On Bifurcation in Finite Elasticity: Buckling of a Rectangular Rod

by

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Abstract. Although there is an extensive literature on the linearization instability of the nonlinear system of partial differential equations that governs an elastic material, there are very few results that prove that a second branch of solutions actually bifurcates from a known solution branch when the known branch becomes unstable. In this paper the implicit function theorem in a Banach space setting is used to prove that the quasistatic compression of a rectangular elastic rod between rigid frictionless plates leads to the buckling of the rod as is observed in experiment and as first predicted by Euler.

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1. Introduction

We suppose that a rectangular rod is composed of a homogeneous, isotropic, hyperelastic material and look for plane-strain equilibrium solutions of the governing nonlinear system of partial differential equations when the rod is subjected to uniaxial compression. It follows from prior results that, under mild constitutive restrictions, there exists a family of homogeneous deformations, one for each value of the compression parameter, each of which is a weak relative minimizer of the elastic energy and, for a wide class of stored energy functions, there are specific values of the compression parameter at which the associated homogeneous deformation becomes unstable, that is, the second variation of the elastic energy is not strictly positive due to a linearization instability. Two distinct modes of instability are possible: a symmetric mode, usually referred to as a barrelling or bulging instability, in which the solution of the linearized problem is symmetric with respect to the loading axis; and an asymmetric mode, usually referred to as a buckling instability, in which the solution of the linearized problem has no such symmetry.

Under the additional hypothesis that the kernel of the linearized elasticity operator is one-dimensional at the parameter value where the homogeneous deformation loses stability and that the *strict crossing condition* is satisfied, i.e., that the smallest eigenvalue of this one-parameter family of linear operators crosses zero at a nontrivial rate, we are then able to adapt a version of the implicit function theorem, due to Crandall and Rabinowitz [23], to show that, locally, there is a second solution branch that bifurcates from the trivial solution branch. Symmetry considerations imply that the bifurcation is pitchfork. The bifurcated branch may be subcritical or supercritical depending on the particular constitutive relation. Moreover, if the bifurcation is supercritical then, locally, each deformation on the bifurcated solution branch is a weak relative minimizer of the elastic energy and has lower energy than the homogeneous solution.

We then verify that our hypotheses are satisfied by the family of compressible neo-Hookean materials

$$W(\mathbf{F}) = c\left(\frac{1}{2}\mathbf{F}:\mathbf{F} + \frac{1}{k}(\det \mathbf{F})^{-k}\right), \qquad c > 0, \quad k > 0. \tag{1.1}$$

Specifically, we determine that the null space of the linearized elasticity operator is indeed one-dimensional, the strict crossing condition is satisfied, and the branch of solutions that first bifurcates from the trivial solution is a buckling instability.

The only other works, of which we are aware, that establish the existence of a bifurcated solution branch in Finite Elasticity are those of Buffoni and Rey [16], Healey and Montes-Pizarro [34], Mielke [40], and Rabier and Oden [48]. In [48] the authors analyze the steady spinning motion of a homogeneous, isotropic, incompressible, hyperelastic, circular cylindrical shell (of finite thickness) as well as that of a circular cylinder. Using the angular velocity as a parameter, they prove that other solution branches bifurcate from the steady solution branch as the angular velocity increases. The mathematical techniques used: linearization instability via separation of variables, linear elliptic estimates, and the implicit function theorem in a Banach space setting, are similar[1] to those utilized in this paper.

