and $\mathbf{P} := (\operatorname{adj}\mathbf{F})(\operatorname{adj}\mathbf{F})^\mathrm{T}/(\det\mathbf{F})$ (with eigenvalues $\alpha\beta/\gamma$, $\beta\gamma/\alpha$, and $\gamma\alpha/\beta$) we conclude, with the aid of (B.4), that for any $(x, y, z) \in \mathcal{C}$

$$\begin{aligned}
\left[\varLambda(\nabla\mathbf{u})\right]^{r/q} &\geq (\mathbf{e}_z \cdot \mathbf{P}\mathbf{e}_z)^r + (\mathbf{e}_x \cdot \mathbf{P}\mathbf{e}_x)^r + (\mathbf{e}_y \cdot \mathbf{P}\mathbf{e}_y)^r \\
&= \frac{|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_z|^{2r}}{|J_\mathbf{u}|^r} + \frac{|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_x|^{2r}}{|J_\mathbf{u}|^r} + \frac{|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_y|^{2r}}{|J_\mathbf{u}|^r},
\end{aligned} \tag{B.16}$$

where $J_\mathbf{u} := \det\nabla\mathbf{u} = \alpha\beta\gamma$. The arithmetic-geometric mean inequality now implies that

$$|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_x|^{2r} + |(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_y|^{2r} \geq 2\left|(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_x \times (\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_y\right|^r. \tag{B.17}$$

However,

$$\begin{aligned}
(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_x \times (\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_y &= (\det\mathbf{F})\mathbf{F}(\mathbf{e}_x \times \mathbf{e}_y) = [J_\mathbf{u}]\mathbf{F}\mathbf{e}_z, \\
J_\mathbf{u} = [\mathbf{F}(\operatorname{adj}\mathbf{F})]^\mathrm{T}\mathbf{e}_z \cdot \mathbf{e}_z &= (\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_z \cdot \mathbf{F}\mathbf{e}_z \leq |(\operatorname{adj}\mathbf{F})^\mathrm{T}\mathbf{e}_z||\mathbf{F}\mathbf{e}_z|,
\end{aligned} \tag{B.18}$$

and $\mathbf{F}\mathbf{e}_z = \mathbf{u}_z$.

If we now combine (B.16), (B.17), and (B.18) we find, with the aid of the monotonicity of

$t \mapsto t^r$, that

$$\begin{aligned}\left[\Lambda(\nabla\mathbf{u})\right]^{p/q} &\geq \left(\frac{|J_{\mathbf{u}}|^r}{|\mathbf{u}_z|^{2r}} + 2|\mathbf{u}_z|^r\right)^{p/r} \\ &= |\mathbf{u}_z|^p \left(\frac{|J_{\mathbf{u}}|^r}{|\mathbf{u}_z|^{3r}} + 2\right)^{p/r} \\ &= |\mathbf{u}_z|^p \left(2 + \left[\frac{|J_{\mathbf{u}}|^{p/3}}{|\mathbf{u}_z|^p}\right]^{3r/p}\right)^{p/r}\end{aligned} \tag{B.19}$$

Define

$$\theta(s) := \left(2 + s^{3r/p}\right)^{p/r}, \qquad G(s,t) := s\theta\left(\frac{t}{s}\right), \qquad H(s,t) := G(s,t)^{q/p}$$

so that (B.19) implies

$$\Lambda(\nabla\mathbf{u}) \geq H\left(|\mathbf{u}_z|^p, |J_{\mathbf{u}}|^{p/3}\right).$$

