A VARIATIONAL APPROACH TO MODELLING INITIATION OF FRACTURE IN NONLINEAR ELASTICITY

J. Sivaloganathan[1] and S. J. Spector[2]

[1] *Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, U.K.*
js@maths.bath.ac.uk

[2] *Department of Mathematics, Southern Illinois University, Carbondale IL 62901, U.S.A.*
sspector@math.siu.edu

Keywords: Nonlinear elasticity, cavitation, energy momentum tensor.

Abstract In this paper we present an overview of a variational approach to modelling the initiation of fracture in a nonlinearly elastic material. The work is motivated by experiments on polymers in which cracks initiate at cavities that form in the polymer sample under sufficiently severe loading.

1. INTRODUCTION

This work is motivated in part by interesting experiments of Gent and Park [6]. Gent and Park took samples of transparent polymers and bonded a rigid bead within each sample. They then subjected the samples to uniaxial tension. The beads acted as stress concentrators and Gent and Park observed that, as the loading increased, small cavities (not previously evident in the sample) would form near the poles of the bead along the axis of tension. These would then enlarge and elongate into a crack-like shape in the direction of loading leading eventually to failure/fracture of the sample (see figure 1).

In this paper we give an overview of new results on a variational approach to modelling such phenomenon that has recently been developed (see e.g. [13], [14], [15]).

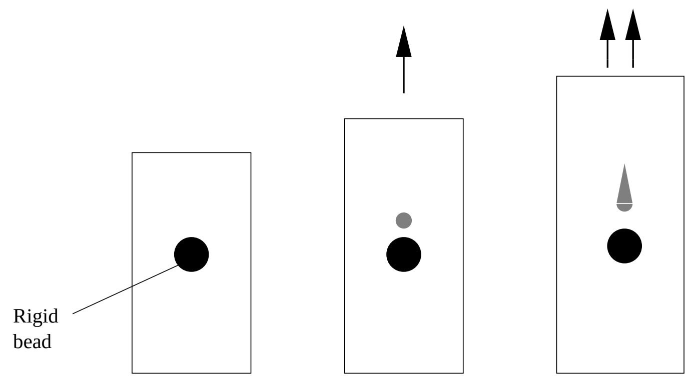

Figure 1 Formation of a cavity leading to a crack.

It is known that, under suitable hypotheses, minimisers of variational problems in elasticity may develop discontinuities which can be interpreted as cavities forming in an initially perfect material. (See [1], [12], [17], [8] for the radial cavitation problem and subsequent development of [10], [13], [14] for general boundary value problems with no assumption of symmetry.) Alternatively, such singular minimisers can be interpreted as the limiting deformation of a body containing a small pre-existing cavity in the reference configuration of diameter $\varepsilon > 0$, centred at the cavitation point, in the limit as $\varepsilon \to 0$ (see [12], [7], [18]).

In elastomers, such as used in the Gent and Park experiments, one expects that the cavities observed result from the expansion of pre-existing (initially non-visible) holes in the material. Thus, in the theoretical variational approach of [10] cavities can form anywhere whereas, in an actual material sample, the cavities are formed by the expansion of tiny pre-existing voids which were created during the manufacturing process.

In this paper both the above viewpoints are adopted simultaneously: the theoretical material is modelled as initially perfect using a homogeneous stored energy function but the flaws in the actual material are incorporated by using deformations whose point discontinuities are constrained to be at pre-specified points (the flaws in the material). If pre-existing voids (flaws) are widespread then one anticipates both approaches should yield similar results. However, we show that if a flaw is not located at an energetically optimal point then this gives rise to a configurational force. Hence, if flaw points are sparse and all such points are located sufficiently far from energetically optimal points, then we conjecture that cavitation will first occur at one of these flaws and that the resulting configurational force may be sufficient to produce a fracture/crack in the material initiating from the cavity.

We would like to acknowledge the fundamental work of Ball [1] on radial cavitation which has motivated our current approach. Our aim in this paper is to give the underlying ideas whilst, for the most part, suppressing technical details.

2. MATHEMATICAL PRELIMINARIES

Let $\Omega \subset \mathrm{I\!R}^3$ denote the region occupied by a nonlinear elastic body in its reference (undeformed) state. By a deformation of the body we mean a map $\mathbf{u} : \Omega \to \mathrm{I\!R}^3$ belonging to the Sobolev space $W^{1,1}(\Omega)$, which is one-to-one (almost everywhere), and which satisfies the local invertibility condition

$$\det \nabla \mathbf{u} > 0. \tag{2.1}$$

As we shall see, this class of deformations includes discontinuous maps. In nonlinear hyper-elasticity we associate with each such deformation of the body an energy given by

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{2.2}$$

where $W : \mathrm{M}^{3\times3}_+ \to \mathrm{I\!R}^+$ is the stored energy function of the material and $\mathrm{M}^{3\times3}_+$ denotes the 3×3 matrices with positive determinant. (See e.g. [4], [2] for further background.) In this paper we consider the displacement boundary value problem in which we require that

$$\mathbf{u}|_{\partial\Omega} = \mathbf{A}\mathbf{x}, \tag{2.3}$$

where $\mathbf{A} \in \mathrm{M}^{3\times3}_+$ is fixed.