Mielke [40, Chapter 10] and Buffoni and Rey [16] study the extension and compression, respectively, of a two-dimensional, infinite elastic rod of finite width. Each obtains the existence of an inhomogeneous solution branch that bifurcates from the trivial homogeneous solution branch by considering the equilibrium equations of Finite Elasticity as a dynamical system, with the infinite direction taken as the "time" variable, followed by a center manifold reduction due to Mielke [40]. Since our analysis utilizes the periodic extension of functions from a rectangle to an infinite strip, our methods would yield the existence of alternative, periodic solutions to the problem in [16] if not for the fact that Buffoni and Rey only consider solutions that are symmetric. In addition, a crucial assumption in the analysis in [16] is the requirement that the load-displacement curve for uniaxial compression fails to be convex.[2] They show that this assumption implies that, at the bifurcation load, the linearized problem at the homogeneous solution must have an infinite number of negative eigenvalues (Agmon's condition fails). Consequently, their deformations are not linearization stable and hence they are not relative minimizers of the elastic energy. (Mielke's [40, Chapter 10] solutions for tension do not have this difficulty.)

Healey and Montes-Pizarro [34] investigate the compression of a three-dimensional cylindrical rod using the same techniques that we employ in this paper. The results we obtain are thus analogous to those in [34] with two interesting differences. Firstly, since for hyperelasticity[3] there is no known analysis of linearization instability due to buckling in this geometry, they are forced to restrict their attention to symmetric (barrelling) instabilities. More significantly, for their problem they are able to adapt results of Rabinowitz [49] and Healey and Simpson [35] in order to continue the barrelled solution branch. They show that each such barrelled solution branch can be continued as long as the linearized problem satisfies the strong-ellipticity and complementing conditions.[4] This part of their analysis could also be adapted to our problem, however, it would involve a reformulation of our work in Hölder spaces rather than the Sobolev spaces we have used.

[1] The problem in [48] has additional technical difficulties: the inverse of the linearized operators are not compact operators, as is the case in [34] and our paper; the complementing condition fails on the trivial solution branch before bifurcation occurs; and, at sufficiently large angular velocities the equations are no longer elliptic.

[2] Their constitutive relations are, however, polyconvex, strictly rank-one convex, and globally strongly-elliptic.

[3] See Chau [20] for both buckling and barrelling of an elastic-plastic circular cylinder.

[4] The rod must also remain in contact with the rigid plates that are compressing it.

The symmetry, or the lack thereof, of the first instability in compression has been addressed in experiments on elastomers by Beatty and Hook [12] and Beatty and Dadras [11]. Their results seem to indicate that a sufficiently long or thin, elastomeric, circular cylinder will first buckle, while a sufficiently short or thick cylinder will instead barrel. However, since the load at which their elastomers barrel is much less than that predicted in any known model, it is suspected that the occurrence of barrelling in these experiments is an edge effect due to friction between the elastomer and the rigid plates used for compression (see the discussions in [38] and [24]). In a rectangular geometry where a mathematical analysis of both buckling and barrelling instabilities has been obtained, all known (globally strongly-elliptic) examples have the asymmetric (buckling) instability preceding the symmetric (barrelling) instability.[5]

Before proceeding with a more detailed outline of the results contained in this manuscript we first mention that the general problem addressed in this paper, that of buckling of an elastic rod, bar, or column, has an extensive history originating in the 18th century with Euler.[6] However, it is only in the last half-century that researchers have systematically addressed the failure of linearization stability for the nonlinear system of partial differential equations that govern the equilibrium behavior of an elastic material. In the particular rectangular geometry herein considered and for general isotropic, compressible, hyperelastic materials the details of the linearization instability of these equations can be found in papers of Davies [24, 25], the book of Ogden [44], and the references therein. More recent interesting work in this area is the postbuckling analysis of Triantafyllidis, Scherzinger, and Huang [61]; the analysis of Grabovsky and Truskinovsky [32] on the comparison of buckling instabilities with rotational (or “flip”) instabilities; and the “safe” load estimates of Del Piero and Rizzoni [27]. (See, also, the references therein.)

