Energy Minimising Properties of the Radial Cavitation Solution in Incompressible Nonlinear Elasticity

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Abstract. Consider an incompressible hyperelastic material, occupying the unit ball $B \subset \mathbb{R}^n$ in its reference state. Suppose that the displacement is specified on the boundary, that is,

$$\mathbf{u}(\mathbf{x}) = \lambda \mathbf{x} \quad \text{for} \quad \mathbf{x} \in \partial B,$$

where $\lambda > 1$ is a given constant.

In this paper, isoperimetric arguments are used to prove that the radial deformation, producing a spherical cavity, is the energy minimiser in a general class of isochoric deformations that are discontinuous at the centre of the ball and produce a (possibly non-symmetric) cavity in the deformed body. This result has implications for the study of cavitation in certain polymers.

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1 Introduction.

Let $B \subset \mathbb{R}^n$ denote the unit ball, the physically relevant values being $n = 2$ or 3. Consider deformations $\mathbf{u} : B \to \mathbb{R}^n$ of an incompressible, nonlinearly elastic material occupying the region B in its reference state. Thus, the admissible deformations $\mathbf{u}$ satisfy the incompressibility constraint[1]

$$\det \nabla\mathbf{u} = 1 \quad \text{for a.e. } \mathbf{x} \in B. \tag{1.1}$$

Deformations satisfying the above condition are known as *isochoric* deformations. In nonlinear hyperelasticity, with each such deformation we associate a corresponding energy

$$E(\mathbf{u}) = \int_B W(\nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.2}$$

where $W : M_1^{n\times n} \to \mathbb{R}$ is the stored-energy function and $M_1^{n\times n}$ denotes the set of $n \times n$ matrices with determinant equal to 1. In the variational approach, we seek equilibrium states by minimising (1.2) on some class of admissible deformations satisfying given boundary conditions (displacement or traction). In this paper we consider a problem which arises from the study of radial cavitation initiated by Ball in the fundamental paper [2]. The work of Ball was, in part, motivated by the work of [3] and subsequently developed by many authors (see, e.g., [5, 6, 13] and the review article [4]). In [2], Ball studies energy minimisers for compressible[2] and incompressible materials in the class of radial deformations of B. It is shown therein that if the imposed boundary tractions or displacements are sufficiently large, then the radial deformation which minimises the energy is discontinuous and corresponds to a hole forming at the centre of the deformed ball. This is the phenomenon of cavitation. To date, little is known about the minimising properties of these radial cavitation solutions in the general class of all (possibly non-symmetric) deformations.[3] The current paper addresses this problem in the incompressible case and proves that the radial incompressible minimiser is the global minimiser of the energy amongst all (possibly non-symmetric) isochoric deformations producing a hole at the centre of the deformed ball. To prove this, we will draw on and refer to a number of key ideas and results from [11].

We will consider the displacement boundary-value problem in which the deformations are required to satisfy

$$\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x} \quad \text{for } \mathbf{x} \in \partial B, \tag{1.3}$$

where $\lambda > 1$ is a given constant and we study energy minimisers of (1.2) in the class of isochoric deformations. Since it is not possible for a smooth isochoric deformation to satisfy (1.3) for any $\lambda > 1$, we must enlarge the class of deformations. We therefore consider those deformations $\mathbf{u}$ that produce a single discontinuity at the centre of the deformed ball. Such discontinuous deformations can be viewed as an idealised limit, as $\epsilon \to 0$, of deformations $\mathbf{u}_\epsilon$ of punctured balls $B_\epsilon = \{\mathbf{x} \in \mathbb{R}^n : \epsilon < |\mathbf{x}| < 1\}$, where $\mathbf{u}_\epsilon(\mathbf{x}) = \lambda\mathbf{x}$ on the outer boundary of B_ϵ and the inner

[1] Vulcanized rubber is often modelled as an incompressible material.
[2] In the compressible case, the condition (1.1) is replaced by $\det \nabla\mathbf{u} > 0$.
[3] In the compressible case, partial results on minimising properties are contained in [7, 8] (see also [10]).

boundary is left free.[4] Thus, such discontinuous equilibria should approximate the behaviour of a ball containing a microvoid of radius ϵ at its centre, provided that ϵ is sufficiently small.

References

[1] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337–403.

[2] J. M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Phil. Trans. R. Soc. Lond.* A **306** (1982), 557–611.

[3] A. N. Gent and P. B. Lindley, Internal rupture of bonded rubber cylinders in tension, *Proc. R. Soc. Lond. A* **249** (1958), 195–205.

[4] C. O. Horgan and D. A. Polignone, Cavitation in nonlinearly elastic solids: A review, *Appl. Mech. Rev.* **48** (1995), 471–485.

[5] F. Meynard, Existence and nonexistence results on the radially symmetric cavitation problem, *Quart. Appl. Math.* **50** (1992), 201–226.

[6] J. Sivaloganathan, Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity, *Arch. Rational Mech. Anal.* **96** (1986), 97–136.

[7] J. Sivaloganathan, Singular minimizers in the calculus of variations: a degenerate form of cavitation, *Anal. non linéaire* **9** (1992), 657–681.

[8] J. Sivaloganathan, On the stability of cavitating equilibria, *Q. Appl. Math.* **53** (1995), 301–313.

[9] J. Sivaloganathan and S. J. Spector, On the existence of minimizers with prescribed singular points in nonlinear elasticity, *J. Elasticity* **59** (2000), 83–113.

[10] J. Sivaloganathan and S. J. Spector, Necessary conditions for a minimum at a radial cavitating singularity in nonlinear elasticity, *Anal. non linéaire* **25** (2008), 201–213.

[11] J. Sivaloganathan and S. J. Spector, On the Symmetry of Energy Minimising Deformations in Nonlinear Elasticity I: Incompressible Materials. Preprint, (2008): (http://www.math.siu.edu/spector/incompressible.pdf).

[12] J. Sivaloganathan, S. J. Spector, and V. Tilakraj, The convergence of regularized minimizers for cavitation problems in nonlinear elasticity, *SIAM J. Appl. Math.* **66**, (2006) 736–757.

[13] C. A. Stuart, Radially symmetric cavitation for hyperelastic materials, *Anal. non linéaire* **2** (1985), 33–66.

[4] See, e.g., [1] for the traction problem for incompressible materials and [4, 6, 12] for the displacement problem for compressible materials.