The equilibrium equations of nonlinear elasticity are formally given by the Euler-Lagrange equations for (2.2):

$$\mathrm{Div}\left(\frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u})\right)_i = \frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla \mathbf{u}(\mathbf{x}))\right] = 0,\ \mathbf{x} \in \Omega,\ i = 1, 2, 3. \tag{2.4}$$

An alternative form of the equilibrium equations is the so called energy-momentum form given by

$$\mathrm{Div}\ \mathbf{M}(\nabla \mathbf{u}) = \mathbf{0}, \tag{2.5}$$

where

$$\mathbf{M}(\nabla \mathbf{u}) := W(\nabla \mathbf{u})\mathbf{I} - (\nabla \mathbf{u})^{\mathrm{T}} \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{u})$$

is the energy-momentum tensor. (See Eshelby [5], and Ball [3] for a rigorous derivation of (2.5) as a necessary condition for a minimiser.) For smooth, invertible equilibria the two forms are equivalent but, for singular/discontinuous equilibria, weak solutions of (2.4) and (2.5) may differ. In the variational approach we seek weak solutions of (2.4), (2.5) by minimising the energy among deformations satisfying (2.1) and (2.3).

Notice that an immediate solution of (2.4), (2.3) is the homogeneous map

$$\mathbf{u}^h(\mathbf{x}) \equiv \mathbf{A}\mathbf{x}.$$

A central question we will address is whether there exist discontinuous deformations satisfying (2.3) with less energy than $\mathbf{u}^h$. The next remark shows that, far from being special, the homogeneous deformations $\mathbf{u}^h$ and the class of boundary conditions (2.3) play an important role in studying local minimisers of the energy (2.2).

Remark. If $\mathbf{u}_0 \in W^{1,1}(\Omega)$ is a strong local minimiser of (2.2) (i.e. for some $\epsilon > 0$, $E(\mathbf{u}) \geq E(\mathbf{u}_0)$ for any $\mathbf{u}$ with $||\mathbf{u} - \mathbf{u}_0||_{L^\infty} < \epsilon$) subject to the conditions (2.1),(2.3) and if $\mathbf{u}_0$ is C^1 at $\mathbf{x}_0 \in \Omega$ then it follows that, for any domain D, the stored energy function satisfies

$$E(\tilde{\mathbf{u}}) = \int_D W(\mathbf{A}_0 + \nabla\mathbf{v}(\mathbf{y}))\,d\mathbf{y} \geq \int_D W(\mathbf{A}_0)\,d\mathbf{y} = E(\mathbf{u}_0^h)$$

for all $\mathbf{v} \in W_0^{1,1}(\Omega)$, where $\tilde{\mathbf{u}}_0(\mathbf{y}) = \mathbf{A}_0\mathbf{y} + \mathbf{v}(\mathbf{y})$, $\mathbf{u}_0^h(\mathbf{y}) = \mathbf{A}_0\mathbf{y}$ and $\mathbf{A}_0 := \nabla\mathbf{u}_0(\mathbf{x}_0)$. In other words the homogeneous deformation $\mathbf{u}_0^h$ has least energy amongst all deformations of $\Omega = D$ satisfying the boundary condition (2.3) with $\mathbf{A} = \mathbf{A}_0$ or, equivalently, the stored energy function W is $W^{1,1}$-quasiconvex at $\mathbf{A}_0$. (See James and Spector [9] for further details.)

Example. A simple class of stored energy functions we will consider is given by

$$W(\mathbf{F}) = \kappa|\mathbf{F}|^p + h(\det\mathbf{F}), \quad \mathbf{F} \in \mathrm{M}_+^{3\times 3}, \quad p \geq 1, \tag{2.6}$$

where $\kappa > 0$, $|\cdot|$ denotes the Euclidean norm ($|\mathbf{A}|^2 = \mathrm{trace}(\mathbf{A}^{\mathrm{T}}\mathbf{A})$), and $h : (0,\infty) \to [0,\infty)$ is C^1, convex and satisfies $h(d) \to +\infty$ as $d \to 0^+$ and $\frac{h(d)}{d} \to +\infty$ as $d \to +\infty$. Such energy functions are examples of polyconvex stored energies introduced by Ball (see e.g. [2]) and include examples of energies proposed by Ogden [11] to fit observed experimental data. For ease of exposition we state the results in this paper for stored energy functions of the form (2.6) (though our methods apply to much more general polyconvex stored energy functions, including those with explicit dependence on the adjugate of $\mathbf{F}$, $\mathrm{Adj}\,\mathbf{F}$).

