An Explicit Radial Cavitation Solution in Nonlinear Elasticity

KATHLEEN A. PERICAK-SPECTOR
Department of Mathematics, Southern Illinois University, Carbondale, IL 62901–4408, USA, E-mail: kpericak@math.siu.edu

JEYABAL SIVALOGANATHAN
Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK, E-mail: js@maths.bath.ac.uk

SCOTT J. SPECTOR
Department of Mathematics, Southern Illinois University, Carbondale, IL 62901–4408, USA, E-mail: sspector@math.siu.edu

(Received 5 October 2000; Final version 28 June 2001)

Abstract: An explicit solution to the ordinary differential equation that governs the radial equilibrium behavior of a particular compressible nonlinearly elastic material is obtained. The resulting deformation exhibits cavitation at the center of a homogeneous isotropic ball in three or more dimensions.

1. INTRODUCTION, PRELIMINARIES, AND RESULTS

We take $\lambda > 0, \mu \geq 0$ and consider the problem of minimizing the energy

$$E(\mathbf{u}) = \int_B W(\nabla_x \mathbf{u}(x;\lambda))\, dx, \quad W(\mathbf{F}) := \mu \|\mathbf{F}\|^2 + h(\det \mathbf{F}), \tag{1.1}$$

of a homogeneous, isotropic elastic material, which occupies the unit ball $B \subset \mathbb{R}^n$, $n \geq 3$, in a homogeneous, stress-free reference configuration, among $\mathbf{u}(\cdot;\lambda)$ in the Sobolev space $W^{1,2}(B;\mathbb{R}^n)$, which satisfy the displacement boundary condition $\mathbf{u}(x;\lambda) = \lambda x$ on ∂B, and which are injective *radial deformations*, i.e.,

$$\mathbf{u}(x;\lambda) = \frac{r(R;\lambda)}{R} x \quad \text{for} \quad x \in B, \quad R := |x|, \tag{1.2}$$

$r(\cdot;\lambda) : [0,1] \to [0,\infty)$, and $r' > 0$ *a.e.* Here $\det \mathbf{F}$ denotes the determinant of the $n \times n$ matrix $\mathbf{F}$, while $\|\mathbf{F}\|^2$ is the sum of the squares of the elements of $\mathbf{F}$.

If $h \in C^2((0,\infty))$ satisfies $h(v) \to +\infty$ when the deformed volume $v \to 0^+$ and $h(v)/v \to +\infty$ when $v \to +\infty$ then results of Ball [1] show that a unique, injective radial minimizer to (1.1) exists and satisfies the Euler–Lagrange (radial equilibrium) equation

$$R\frac{d}{dR}\left[2\mu r' + \left(\tfrac{r}{R}\right)^{n-1} h'(V)\right] = -(n-1)\left(r' - \tfrac{r}{R}\right)\left[2\mu - \left(\tfrac{r}{R}\right)^{n-2} h'(V)\right], \qquad (1.3)$$

where

$$r' = r'(R;\lambda) := \tfrac{d}{dR} r(R;\lambda), \quad V(R;\lambda) := r'(R;\lambda)\left(\tfrac{r(R;\lambda)}{R}\right)^{n-1}.$$

Moreover, there exists a critical value $\lambda_{cr} > 1$ with the property that:

(i) if $\lambda \le \lambda_{cr}$ the radial minimizer is the homogeneous map $\mathbf{u}^h(\mathbf{x}) := \lambda x$;

(ii) if $\lambda > \lambda_{cr}$ the radial minimizer corresponds to a map of the form (1.2) that satisfies $r(0;\lambda) > 0$ and the natural boundary condition

$$T_{RR}(R;\lambda) := \left(\tfrac{R}{r(R;\lambda)}\right)^{n-1}\left[2\mu r'(R;\lambda) + \left(\tfrac{r(R;\lambda)}{R}\right)^{n-1} h'(V(R;\lambda))\right] \to 0 \qquad (1.4)$$

as $R \to 0^+$ for each fixed λ. Thus in this case the minimizer is discontinuous; it produces a hole of radius $r(0;\lambda)$ at the center of the ball (this is the phenomenon of cavitation; it can also be interpreted as the rapid growth of a preexisting microvoid (see 2, 3)); and the material at the surface of this hole experiences no normal stresses.