We now present a more detailed outline of our work. We start in §2 with a presentation of many of our notations together with an identification of the body with the rectangle $\mathcal{R} := [-R, R] \times [0, L] \subset \mathbb{R}^2$. Deformations, $\mathbf{f} : \mathcal{R} \to \mathbb{R}^2$, of such a two-dimensional body are known to be equivalent (see, e.g., [34], [44, p. 415]) to the consideration of plane strain of a three-dimensional rectangular solid. We also list the versions of Korn’s inequality that we use in this paper.

In §3 we consider a two-dimensional, homogeneous, isotropic, hyperelastic material with stored energy function $W(\mathbf{F}) = \boldsymbol{\Phi}(\boldsymbol{\nu_1}, \boldsymbol{\nu_2})$, where $\mathbf{F}$ is a two-by-two matrix (which is equal to the gradient of a deformation $\mathbf{f} : \mathcal{R} \to \mathbb{R}^2$ at a point $\mathbf{x} \in \mathcal{R}$) and ν_1 and ν_2 are the principal stretches: the eigenvalues of $\sqrt{\mathbf{F}\mathbf{F}^{\mathrm{T}}}$. We next recall some of the representation theorems

[5]However, see Pence and Song [47] who analyze a composite, rectangular, (incompressible) neo-Hookean material and show that the first instability for such a material may be barrelling, buckling, or failure of the complementing condition. See Davies [25] for a comparison of the critical load for barrelling of a cylinder with that of buckling for a rectangle, and Chau [20] for a comparison of the barrelling and buckling loads of an elastic-plastic cylinder.

[6]See, for example, the introduction in the recent paper [32].

of Ball [9] and Sylvester [59] (see also Chadwick and Ogden [19, 18] and Šilhavý [52]) that establish the equivalence of differentiating the stored energy with respect to deformation gradient and principal stretches. We describe the strong-ellipticity condition (SE) and the complementing condition (CC) of Agmon, Douglis and Nirenberg [5]. We also present an algebraic condition of Agmon (see, e.g., [56] or Friedman [30, p. 77]) which, given (SE) and (CC), is equivalent to a lower bound on the spectrum of a linear elliptic system of partial differential equations. Finally, we examine deformations of the rectangle with constant, diagonal deformation gradient and there simplify results of Davies [24] on necessary and sufficient conditions for both the complementing condition and Agmon's condition to be satisfied at such deformations.

In §4 we pose our nonlinear mathematical problem: find a deformation $\mathbf{f}$ that, for some $\lambda \in (0,\infty)$, satisfies

$$\operatorname{div} \mathbf{S}(\nabla \mathbf{f}) = \mathbf{0} \text{ in } \mathcal{R}, \qquad \begin{array}{ll} f_2 = \lambda L \ \text{ on } \mathcal{R}_T, & (\mathbf{S}(\nabla \mathbf{f})\mathbf{n})_1 = 0 \ \text{ on } \mathcal{R}_T \cup \mathcal{R}_B, \\ f_2 = 0 \quad \text{ on } \mathcal{R}_B, & \mathbf{S}(\nabla \mathbf{f})\mathbf{n} = \mathbf{0} \ \text{ on } \mathcal{S}, \end{array}$$

where $\mathcal{R}_T$ and $\mathcal{R}_B$ are the top $(y = L)$ and bottom $(y = 0)$ of the rectangle, respectively, $\mathcal{S}$ denotes the sides $(x = \pm R)$ of the rectangle, $\mathbf{n}$ is the outward unit normal to the boundary of $\mathcal{R}$, and $S_{ij} = \partial W/F_{ij}$ is the Piola-Kirchhoff stress tensor. For $\lambda \in (0, 1]$ this corresponds to the material being placed between two lubricated, rigid plates whose distance, λL, is then prescribed. In addition we require that our solution satisfy $\int_{\mathcal{R}} f_1(x, y)\, dxdy = 0$ in order to eliminate the trivial nonuniqueness of solutions that are translates of a given solution. Moreover, since the specimen must remain in contact with the lubricated plates we restrict our solutions to those that satisfy $S_{22}(\nabla \mathbf{f}(x, y)) \leq 0$ for $(x, y) \in \mathcal{R}_T \cup \mathcal{R}_B$, i.e., for $y = 0$ and $y = L$.