It is a mathematical fact that, for stored energy functions of the form (2.6), if $p > 3$ and $E(\mathbf{u}) < +\infty$ then $\mathbf{u} \in W^{1,p}(\Omega)$ and is continuous by the Sobolev embedding theorem (a more sophisticated argument yields the same result under condition (2.1) in the case $p = 3$). Hence to model the formation of discontinuities we work in the regime $p < 3$.

The Distributional Jacobian. The key to modelling the formation of discontinuities lies in the use of the distributional Jacobian; first note that, for C^2 maps, $\det \nabla \mathbf{u}$ is expressible as a divergence:

$$\det \nabla \mathbf{u} = \frac{\partial}{\partial x^\alpha}\left(\frac{1}{3}(\mathrm{Adj}\nabla \mathbf{u})_i^\alpha u^i\right) = \mathrm{div}\big(\tfrac{1}{3}(\mathrm{Adj}\nabla \mathbf{u})\mathbf{u}\big). \tag{2.7}$$

where $\mathrm{Adj}\nabla\mathbf{u}$ denotes the adjugate matrix of $\nabla \mathbf{u}$. Next let $\varphi \in C_0^\infty(\Omega)$ (the infinity differentiable functions on Ω with compact support), multiply (2.7) by φ and integrate by parts to obtain

$$\int_\Omega \varphi \det \nabla \mathbf{u}\, d\mathbf{x} = -\int_\Omega \nabla\varphi \cdot \left[\frac{1}{3}(\mathrm{Adj}\nabla \mathbf{u})\mathbf{u}\right] d\mathbf{x}. \tag{2.8}$$

This motivates the definition of the distributional Jacobian, $\mathrm{Det}\, \nabla \mathbf{u}$, which is the functional defined by

$$(\mathrm{Det}\, \nabla \mathbf{u})(\varphi) := -\frac{1}{3}\int_\Omega \nabla\varphi \cdot [(\mathrm{Adj}\nabla \mathbf{u})\mathbf{u}]\, d\mathbf{x}, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.9}$$

It follows from (2.8) that for C^2 maps $\mathbf{u}$

$$(\mathrm{Det}\, \nabla \mathbf{u})(\varphi) = \int_\Omega \varphi \det \nabla \mathbf{u}\, d\mathbf{x}, \quad \forall \varphi \in C_0^\infty(\Omega),$$

and we denote this by

$$\mathrm{Det}\, \nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^3, \tag{2.10}$$

where $\mathcal{L}^3$ denotes 3-dimensional Lebesgue measure. For discontinuous deformations $\mathbf{u} \in W^{1,1}(\Omega)$ the distributional Jacobian is not in general given by (2.10) as illustrated by the next example.

Example. Let $\Omega = B$ the unit ball in $\mathbb{R}^3$ and let $\mathbf{u} : \Omega \longrightarrow \mathbb{R}^3$ be given by $\mathbf{u}(\mathbf{x}) = \mathbf{x} + c\frac{\mathbf{x}}{|\mathbf{x}|}$ where $c > 0$. Then $\det \nabla \mathbf{u}(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{u}$ produces a hole of radius c at the centre of B. The distributional Jacobian of this map is given by

$$\mathrm{Det}\, \nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^3 + \frac{4}{3}\pi c^3 \delta_{\mathbf{0}}\ , \tag{2.11}$$

where $\delta_{\mathbf{0}}$ denotes the Dirac measure supported at the origin. In analogy with (2.10), (2.11) is to be interpreted in the sense (of distributions) that

$$(\mathrm{Det}\,\nabla\mathbf{u})(\varphi) = \int_\Omega \varphi \det \nabla\mathbf{u}\, d\mathbf{x} + \frac{4}{3}\pi c^3 \varphi(0), \quad \forall \varphi \in C_0^\infty(\Omega).$$

Modelling flaws in a material. The above discussion motivates our modelling of (initially non-visible) flaws in a material by restricting attention to maps satisfying

$$\mathrm{Det}\,\nabla\mathbf{u} = (\det \nabla\mathbf{u})\mathcal{L}^3 + \sum_{i=1}^{m} \alpha_i \delta_{\mathbf{x}_i}\ , \tag{2.12}$$

where $\alpha_i \geq 0$. Hence if $\alpha_k > 0$ for some k, then the map $\mathbf{u}$ produces a hole of volume α_k at $\mathbf{x}_k \in \Omega$. (Note that the holes produced need not be spherical.)