Although there is an extensive literature on the existence of radial minimizers to elastic energies and the existence of solutions to the corresponding Euler–Lagrange equation (see the review article by Horgan and Polignone [4]) very few *compressible* constitutive relations permit an explicit solution of the Euler–Lagrange equation. The only (modulo a radial null-Lagrangian, see Horgan [5] and Steigmann [6]) such solutions that appear in the literature are for an elastic fluid ($\mu = 0$ in (1.1), see, e.g., [5] or [7]), where the minimizer is

$$r(R;\lambda) = \left[\lambda_{cr}^n R^n + \lambda^n - \lambda_{cr}^n\right]^{1/n}; \qquad (1.5)$$

for the Blatz–Ko constitutive relation for foam rubbers, $W(\mathbf{F}) = \mu(||\mathbf{F}^{-1}||^2 + 2\det\mathbf{F})$, which was obtained, in two dimensions, by Horgan and Abeyaratne [2] and in three dimensions, by Tian-hu [8]; and for the generalized Carroll material (a convex function of a radial null Lagrangian), which was obtained by Murphy and Biwa [9].

We believe that such explicit solutions can help to develop one's intuition concerning the behavior of cavitating solutions and it is the purpose of this note to present another such solution. Accordingly, we take $\mu > 0$ and restrict our attention to functions h that are quadratic on the interval $[1,\infty)$. Thus for $v \ge 1$,

$$h(v) = \mu(av^2 - bv + c)$$

with $a > 0$. In addition, the requirement that the reference configuration be stress free yields $b = 2(a+1) > 0$. For this constitutive relation Ball [1, p. 605] showed further that λ_{cr} is the unique solution of the equation

$$\lambda_{cr}\left[1 + a\lambda_{cr}^{2(n-1)}\right]^{1/2} = \frac{1+a}{\sqrt{a}} + (n-1)\int_{\lambda_{cr}}^{\infty} \frac{d\theta}{\left[1 + a\theta^{2(n-1)}\right]^{1/2}} \tag{1.6}$$

and that the radial component of the Cauchy stress T_{RR} (force per unit deformed area) can be explicitly determined as a function of the circumferential strain $\epsilon(R;\lambda) := r(R;\lambda)/R$, namely,

$$\frac{T_{RR}(\epsilon) + 2\mu(1+a)}{\left[\epsilon^{-2(n-1)} + a\right]^{1/2}} = 2\mu\frac{1+a}{\sqrt{a}} + 2\mu(n-1)\int_0^{1/\epsilon} \frac{t^{n-3}\,dt}{\left[a + t^{2(n-1)}\right]^{1/2}}.$$

(Note that the above equation in [1] is expressed using the reciprocal of the circumferential strain $w = 1/\epsilon$.)

The main result of the paper is the following theorem.

Theorem. *Let $n \geq 3$ and $\lambda > \lambda_{cr}$. Then the unique injective radial minimizer of (1.1) that satisfies the boundary condition $r(1;\lambda) = \lambda$ is given by*

$$P(\lambda)R^{-n} = \int_{\lambda_{cr}}^{\frac{r(R;\lambda)}{R}} \left[1 + a\theta^{2(n-1)}\right]^{1/2} d\theta, \tag{1.7}$$

$$P(\lambda) := \int_{\lambda_{cr}}^{\lambda} \left[1 + a\theta^{2(n-1)}\right]^{1/2} d\theta. \tag{1.8}$$

2. PROOF OF THE THEOREM

Results in [1] show that the radial energy minimizer is the only solution of (1.3) that satisfies $r(1;\lambda) = \lambda$ and the traction-free boundary condition (1.4). Thus all we need show is that the function given by (1.7)–(1.8) satisfies (1.3), (1.4), and $r(1;\lambda) = \lambda$. However, since the integrand in (1.7) is strictly positive, the boundary condition at the outer boundary must be satisfied in order that (1.8) be compatible with (1.7) at $R = 1$.