Then

$$\theta''(s) = \tfrac{6}{p} s^{[-2+3r/p]}\left(2 + s^{3r/p}\right)^{[-2+p/r]}\left[ps^{3r/p} + (3r-p)\right] > 0$$

for $s > 0$ and $3r \geq p$ and hence (see, e.g., [30, Lemma A.1]) H is strictly convex. Moreover,

$$\frac{\partial H}{\partial s} = \theta(\tau) - \tau\theta'(\tau) = 2\left(1 - \tau^{3r/p}\right)\left(2 + \tau^{3r/p}\right)^{[-1+p/r]} > 0$$

for $\tau := t/s < 1$; thus for any $t > 0$ the mappings $s \mapsto G(s,t)$ and hence $s \mapsto H(s,t)$ are strictly increasing on $[t, \infty)$. The remainder of the proof of (B.6) is then similar to the proof of (3.3) in Proposition 3.1. □

C The Homogeneity of Energy-Minimizing Deformations II.

Fix $N \in \mathbb{N}$ and define $\mathbf{\Lambda} : \mathrm{M}^{3\times3}_+ \to \mathbb{R}^N$ by

$$\mathbf{\Lambda}(\mathbf{F}) := \left(\Lambda_1(\mathbf{F}), \Lambda_2(\mathbf{F}), \Lambda_3(\mathbf{F}), \ldots, \Lambda_N(\mathbf{F})\right), \tag{C.1}$$

where each component $\Lambda_i : \mathrm{M}^{3\times3}_+ \to \mathbb{R}$ is one of the functions, $\Lambda(\mathbf{F})$, given in this section and $p^i_{\max}$ is the corresponding maximum value of the parameter p.

Lemma C.1. *Let* $\mathbf{\Lambda}$ *be given by* (C.1)*. Suppose that the stored energy* $W \in C^1(\mathrm{M}^{3\times3}_+; [0,\infty))$ *satisfies*

$$W(\mathbf{F}) = \Psi(\mathbf{\Lambda}(\mathbf{F}), (\det\mathbf{F})^{p/3}),$$

where $p \in [1, p_{\max}]$ *with* $p_{\max} := \min\{p^i_{\max}, 1 \leq i \leq n\}$, $\Psi : (0,\infty)^{N+1} \to \mathbb{R}$ *is (strictly) convex, and* $\mathbf{\Lambda}$ *is given by* (C.1)*. Then*

$$t \mapsto W(\sqrt[p]{t}\,\mathbf{I}) \quad \textit{is (strictly) convex on } \mathbb{R}^+.$$

Moreover, if in addition the reference configuration is stress free, then

$$\lambda \mapsto W(\lambda \mathbf{I}) \quad \text{is (strictly) increasing on } [1,\infty).$$

Proof. To obtain convexity define

$$\xi(t) := W(\sqrt[p]{t}\,\mathbf{I}) = \Psi\big(\boldsymbol{\Lambda}(\sqrt[p]{t}\,\mathbf{I}), t\big).$$

Then Proposition B.5 together with Proposition A.1 yield the (strict) convexity of ξ.

Next, if we take the derivative of ξ we find that

$$\dot{\xi}(t) = \tfrac{\sqrt[p]{t}}{pt}\,\mathbf{S}(\sqrt[p]{t}\,\mathbf{I}) : \mathbf{I},$$

where $\mathbf{S}$ is the Piola-Kirchhoff stress tensor (2.6). Thus, if the reference configuration is stress free, $\dot{\xi}(1) = 0$. The (strict) convexity of ξ then implies that ξ and hence the mapping $\lambda \mapsto W(\lambda\mathbf{I})$ is (strictly) increasing on $[1,\infty)$. □