Condition INV. It turns out that maps $\mathbf{u} \in W^{1,1}(\Omega)$ satisfying (2.1) and (2.3) need not be one-to-one almost everywhere. Take, for example, the discontinuous deformation of B given above and this time let $-1 < c < 0$ (a hole of "negative radius"). It is easily verified that the corresponding deformation $\mathbf{u}$ still satisfies $\det \nabla\mathbf{u} > 0$ almost everywhere, however, $\mathbf{u}$ is no longer one-to-one in a neighbourhood of the origin. Moreover, in this case

$$\int_B \det \nabla\mathbf{u}\, d\mathbf{x} = \frac{4}{3}\pi((1+c)^3 - c^3) > \mathrm{vol}(\mathbf{u}(B)),$$

so the classical change of variables formula clearly fails.

The (invertibility) condition INV introduced by Müller and Spector [10] in particular excludes the last example and is well suited for proving the existence of minimisers of (2.2) in classes of deformations that allow cavitation to occur. Essentially, INV is the requirement that holes produced within one part of the body are not filled by material from other parts (see [10] for further details).

Remark. In particular, it is shown in [10] that if $\mathbf{u}$ satisfies INV on Ω and $\mathbf{u}|_{\partial\Omega} = \mathbf{h}$, where $\mathbf{h} : \bar{\Omega} \to \mathbb{R}^3$ is a homeomorphism then vol($\mathbf{h}(\Omega)$)= $(\mathrm{Det}\,\nabla\mathbf{u})(\Omega)$.

Theorem 1 (Existence of minimisers [13]). *Let* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \Omega$, $p > 2$, $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$, *and define the set of admissible deformations*

$\mathcal{A} = \mathcal{A}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ *by*

$$\mathcal{A} := \{\mathbf{u} \in W^{1,p}(\Omega) : \mathbf{u}|_{\partial\Omega} = \mathbf{A}\mathbf{x},\ \mathbf{u}^e \text{ satisfies INV on } \Omega_0,$$
$$\operatorname{Det} \nabla\mathbf{u} = (\det \nabla\mathbf{u})\mathcal{L}^3 + \sum_{i=1}^{m} \alpha_i \delta_{\mathbf{x}_i},\ \det \nabla\mathbf{u} > 0 \ a.e.\},$$

where Ω_0 *is a bounded domain with* $\Omega_0 \supset \bar{\Omega}$ *and* $\mathbf{u}^e$ *is the homogeneous extension to* Ω_0 *of* $\mathbf{u}$ *defined by*[1]

$$\mathbf{u}^e(\mathbf{x}) := \begin{cases} \mathbf{u}(\mathbf{x}), & \text{if } \mathbf{x} \in \bar{\Omega}, \\ \mathbf{A}\mathbf{x}, & \text{if } \mathbf{x} \notin \bar{\Omega}. \end{cases}$$

Let E *be defined by (2.2) and (2.6). Then* E *has a minimum on* $\mathcal{A}$.

The next result shows that any minimiser given by the above theorem must produce a discontinuity if the boundary condition (2.3) is sufficiently severe.

Theorem 2. *Let* $\tilde{\mathbf{A}} \in \mathrm{M}_{+}^{3\times 3}$ *be fixed and define* $\mathbf{A} = t\tilde{\mathbf{A}}$ *where* $t > 0$. *Then for sufficiently large* t *any minimiser of* E *on* $\mathcal{A}$ *(whose existence is given by the last theorem) must satisfy* $\alpha_i > 0$ *for some* i.

Proof. The proof proceeds in two stages.

Step 1. We first prove that if the deformation $\mathbf{u}$ is such that $\alpha_i = 0,\ i = 1, 2, \ldots, m$, then

$$E(\mathbf{u}) \geq E(\mathbf{u}^h),$$

where $\mathbf{u}^h$ is the homogeneous deformation. This follows easily from the definition of E and the convexity of h in (2.6) since

$$\begin{aligned} E(\mathbf{u}) &= \int_\Omega [\kappa|\nabla\mathbf{u}|^p + h(\det \nabla\mathbf{u})]\, d\mathbf{x} \\ &\geq \int_\Omega \big[\kappa|\nabla\mathbf{u}|^p + h(\det \mathbf{A}) + h'(\det \mathbf{A})(\det \nabla\mathbf{u} - \det \mathbf{A})\big]\, d\mathbf{x}. \end{aligned}$$

It follows from our assumptions that $\operatorname{Det} \nabla\mathbf{u} = (\det \nabla\mathbf{u})\mathcal{L}^3$ and hence, by an earlier remark (see discussion on condition INV), it follows that

$$\int_\Omega \det \nabla\mathbf{u}\, d\mathbf{x} = \operatorname{vol}(\mathbf{u}^h(\Omega)) = \int_\Omega \det \mathbf{A}\, d\mathbf{x}.$$

[1]The use of $\mathbf{u}^e$ is a technical requirement which prevents the formation of cavities at the boundary of Ω.