We next show that any function $r(R;\lambda)$ that satisfies (1.7) will also satisfy the radial equilibrium equation (1.3). To accomplish this we first follow [1] and change both the independent and the dependent variables in (1.3); we suppress the dependence upon λ and let

$$s := \ln R, \quad z(s) := \frac{r(e^s;\lambda)}{e^s} = \frac{r(R;\lambda)}{R} \tag{2.1}$$

$$\dot{z}(s) := \frac{d}{ds}z(s) = r'(e^s;\lambda) - \frac{r(e^s;\lambda)}{e^s}, \tag{2.2}$$

which yields the autonomous equation

$$\frac{d}{ds}\left[2\mu(\dot{z}+z) + z^{n-1}h'\left((\dot{z}+z)z^{n-1}\right)\right] = -(n-1)\dot{z}\left[2\mu - z^{n-2}h'\left((\dot{z}+z)z^{n-1}\right)\right]$$

for $s \in (-\infty, 0)$. However, $h'(v) = 2\mu(av - (a+1))$ and so the last equation simplifies:

$$[\ddot{z} + n\dot{z}]\left[1 + az^{2(n-1)}\right] + a(n-1)\dot{z}^2 z^{2n-3} = 0. \tag{2.3}$$

Now consider our proposed solution (1.7). The change of variables (2.1) yields

$$Pe^{-sn} = \int_{\lambda_{cr}}^{z(s)} \left[1 + a\theta^{2(n-1)}\right]^{1/2} d\theta. \tag{2.4}$$

Before proceeding further we note that, by (1.8), $P > 0$. Therefore, in view of the strict positivity and continuity of the integrand it follows (intermediate-value theorem) that for each s there is a unique $z = z(s) \in (\lambda_{cr}, \infty)$ such that (2.4) is satisfied. Moreover, the implicit function theorem can be used to show that the resulting function $s \mapsto z(s)$ is C^∞. In addition, it is clear from (2.4) that $s \mapsto z(s)$ *is strictly monotone decreasing and therefore one-to-one.*

We next differentiate (2.4) with respect to s to get

$$-nPe^{-sn} = \dot{z}(s)\left[1 + a[z(s)]^{2(n-1)}\right]^{1/2} \tag{2.5}$$

and hence, if we substitute for Pe^{-sn} from (2.4), we conclude

$$-n\int_{\lambda_{cr}}^{z(s)} \left[1 + a\theta^{2(n-1)}\right]^{1/2} d\theta = \dot{z}(s)\left[1 + a[z(s)]^{2(n-1)}\right]^{1/2}. \tag{2.6}$$

If we then differentiate (2.6) with respect to s we discover that

$$-n\dot{z}\left[1 + az^{2(n-1)}\right]^{1/2} = \ddot{z}\left[1 + az^{2(n-1)}\right]^{1/2} + a(n-1)\dot{z}^2 z^{2n-3}\left[1 + az^{2(n-1)}\right]^{-1/2},$$

which when multiplied by the square root of $[1+az^{2(n-1)}]$ is (2.3). Thus any function $r(R;\lambda)$ that satisfies (1.7) will also satisfy (1.3).

Finally, we consider the traction-free boundary condition, (1.4), on the surface of the newly formed cavity. The change of variables (2.1)–(2.2) yields the equivalent formulation:

$$2\mu(\dot{z}(s) + z(s))[z(s)]^{1-n} + h'\left((\dot{z}(s) + z(s))[z(s)]^{n-1}\right) \to 0 \quad \text{as } s \to -\infty. \tag{2.7}$$