C.1 Energy Reduction by Symmetrization

Theorem C.2. *Assume that the reference configuration is stress free. Suppose that the stored energy* $W \in C^1(\mathrm{M}^{3\times 3}_+; [0,\infty))$ *satisfies*

$$W(\mathbf{F}) = \Psi\big(\boldsymbol{\Lambda}(\mathbf{F}), (\det \mathbf{F})^{p/3}\big), \tag{C.2}$$

where $p \in [1, p_{\max}]$ *with* $p_{\max} := \min\{p^1_{\max}, p^2_{\max}, p^3_{\max}, \ldots, p^N_{\max}\}$, $\boldsymbol{\Lambda}$ *is given by* (C.1), *and* $\Psi : (0,\infty)^{N+1} \to \mathbb{R}$ *is increasing in each of its first N-arguments and convex. Let* $\lambda \geq 1$ *and* $\mathbf{u} \in \mathcal{A}_\lambda$. *Define* $\mu = \mu(\lambda, p, \mathbf{u}) > 0$ *to be the unique positive real number that satisfies* (3.2). *Then*

$$⨍_{\mathcal{C}} W\big(\nabla \mathbf{u}\big)\mathrm{d}V \geq ⨍_{\mathcal{C}} W\big(\nabla \mathbf{u}^\nu_\lambda\big)\mathrm{d}V, \tag{C.3}$$

where $\mathbf{u}^\nu_\lambda$ *is given by* (3.1) *with* $\nu := \mu$ *if* $\mu \leq \lambda$ *and* $\nu := \lambda$ *if* $\mu \geq \lambda$. *Moreover, if* $\mu \leq \lambda$, $p > 1$, *and* Ψ *is strictly increasing in one of its first N-arguments, then* (C.3) *is a strict inequality unless* $\mathbf{u} \in \mathrm{PS}_\lambda$.

Proof. Fix $\lambda \geq 1$, $\mathbf{u} \in \mathcal{A}_\lambda$, and define μ by (3.2). If we now let $\mathbf{F} = \nabla\mathbf{u}$ in (C.2), integrate over the cylinder $\mathcal{C}$, and apply Jensen's inequality to the convex function Ψ we conclude, with the aid of (3.2), that

$$\begin{aligned} ⨍_{\mathcal{C}} W(\nabla \mathbf{u})\,\mathrm{d}V &= ⨍_{\mathcal{C}} \Psi\Big(\boldsymbol{\Lambda}(\nabla\mathbf{u}), [J_{\mathbf{u}}]^{p/3}\Big)\mathrm{d}V \\ &\geq \Psi\bigg(⨍_{\mathcal{C}} \boldsymbol{\Lambda}(\nabla\mathbf{u})\,\mathrm{d}V, ⨍_{\mathcal{C}} [J_{\mathbf{u}}]^{p/3}\,\mathrm{d}V\bigg) \\ &= \Psi\bigg(⨍_{\mathcal{C}} \boldsymbol{\Lambda}(\nabla\mathbf{u})\,\mathrm{d}V, \big(\mu^2\lambda\big)^{p/3}\bigg). \end{aligned} \tag{C.4}$$

Now suppose that $\mu \le \lambda$. Then we can make use of Proposition B.3, the monotonicity of Ψ in each of its first N-arguments, (3.1), and (C.2), to get

$$\begin{aligned}\Psi\left(\fint_{\mathcal{C}} \mathbf{\Lambda}(\nabla \mathbf{u})\,\mathrm{d}V, \left(\mu^2\lambda\right)^{p/3}\right) &\ge \Psi\left(\fint_{\mathcal{C}} \mathbf{\Lambda}(\nabla \mathbf{u}_\lambda^\mu)\,\mathrm{d}V, \left(\mu^2\lambda\right)^{p/3}\right) \\ &= \fint_{\mathcal{C}} W\left(\nabla \mathbf{u}_\lambda^\mu\right)\mathrm{d}V,\end{aligned} \tag{C.5}$$

which when combined with (C.4) yields the desired inequality, (C.3) with $\nu = \mu$. Moreover, if Ψ is strictly increasing in, say, its i-argument, then the inequality in (C.5) is a strict inequality unless

$$\fint_{\mathcal{C}} \Lambda_i(\nabla \mathbf{u})\,\mathrm{d}V = \fint_{\mathcal{C}} \Lambda_i(\nabla \mathbf{u}_\lambda^\mu)\,\mathrm{d}V.$$

Proposition B.3 now yields $\mathbf{u} \in \mathrm{PS}_\lambda$ when $p > 1$.