Thus

$$E(\mathbf{u}) \geq \int_\Omega \kappa|\nabla\mathbf{u}|^p + h(\det\mathbf{A})\,d\mathbf{x} \geq \int_\Omega \kappa|\mathbf{A}|^p + h(\det\mathbf{A})\,d\mathbf{x} = E(\mathbf{u}^h),$$

where the last inequality is a result of Jensen's inequality.

Step 2. It now suffices to demonstrate that for sufficiently large t, there exists a deformation $\tilde{\mathbf{u}} \in \mathcal{A}$ satisfying

$$E(\tilde{\mathbf{u}}) < E(\mathbf{u}^h).$$

It will then follow that any minimiser must satisfy $\alpha_i > 0$ for some i (i.e. the minimiser must be discontinuous). We first assume that $\Omega = B$ and define

$$\tilde{\mathbf{u}}(\mathbf{x}) = \tilde{\mathbf{A}}(d|\mathbf{x}|^3 + (1-d))^{\frac{1}{3}}\frac{\mathbf{x}}{|\mathbf{x}|}\ , \tag{2.13}$$

where $d \in (0,1)$ is a constant. An easy calculation yields

$$\det\nabla\tilde{\mathbf{u}} = (\det\tilde{\mathbf{A}})\,d$$

and clearly

$$\tilde{\mathbf{u}}|_{\partial B} = \tilde{\mathbf{A}}\mathbf{x}.$$

For $t > 0$ we next compare the energies of the maps

$$\mathbf{u} = t\tilde{\mathbf{u}}, \qquad \mathbf{u}^h = t\tilde{\mathbf{u}}^h,$$

where $\tilde{\mathbf{u}}^h(\mathbf{x}) \equiv \tilde{\mathbf{A}}\mathbf{x}$. Define $\triangle E := E(\mathbf{u}) - E(\mathbf{u}^h)$. Then

$$\begin{aligned}
\triangle E &= \int_\Omega \Big[t^p[|\nabla\tilde{\mathbf{u}}|^p - |\tilde{\mathbf{A}}|^p] + h(t^3\det\nabla\tilde{\mathbf{u}}) - h(t^3\det\tilde{\mathbf{A}})\Big]d\mathbf{x} \\
&= t^p\int_\Omega \Big[|\nabla\tilde{\mathbf{u}}|^p - |\tilde{\mathbf{A}}|^p\Big]d\mathbf{x} + |\Omega|\Big[h(t^3 d\det\tilde{\mathbf{A}}) - h(t^3\det\tilde{\mathbf{A}})\Big] \\
&= t^p\int_\Omega \Big[|\nabla\tilde{\mathbf{u}}|^p - |\tilde{\mathbf{A}}|^p\Big]d\mathbf{x} + |\Omega|\left[\int_{t^3 d\det\tilde{\mathbf{A}}}^{t^3\det\tilde{\mathbf{A}}} -h'(s)\,ds\right] \\
&\leq t^p\int_\Omega \Big[|\nabla\tilde{\mathbf{u}}|^p - |\tilde{\mathbf{A}}|^p\Big]d\mathbf{x} - |\Omega|t^3(1-d)(\det\tilde{\mathbf{A}})h'(t^3 d\det\tilde{\mathbf{A}}).
\end{aligned}$$

Next note that, since $\frac{h(s)}{s} \to \infty$ by assumption, it follows that $h'(s) \to \infty$ as $s \to \infty$ and thus, since $p < 3$, it follows that $\triangle E < 0$ for all sufficiently large t. This completes the proof in the case $\Omega = B$. If $\Omega \neq B$ then we simply rescale and translate the deformation (2.13) onto a small ball centred at one of the flaw points and then extend it to $\Omega\backslash B$ by the homogeneous deformation $\tilde{\mathbf{A}}\mathbf{x}$. Then exactly analogous arguments yield the same result.

Remark. It is a consequence of arguments in [16] that if $\mathbf{A} \in \mathrm{M}_+^{3\times 3}$ is such that $h'(\det \mathbf{A}) \leq 0$ then

$$E(\mathbf{u}^h) \leq E(\mathbf{u}) \text{ for all } \mathbf{u} \in \mathcal{A}.$$

So discontinuous deformations satisfying (2.3) do not have less energy than the homogeneous deformation $\mathbf{u}^h$ if this condition holds.