We assume that the body initially occupies a stress-free reference configuration. The existence of a unique minimizer $\mathbf{f}_\lambda$ of the energy among homogeneous deformations follows from the tension-extension inequalities together with standard growth conditions on W at zero and infinity. The smoothness in λ of the path $\mathbf{f}_\lambda$ is a consequence of the implicit function theorem (in $\mathbb{R}^2$). After noting that the boundary condition $(\mathbf{S}(\nabla \mathbf{f})\mathbf{n})_1 = 0$ on $\mathcal{R}_T \cup \mathcal{R}_B$ is equivalent to $\partial f_1/\partial y = 0$ on these lines, we make use of results of Valent [63] to reformulate the above problem as that of finding the zeros of a nonlinear mapping $\mathfrak{F}(\lambda, \mathbf{u})$ for $\mathbf{u}$ in an appropriate Banach space. The Banach spaces we use are the standard Sobolev spaces $W^{m,p}$ of functions whose first m weak derivatives to the p-th power are integrable. An essential point is that, following Davies [24], the boundary conditions $f_2 = 0$ and $\partial f_1/\partial y = 0$ allow one to eliminate the corners of the rectangle by periodically extending all of the functions to the infinite strip $[-R, R] \times \mathbb{R}$.

In §5 we analyze the linearized operator $\partial_{\mathbf{u}}\mathfrak{F}(\lambda,\mathbf{0})[\mathbf{v}]$, i.e.,

$$\operatorname{div}\mathsf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{v}]=\mathbf{g}\ \text{in}\ \mathcal{R},\qquad \begin{array}{ll} v_2=0\ \text{ on }\mathcal{R}_T, & (\mathsf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{v}]\mathbf{n})_1=0\ \text{ on }\mathcal{R}_T\cup\mathcal{R}_B,\\ v_2=0\ \text{ on }\mathcal{R}_B, & \mathsf{C}(\nabla\mathbf{f}_\lambda)[\nabla\mathbf{v}]\mathbf{n}=\mathbf{h}\ \text{ on }\mathcal{S}, \end{array} \tag{1.2}$$

for $\mathbf{g}$ and $\mathbf{h}$ each in an appropriate Sobolev space. Assuming that at each λ the elasticity tensor, $\mathsf{C}(\nabla\mathbf{f}_\lambda)$, is strongly elliptic and satisfies the complementing condition, the standard estimates of Agmon, Douglis, and Nirenberg [5] imply that, for such λ, this linear system of partial differential equations and linear boundary conditions can be viewed as a semi-Fredholm operator. If the elasticity tensor at the reference configuration is positive definite (on symmetric tensors) then standard results from the theory of elliptic partial differential equations (see, e.g., Fichera [28]) show that, at reference, $\partial_{\mathbf{u}}\mathfrak{F}(1,\mathbf{0})$ is a Fredholm operator of index zero. In particular, since this operator is injective it is also surjective. We next note that the one-parameter family of homogeneous solutions to the nonlinear problem induces a homotopy for the linearized operators; thus, it follows that (see, e.g., Kato [36, p. 235]), as long as each of the linearized operators along the solution path $\mathbf{f}_\lambda$ is strongly elliptic and satisfies the complementing condition, each is a Fredholm operator of index zero.

In §6 we use separation of variables to solve the linearized problem (1.2), with $\mathbf{g}=\mathbf{0}$ and $\mathbf{h}=\mathbf{0}$. We first recall the relevant results of Davies [24] and then use a different technique to give an alternate proof of the result in [24] that is most pertinent to our work, Theorem 6.3. This theorem demonstrates, in particular, that if the constitutive relation satisfies:

- The body becomes wider as it is compressed;[7]
- $\sqrt{\Phi_{,11}\,\Phi_{,22}}\ \geq\ \Phi_{,12}+\dfrac{\nu_1\Phi_{,1}-\nu_2\Phi_{,2}}{\nu_1^2-\nu_2^2}+\dfrac{\nu_2\Phi_{,1}-\nu_1\Phi_{,2}}{\nu_1^2-\nu_2^2}$ for all[8] ν_1 and ν_2.