In order to prove (2.7) we first take the limit as $s \to -\infty$ in (2.4); since z is continuous and strictly monotone decreasing we must have

$$z(s) \to +\infty \quad \text{as } s \to -\infty. \tag{2.8}$$

We claim that

$$\lim_{s\to-\infty} (\dot{z}(s) + z(s))[z(s)]^{n-1} = \tfrac{1+a}{a} \tag{2.9}$$

and, moreover, that (2.7) is a consequence of (2.9). The latter is clear since $h'((1+a)/a) = 0$ and

$$\begin{aligned}(\dot{z}(s) + z(s))[z(s)]^{1-n} &= \left[(\dot{z}(s) + z(s))[z(s)]^{n-1}\right]\left[[z(s)]^{2(1-n)}\right] \\ &\to \left[\tfrac{1+a}{a}\right][0] = 0 \quad \text{as} \quad s \to -\infty.\end{aligned}$$

To obtain (2.9) we recall that the mapping $s \mapsto z(s)$ is one-to-one and hence, in view of (2.8), the limit in (2.9) can be replaced by the limit as $z \to +\infty$ provided $\dot{z}$ is considered as a function of z through equation (2.6); thus,

$$\begin{aligned}\lim_{s\to-\infty}\left(z^{n-1}[z+\dot{z}]\right) &= \lim_{z\to+\infty} z^{n-1}\left(z - \frac{n\int_{\lambda_{cr}}^{z}\left[1+a\theta^{2(n-1)}\right]^{1/2}d\theta}{\left[1+az^{2(n-1)}\right]^{1/2}}\right) \\ &= \lim_{z\to+\infty}\left(\frac{z\left[1+az^{2(n-1)}\right]^{1/2} - n\int_{\lambda_{cr}}^{z}\left[1+a\theta^{2(n-1)}\right]^{1/2}d\theta}{\left[z^{-2(n-1)}+a\right]^{1/2}}\right). \end{aligned} \tag{2.10}$$

Now,

$$\begin{aligned} z\left[1+az^{2(n-1)}\right]^{\frac{1}{2}} &- \lambda_{cr}\left[1+a\lambda_{cr}^{2(n-1)}\right]^{\frac{1}{2}} \\ &= \int_{\lambda_{cr}}^{z} \frac{d}{d\theta}\left(\theta\left[1+a\theta^{2(n-1)}\right]^{\frac{1}{2}}\right)d\theta \\ &= \int_{\lambda_{cr}}^{z}\left(\left[1+a\theta^{2(n-1)}\right]^{\frac{1}{2}} + \frac{a(n-1)\theta^{2(n-1)}}{\left[1+a\theta^{2(n-1)}\right]^{\frac{1}{2}}}\right)d\theta\end{aligned}$$

and so a straightforward computation shows that the numerator in the right-hand side of (2.10) is equal to

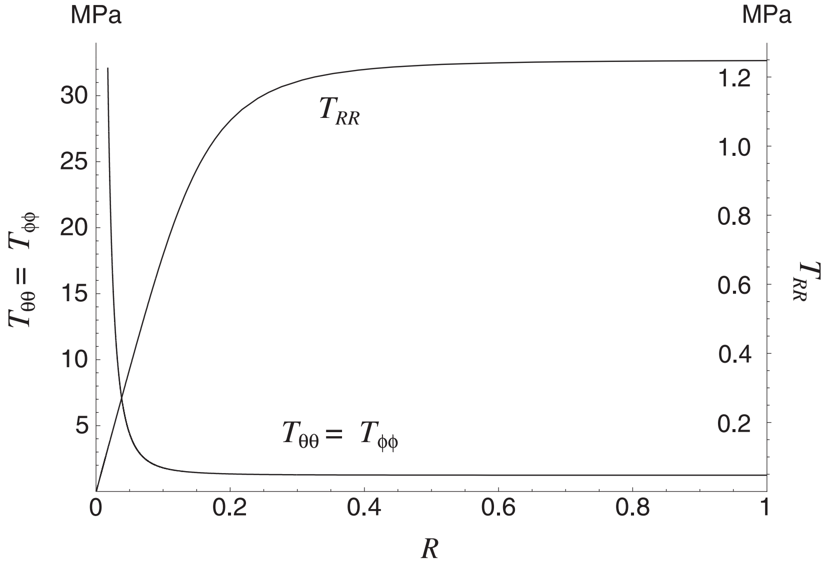

Figure 1. Typical values of the Cauchy stress.

$$\lambda_{cr}\left[1 + a\lambda_{cr}^{2(n-1)}\right]^{1/2} - (n-1)\int_{\lambda_{cr}}^{z}\frac{d\theta}{\left[1 + a\theta^{2(n-1)}\right]^{1/2}}. \tag{2.11}$$

Therefore, if we make use of (2.11) to take the limit in the right-hand side of (2.10) we conclude, with the aid of (1.6), that (2.9) is satisfied, which concludes the proof.