3. OPTIMAL LOCATION FOR A DISCONTINUITY

We next consider the case where there is only one flaw point in the material (so that deformations satisfy (2.12) with $m = 1$, i.e. there is only one flaw at $\mathbf{x}_1 \in \Omega$). It is a consequence of a simple scaling argument that if there exists a deformation $\mathbf{u} \in \mathcal{A}(\mathbf{x}_1)$ with a discontinuity at $\mathbf{x}_1 \in \Omega$ and such that $E(\mathbf{u}) < E(\mathbf{u}^h)$ then given *any* point $\mathbf{x}_2 \in \Omega$ there exists a map $\tilde{\mathbf{u}} \in \mathcal{A}(\mathbf{x}_2)$ with discontinuity at $\mathbf{x}_2$ and with $E(\tilde{\mathbf{u}}) < E(\mathbf{u}^h)$ (see [14]). We now address the problem of determining the energetically optimal location for the flaw point.

Inner variations; moving a flaw; the energy momentum tensor. Let $\mathbf{u} = \mathbf{u}(\cdot, \mathbf{x}_1, \mathbf{A})$ denote a minimiser of E on $\mathcal{A}(\mathbf{x}_1)$. We consider an inner variation of $\mathbf{u}$ i.e. a variation of $\mathbf{u}$ of the form:

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}(\mathbf{x} + \varepsilon \mathbf{v}(\mathbf{x})), \qquad \varepsilon \in \mathbb{R},$$

where $\mathbf{v} \in C_0^1(\Omega)$ satisfies $\mathbf{v} \equiv -\mathbf{m}$ (a constant vector in a neighbourhood $B_\delta(\mathbf{x}_1)$ of $\mathbf{x}_1$). Note that the variations move the potential discontinuity point $\mathbf{x}_1$ in the direction of $\mathbf{m}$. It follows from arguments of, for example, Ball [3] that if h satisfies $|sh'(s)| \leq const[h(s) + 1]$ $\forall s$ then

$$\delta E := \frac{d}{d\varepsilon} E(\mathbf{u}_\varepsilon)\bigg|_{\varepsilon=0} = \int_{\Omega \backslash B_\delta(\mathbf{x}_1)} \nabla \mathbf{v} : \mathbf{M}(\nabla \mathbf{u})\, d\mathbf{x},$$

where

$$\mathbf{M}(\mathbf{F}) = \left[W(\mathbf{F})\mathbf{I} - \mathbf{F}^{\mathrm{T}} \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}) \right]$$

is the energy-momentum tensor.

Next assume that $\mathbf{u}(\cdot, \mathbf{x}_1, \mathbf{A})$ is smooth in $\Omega \backslash B_\delta(x_1)$ (so that (2.5) holds in this region) to obtain

$$\delta E = \int_{\Omega \backslash B_\delta(\mathbf{x}_1)} \mathrm{div}[\mathbf{M}(\nabla \mathbf{u})^{\mathrm{T}} \mathbf{v}]\, d\mathbf{x} = \int_{\partial B_\delta(x_1)} \mathbf{v} \cdot \mathbf{M}(\nabla \mathbf{u}) \mathbf{n}\, ds$$

$$= \int_{\partial B_\delta(\mathbf{x}_1)} -\mathbf{m}\cdot\mathbf{M}(\nabla\mathbf{u})\mathbf{n}\,ds = \int_{\partial\Omega} \mathbf{m}\cdot\mathbf{M}(\nabla\mathbf{u})\mathbf{n}\,ds. \tag{3.1}$$

To continue with our analysis we need to make assumptions on the regularity of minimisers given by Theorem 1. These are motivated by corresponding results that are known to hold in the case of radial cavitation.

Hypotheses. We assume that for each $\mathbf{x}_1 \in \Omega$ there is a unique minimiser of E on $\mathcal{A}(\mathbf{x}_1)$ denoted $\mathbf{u}(\cdot, \mathbf{x}_1, \mathbf{A})$. We further assume that $\mathbf{A}(t)$, $t \in \mathbb{R}$ is a smooth one-parameter family of matrices such that for $t > 0$ the minimiser $\mathbf{u}(\cdot, \mathbf{x}_1, \mathbf{A}) \in \mathcal{A}(\mathbf{x}_1)$ is discontinuous and for $t \leq 0$ the minimiser is $\mathbf{u}(\mathbf{x}, \mathbf{x}_1, \mathbf{A}) \equiv \mathbf{A}\mathbf{x}$. We define $\mathbf{A}_{crit} = \mathbf{A}(0)$. Finally we assume that for any $\delta > 0$, $\mathbf{u}(\cdot, \mathbf{x}_1, \mathbf{A}) \to \mathbf{u}^h_{crit} \equiv \mathbf{A}_{crit}\mathbf{x}$ in $C^2(\bar{\Omega}\backslash B_\delta(\mathbf{x}_1))$ as $t \to 0$.