Then at the largest value of $\lambda\in(0,1)$ at which instability occurs the linearized problem, (1.2) with $\mathbf{g}=\mathbf{0}$ and $\mathbf{h}=\mathbf{0}$, satisfies:

- The linear operator has a one-dimensional null space;
- The instability is asymmetric, that is, it is a buckling instability;
- The mode number is one, i.e., the horizontal displacement is of the form $\phi(x)\cos(\pi y/L)$.

In §7 we establish our bifurcation results. In Theorem 7.1 we prove that if the linearized problem is strongly elliptic and satisfies both Agmon's condition and the complementing

[7] Equivalently, the material must have a positive Poisson's ratio at each finite deformation $\mathbf{f}_\lambda$.

[8] Here, e.g., $\Phi_{,12}:=\Phi_{,12}(\nu_1,\nu_2):=\partial_{\nu_2}\partial_{\nu_1}\Phi(\nu_1,\nu_2)$. This hypothesis need only be satisfied at the eigenvalues of the homogeneous deformation $\mathbf{f}_\lambda$. Strong ellipticity implies that the roots of a certain biquadratic, (3.16) with $\tau=0$, have nonzero real part; this hypothesis implies that the biquadratic has real roots.

condition at a value of λ where an eigenvalue of the linearized problem has a one-dimensional eigenspace and, furthermore, this eigenvalue satisfies the strict crossing condition, (7.2), then a second solution branch bifurcates from the homogeneous branch. In Proposition 7.2 we show[9] that, in most cases, the second, third, and fourth derivatives of the stored energy, evaluated at the bifurcation point, determine whether the bifurcation is supercritical or subcritical. In Proposition 7.3 we establish that the sign of the energy difference between the bifurcated branch and the homogeneous branch is, in most cases, completely determined by the direction of the change of sign in the crossing condition and the direction of bifurcation (subcritical or supercritical).

In §8 we study the stored energy given by (1.1). We verify that for this constitutive relation the body becomes wider as it is compressed, the roots of the ellipticity biquadratic, (3.16) with $\tau = 0$, are real and the crossing condition is satisfied. Therefore, by Theorem 6.3 and Theorem 7.1, for these constitutive relations a second solution branch does indeed bifurcate from the homogeneous branch when the material is sufficiently compressed. Moreover, this new branch is associated with mode-one buckling.

In the Appendix for the convenience of the reader we gather together all of the assumptions used in the paper. We then present the proofs of the results in the paper that are of a purely technical nature.

An interesting problem that we have not addressed is whether the equilibrium solutions we construct are global minimizers of the elastic energy. The existence theory of Ball [7, 8] yields one or more deformations that are absolute minimizers of this energy. Except for the stress-free, linearization-stable reference configuration, it has not been determined[10] whether either the homogeneous solutions or the solutions given by the implicit function theorem coincide with the absolute minimizers of Ball.

Finally, we remark that this manuscript, sans introduction, was for the most part written in the late 1980's. Since that time it has gathered dust on our desk, except for the few times it was photocopied in answer to a request. At the urging of Tim Healey, Errol Montes Pizarro, and Pablo Negrón-Marrero we have finally finished the manuscript and also updated the references to include some of the interesting work that has taken place since that time.