3. NUMERICAL RESULTS IN THREE DIMENSIONS

In the case of physical interest, $n = 3$, the integral cannot be computed explicitly since it is an elliptic integral. We have however used z as a parameter to obtain numerical results (see Figure 1). The radial and tangential components of the Cauchy stress are given by

$$\begin{aligned} T_{RR}(z) &= 2\mu\left[\frac{\dot{z}+z}{z^2} - 1 + a(z^2[\dot{z}+z] - 1)\right], \\ T_{\theta\theta}(z) &= T_{\phi\phi}(z) = 2\mu\left[\frac{1}{\dot{z}+z} - 1 + a(z^2[\dot{z}+z] - 1)\right], \end{aligned}$$

respectively, where (by (2.4), (2.10), and (2.11))

$$\dot{z} = -z + \frac{\lambda_{cr}\sqrt{1+a\lambda_{cr}^4} - 2\int_{\lambda_{cr}}^{z} \frac{d\theta}{\sqrt{1+a\theta^4}}}{\sqrt{1+az^4}},$$

$$R(z,\lambda)^{-3} = 1 + \frac{\int_{\lambda}^{z} \sqrt{1+a\theta^4}\, d\theta}{P(\lambda)}, \quad P(\lambda) = \int_{\lambda_{cr}}^{\lambda} \sqrt{1+a\theta^4}\, d\theta.$$

A simple computation shows that Young's modulus and Poisson's ratio at the stress-free reference configuration are given by $E = 2\mu(3 - \frac{1}{a})$ and $\nu = .5 - \frac{1}{2a}$, respectively. The choice $a = 10,000$ will make the material nearly incompressible and the choice $\mu = \frac{1}{4}$ MPa then yields a Young's modulus of approximately 1.5 MPa, which is typical of an elastomer that cavitates. The critical value of the load parameter is $\lambda_{cr} \approx 1.000\,083$ and the value at which the stresses have been graphed is $\lambda = 1.001$.

Acknowledgement. The work of SJS was supported by the National Science Foundation under Grant No. 0072414.

REFERENCES

[1] Ball, J. M.: Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. *Phil. Trans. R. Soc. Lond.*, A 306, 557–611 (1982).

[2] Horgan, C. O. and Abeyaratne, R.: A bifurcation problem for a compressible nonlinearly elastic medium: growth of a microvoid. *J. Elasticity*, 16, 189–200 (1986).

[3] Sivaloganathan, J.: Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity. *Arch. Rational Mech. Anal.*, 96, 97–136 (1986).

[4] Horgan, C. O. and Polignone, D. A.: Cavitation in nonlinearly elastic solids: A review. *Appl. Mech. Rev.*, 48, 471–485 (1995).

[5] Horgan, C. O.: Void nucleation and growth for compressible non-linearly elastic materials: an example. *Int. J. Solids Structures*, 29, 279–291 (1992).

[6] Steigmann, D. J.: Cavitation in membranes – an example. *J. Elasticity*, 28, 277–287 (1992).

[7] Lancia, M. R., Podio-Guidugli, P. and Vergara-Caffarelli, G.: Gleanings of radial cavitation. *J. Elasticity*, 44, 183–192 (1996).

[8] Tian-hu, H.: A theory of the appearance and growth of the micro-spherical void. *Int. J. Fracture*, 43, R51–R55 (1990).

[9] Murphy, J. G. and Biwa, S.: Nonmonotonic cavity growth in finite, compressible elasticity. *Int. J. Solids Structures*, 34, 3859–3872 (1997).