We write

$$\mathbf{u}(\mathbf{x}, \mathbf{x}_1, \mathbf{A}) = \mathbf{A}\mathbf{x} + \mathbf{w}(\mathbf{x}, \mathbf{x}_1, \mathbf{A}), \tag{3.2}$$

where $\mathbf{w} = \mathbf{0}$ for $\mathbf{x} \in \partial\Omega$, and expand (3.1) as a series for $t > 0$ at $t = 0$. First, we rewrite the right hand side of (3.1) using (3.2) as

$$\mathbf{m}\cdot\int_{\partial\Omega} \left[(W(\mathbf{A} + \nabla_{\mathbf{x}}\mathbf{w}) - W(\mathbf{A}))\mathbf{I} - (\nabla_{\mathbf{x}}\mathbf{w})^{\mathrm{T}}\frac{\partial W}{\partial \mathbf{F}}(\mathbf{A} + \nabla_{\mathbf{x}}\mathbf{w})\right]\mathbf{n}\,ds, \tag{3.3}$$

where we have used the fact that

$$\int_{\partial\Omega} \mathbf{A}^{\mathrm{T}}\frac{\partial W}{\partial \mathbf{F}}(\mathbf{A} + \nabla_{\mathbf{x}}\mathbf{w})\mathbf{n}\,ds = \mathbf{0}$$

since $\mathbf{u}(\cdot, \mathbf{x}_1, \mathbf{A})$ is a minimiser of E on $\mathcal{A}(\mathbf{x}_1)$ (see [14] for details) and also that

$$\int_{\partial\Omega} W(\mathbf{A})\mathbf{n}\,ds = \mathbf{0}.$$

Expanding (3.3) as a series in t for $t > 0$ now yields

$$\delta E = -\frac{t^2}{2}\mathbf{m}\cdot\int_{\partial\Omega} \mathbf{n}\left(\nabla_{\mathbf{x}}\dot{\mathbf{w}} : \frac{\partial^2 W}{\partial \mathbf{F}^2}(\mathbf{A}_{crit})[\nabla_{\mathbf{x}}\dot{\mathbf{w}}]\right)ds + o(t^2)$$

$$= -\frac{t^2}{2}\int_{\partial\Omega} m^\gamma n^\gamma \frac{\partial \dot{w}^i}{\partial x^\alpha}\frac{\partial^2 W}{\partial F^i_\alpha \partial F^j_\beta}(\mathbf{A}_{crit})\frac{\partial \dot{w}^j}{\partial x^\beta}\,ds + o(t^2), \tag{3.4}$$

where $\dot{\mathbf{w}} = \frac{d}{dt}\mathbf{w}(\mathbf{x}, \mathbf{x}_1, \mathbf{A}(t))\big|_{t=0}$ is the right derivative of $\mathbf{w}$ at $t = 0$. The next lemma shows that $\dot{\mathbf{w}}$ solves a linear system.

Lemma 1. $\dot{\mathbf{w}}$ *is a (singular) solution of the linear system*

$$\frac{\partial}{\partial x^\gamma}\left[\frac{\partial^2 W}{\partial F^i_\gamma \partial F^j_\beta}(\mathbf{A}_{crit})\frac{\partial \dot{w}^j}{\partial x^\beta}\right] = 0 \text{ in } \Omega\backslash\{\mathbf{x}_1\},\ i = 1, 2, 3, \tag{3.5}$$

and satisfies the boundary condition $\dot{\mathbf{w}} = \mathbf{0}$ *on* $\partial\Omega$.

The proof of this follows from the assumptions on $\mathbf{w}$: in particular $\mathbf{w}$ satisfies

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\mathbf{A}+\nabla\mathbf{w})\right] = 0 \text{ in } \Omega\backslash\{\mathbf{x}_1\},\ i = 1,2,3$$

and $\mathbf{w} = \mathbf{0}$ on $\partial\Omega$. Then, taking a right hand derivative with respect to t at $t = 0$ yields the required result.

Remark. To evaluate the integral in (3.4) it is helpful to first note that $\dot{\mathbf{w}} = 0$ on $\partial\Omega$ and to then rewrite it in the form

$$\int_{\partial\Omega} m^\gamma n^\gamma \frac{\partial \dot{w}^i}{\partial x^\alpha}\Lambda^{ij}_{\alpha\beta}\frac{\partial \dot{w}^j}{\partial x^\beta} - 2\dot{w}^i\Lambda^{ij}_{\alpha\beta}\frac{\partial^2 \dot{w}^j}{\partial x^\beta \partial x^\gamma} m^\gamma n^\alpha \, ds, \tag{3.6}$$

where $\Lambda^{ij}_{\alpha\beta} = \frac{\partial^2 W}{\partial F^i_\alpha \partial F^j_\beta}(\mathbf{A}_{crit})$. The second term is included since, by lemma 1, the corresponding integrand is then divergence free in $\Omega\backslash\{\mathbf{x}_1\}$ i.e. we have the quadratic conservation law

$$\frac{\partial}{\partial x^\gamma}\left(m^\gamma \frac{\partial \dot{w}^i}{\partial x^\alpha}\Lambda^{ij}_{\alpha\beta}\frac{\partial \dot{w}^j}{\partial x^\beta}\right) - \frac{\partial}{\partial x^\alpha}\left(2\dot{w}^i\Lambda^{ij}_{\alpha\beta}\frac{\partial^2 \dot{w}^j}{\partial x^\beta \partial x^\gamma} m^\gamma\right) = 0. \tag{3.7}$$

in $\Omega\backslash\{x_1\}$. In evaluating the integral (3.6) it is sometimes helpful to use (3.7) to transform it to an integral on $\partial B_\varepsilon(x_1)$ (see [15]).