2. Preliminaries; Korn's Inequality

We let $\mathrm{Lin} := \mathrm{Lin}(\mathbb{R}^2;\mathbb{R}^2)$ denote the space of all linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$ with inner product and norm, respectively, given by:

$$\mathbf{G}:\mathbf{H} := \mathrm{trace}\,(\mathbf{G}\mathbf{H}^{\mathrm{T}}), \qquad |\mathbf{G}|^2 := \mathbf{G}:\mathbf{G},$$

[9] See Triantafyllidis, Scherzinger, and Huang [61] for similar results.

[10] However, for the pure displacement problem results of Zhang [66] show that, in a neighborhood of a stress-free, linearization-stable reference configuration, these solutions coincide.

where $\mathbf{H}^{\mathrm{T}}$ denotes the transpose of $\mathbf{H}$. We write

$$\mathrm{Lin}^+ := \{\mathbf{H} \in \mathrm{Lin} \,:\, \det \mathbf{H} > 0\}\,,$$

where det denotes the determinant;

$$\mathrm{Orth}^+ := \left\{\mathbf{Q} \in \mathrm{Lin}^+ \,:\, \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}\right\}$$

is the set of orthogonal matrices with positive determinant, and $\mathbf{I}$ is the identity matrix. We denote by $\mathbf{a}\otimes\mathbf{b}$ the tensor product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$; in components $(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j$. We write (see Del Piero [26]) LinLin = Lin(Lin; Lin) for the space of all linear transformations from Lin into Lin; thus in components if $\mathsf{C} \in$ LinLin and $\mathbf{A} \in$ Lin

$$(\mathsf{C}[\mathbf{A}])_{ij} = \sum_{k,l=1}^{2} \mathsf{C}_{ijkl} A_{kl}.$$

We write ∇ and div for the gradient and divergence operators in $\mathbb{R}^2$; for a vector field $\mathbf{u}$, $\nabla\mathbf{u}$ is the tensor field with components

$$(\nabla\mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}.$$

For a tensor field $\mathbf{S}$, $\operatorname{div}\mathbf{S}$ is the vector field with components

$$(\operatorname{div}\mathbf{S})_i = \frac{\partial S_{i1}}{\partial x_1} + \frac{\partial S_{i2}}{\partial x_2}.$$

Also, with the notation $x = x_1$ and $y = x_2$ we denote by ∂_x and ∂_y the partial derivatives with respect to x and y, respectively. Given any function $\mathsf{A} : \mathcal{U} \to \mathrm{Lin}$, defined on an open set $\mathcal{U} \subset \mathrm{Lin}$, we denote by $\frac{\mathrm{d}}{\mathrm{d}\mathbf{F}}\mathsf{A}(\mathbf{F})[\mathbf{H}]$ the Frechét derivative of A at $\mathbf{F} \in \mathcal{U}$ in the direction of $\mathbf{H} \in$ Lin, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{F}}\mathsf{A}(\mathbf{F})[\mathbf{H}] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{A}(\mathbf{F} + t\mathbf{H})\right|_{t=0}.$$

More generally, given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ and a mapping $\mathfrak{F} : \mathcal{U} \to \mathcal{Y}$, defined on an open set $\mathcal{U} \subset \mathcal{X}$, we denote by $(\partial_{\mathbf{u}}\mathfrak{F})(\mathbf{u})[\mathbf{v}]$ the Frechét derivative of $\mathfrak{F}$ at $\mathbf{u} \in \mathcal{U}$ in the direction of $\mathbf{v} \in \mathcal{X}$, i.e.,

$$(\partial_{\mathbf{u}}\mathfrak{F})(\mathbf{u})[\mathbf{v}] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{F}(\mathbf{u} + t\mathbf{v})\right|_{t=0}.$$

The space of *bounded linear* maps from the Banach space $\mathcal{X}$ into the Banach space $\mathcal{Y}$ will be denoted by $\mathrm{BL}(\mathcal{X};\mathcal{Y})$.