Finally, suppose that $\mu \ge \lambda$ so that $\nu = \lambda$. Then we can make use of Proposition B.3, the monotonicity of Ψ in each of its first N-arguments, (3.1), and (C.2), to obtain

$$\begin{aligned}\Psi\left(\fint_{\mathcal{C}} \mathbf{\Lambda}(\nabla \mathbf{u})\,\mathrm{d}V, \left(\mu^2\lambda\right)^{p/3}\right) &\ge \Psi\left(\mathbf{\Lambda}\left(\left[\mu^2\lambda\right]^{1/3}\mathbf{I}\right), \left(\mu^2\lambda\right)^{p/3}\right) \\ &= W(\eta\mathbf{I}), \quad \text{where } \eta := (\mu^2\lambda)^{1/3} \ge \lambda \ge 1\end{aligned} \tag{C.6}$$

and we have made use of (C.2). However, Lemma C.1 then implies that $W(\eta\mathbf{I}) \ge W(\lambda\mathbf{I})$, which together with (C.4) and (C.6) yields the desired result, (C.3) with $\nu = \lambda$. □

C.2 Existence of a Homogeneous Minimizer

Theorem C.3. *Let W, Ψ, $\mathbf{\Lambda}$, and p satisfy the hypotheses of Theorem C.2. Suppose, in addition, that*

$$\lim_{|\mathbf{F}|\to\infty} W(\mathbf{F}) = +\infty, \qquad \lim_{\det \mathbf{F}\to 0^+} W(\mathbf{F}) = +\infty. \tag{C.7}$$

Then for each $\lambda \ge 1$ there exists a $\kappa = \kappa(\lambda) > 0$ such that the deformation

$$\mathbf{u}_\lambda^\kappa(x,y,z) := \begin{bmatrix}\kappa x\\ \kappa y\\ \lambda z\end{bmatrix}. \tag{C.8}$$

is an absolute minimizer of the energy among deformations in $\mathcal{A}_\lambda$.

Proof. Fix $\lambda \ge 1$. Then (C.7) and Proposition 2.5 yield $\alpha > 0$ and $\beta > 0$ such that

$$\Phi(s,t,\lambda) \ge \Phi(\alpha,\beta,\lambda) \quad \text{for every } s > 0 \text{ and } t > 0. \tag{C.9}$$

Now apply Theorem C.2 with $\mathbf{u} = \mathbf{u}^*$, where

$$\mathbf{u}^*(x, y, z) = \begin{bmatrix} \alpha x \\ \beta y \\ \lambda z \end{bmatrix},$$

to get a $\kappa := \nu > 0$ such that

$$\Phi(\alpha, \beta, \lambda) = \bar{E}(\mathbf{u}^*) \geq \bar{E}(\mathbf{u}_\lambda^\kappa) = \Phi(\kappa, \kappa, \lambda), \tag{C.10}$$

where we have made use of (3.12) and (C.8).

We claim that the homogeneous deformation given by (C.8) with this value of κ is an absolute minimizer of E among deformations in $\mathcal{A}_\lambda$. To see this, let $\mathbf{u} \in \mathcal{A}_\lambda$. Then, by Theorem C.2, there is $\nu > 0$ such that the homogeneous deformation $\mathbf{u}_\lambda^\nu$ given by (3.1) satisfies (see (3.12))

$$\bar{E}(\mathbf{u}) \geq \bar{E}(\mathbf{u}_\lambda^\nu) = \Phi(\nu, \nu, \lambda). \tag{C.11}$$

The desired minimality of $\mathbf{u}_\lambda^\kappa$ now follows from (C.10), (C.11), and (C.9) with $s = t = \nu$. $\square$

Acknowledgement. The work of SJS was partially supported by the National Science Foundation under Grant No. DMS–1107899.

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