Concluding Remarks. Our approach is to next derive an expansion on $\partial\Omega$ for the singular solution $\dot{\mathbf{w}}$ of the system (3.5) and hence calculate the sign of δE given by (3.1). If $\delta E \neq 0$ then we can conclude in particular that the location of the flaw point is not optimal. This procedure has been carried out in [15] in the case when $\Omega = B$, $\mathbf{A} = \lambda\mathbf{I}$ and it is shown that $\delta E \neq 0$ unless $\mathbf{x}_1$ is located at the centre of the ball B. Hence a cavity forming at a non-central location in the ball induces a non-zero configurational force (which is in fact directed towards the centre of the ball). We conjecture that this may be sufficient to cause the formation of a crack initiated from the cavity and directed towards the centre of the ball. For further details and a discussion of related issues we refer the interested reader to [13], [14], [15].

Acknowledgements. The work of JS was supported by a Leverhulme Fellowship. The work of SJS was supported by the National Science Foundation under Grant No. 0072414.

References

[1] J. M. Ball, “Discontinuous equilibrium solution and cavitation in nonlinear elasticity”, Phil. Trans. Roy. Soc. London A **306** (1982), 557–611.

[2] J. M. Ball, “Constitutive inequalities and existence theorems in nonlinear elastostatics” in Nonlinear Analysis and Mechanics, Heriot-Watt Symposium Vol. 1, 187–241, Ed. R. J. Knops, Pitman, 1977.

[3] J. M. Ball, “Minimisers and the Euler-Lagrange equations” in Trends and Applications of Pure Mathematics to Mechanics, 1–4, Eds. P. G. Ciarlet and M. Roseau, Springer, 1984.

[4] P. G. Ciarlet, Mathematical Elasticity Vol. 1: Three-Dimensional Elasticity, North Holland, Amsterdam, 1988.

[5] J. D. Eshelby, “The elastic energy-momentum tensor”, J. Elasticity **5** (1975), 321–335.

[6] A. Gent and B. Park, “Failure processes in elastomers at or near a rigid spherical inclusion”, J. Materials Sci. **19** (1984), 1947–1956.

[7] C. O. Horgan and R. Abeyaratne, “A bifurcation problem for a compressible nonlinearly elastic medium: growth of a microvoid”, J. Elasticity **16** (1986), 189–200.

[8] C. O. Horgan and D. A. Polignone, “Cavitation in nonlinearly elastic solids: A review”, Appl. Mech. Rev. **48** (1995), 471–485

[9] R. D. James and S. J. Spector, “Remarks on $W^{1,p}$-quasiconvexity, interpenetration of matter, and function spaces for elasticity”, Anal. non linéaire **9** (1992), 263–280.

[10] S. Müller and S. J. Spector, “An existence theorem for nonlinear elasticity that allows for cavitation” Arch. Rational Mech. Anal. **131** (1995), 1–66.

[11] R. Ogden, “Large deformation isotropic elasticity: on the correlation of theory and experiment for compressible rubberlike solids”, Proc. Roy. Soc. London, A**328** (1972), 567–583.

[12] J. Sivaloganathan, “Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity”, Arch. Rational Mech. Anal. **96** (1986), 97–136.

[13] J. Sivaloganathan and S. J. Spector, “On the existence of minimisers with prescribed singular points in nonlinear elasticity”, J. Elasticity **59** (2000), 83–113

[14] J. Sivaloganathan and S. J. Spector, “On the optimal location of singularities arising in variational problems of nonlinear elasticity”, J. Elasticity **58** (2000), 191–224.

[15] J. Sivaloganathan and S. J. Spector, “On cavitation, configurational forces and implications for fracture in a nonlinearly elastic material”, J. Elasticity, **67** (2002), 25-49.

[16] S. J. Spector, “Linear deformations as global minimisers in nonlinear elasticity”, Q. Appl. Math. **52** (1994), 59–64.

[17] C. A. Stuart, “Radially symmetric cavitation for hyperelastic materials”, Analyse Nonlineaire **2** (1985), 33–66.

[18] J. Sivaloganathan, S. J. Spector, and V. Tilakraj, “On the convergence of regularised minimisers of variational problems in nonlinear elasticity”, preprint (2003).