If $\mathcal{B} \subset \mathbb{R}^2$ is a locally Lipschitz, bounded, open region we let $C^m(\overline{\mathcal{B}};\mathbb{R}^2)$, $m \in \mathbb{N}$, denote the set of $\mathbb{R}^2$-valued functions with m continuous derivatives in $\overline{\mathcal{B}}$. $C^m(\overline{\mathcal{B}};\mathbb{R}^2)$ is a Banach space under the supremum norm $\|\cdot\|_{C^m}$. The corresponding Sobolev spaces for vector-valued functions $\mathbf{u} : \mathcal{B} \to \mathbb{R}^2$ are denoted $W^{m,p}(\mathcal{B};\mathbb{R}^2)$, $1 \le p < \infty$, $m \in \mathbb{N}$ where each component of $\mathbf{u}$ is an element of $W^{m,p}(\mathcal{B})$; the norm is

$$\|\mathbf{u}\|_{m,p,\mathcal{B}} = \left(\|u_1\|^p_{W^{m,p}(\mathcal{B})} + \|u_2\|^p_{W^{m,p}(\mathcal{B})}\right)^{1/p}.$$

Note $W^{0,p}(\mathcal{B};\mathbb{R}^2) = L^p(\mathcal{B};\mathbb{R}^2)$. Similarly, for matrix-valued functions $\mathbf{F} : \mathcal{B} \to \mathrm{Lin}$, we denote by $W^{m,p}(\mathcal{B};\mathrm{Lin})$ the Sobolev space with each component of $\mathbf{F}$ in $W^{m,p}(\mathcal{B})$; $C^m(\overline{\mathcal{B}};\mathrm{Lin})$ is similarly defined. We note the trace and embedding theorems (see, e.g., [1, 2] or [42]): the trace maps $W^{m,p}(\mathcal{B};\mathbb{R}^2)$ into $W^{m-1/p,p}(\partial\mathcal{B};\mathbb{R}^2)$ for $p \in (1,\infty)$ and integers $m \ge 1$, where $\partial\mathcal{B}$ is the boundary of $\mathcal{B}$; if $p \in [1,\infty)$ and $m \in \mathbb{Z}^+$ satisfies $m > 2/p$, $W^{m,p}(\mathcal{B};\mathbb{R}^2)$ is continuously embedded in $C^0(\overline{\mathcal{B}};\mathbb{R}^2)$.

We fix $R > 0$. The infinite strip of width $2R$ surrounding the y-axis will be denoted by

$$\Omega := (-R,R) \times (-\infty,\infty) \subset \mathbb{R}^2.$$

We use the notation $W^{m,p}_{\mathrm{loc}}(\Omega;A)$ for the equivalence class of functions $\mathbf{u} : \Omega \to A$ that satisfy $\mathbf{u} \in W^{m,p}(\Omega \cap D(\mathbf{x},\rho);A)$ for every open disk $D(\mathbf{x},\rho) \subset \mathbb{R}^2$, where A will be $\mathbb{R}$, $\mathbb{R}^2$, Lin, Lin^+, or LinLin, as needed. Similarly, we write $W^{m-1/p,p}_{\mathrm{loc}}(\partial\Omega;A)$ for the local Sobolev spaces on the boundary. We fix $L > 0$ and define the *rectangle* $\mathcal{R} := (-R,R) \times (0,L)$. We denote by

$$\mathbf{e}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

the unit coordinate vectors in $\mathbb{R}^2$. Finally we quote the versions of Korn's inequality that we use in this paper.

Proposition 2.1 (Korn's Inequalities). *Let $\mathcal{B} \subset \mathbb{R}^2$ be a locally Lipschitz, bounded, open region. Let*

$$(\nabla\mathbf{v})_s := \frac{1}{2}\left(\nabla\mathbf{v} + [\nabla\mathbf{v}]^{\mathrm{T}}\right)$$

denote the symmetric part of the gradient of $\mathbf{v}$. Then there exists a constant $k = k(\mathcal{B}) > 0$ such that

$$\int_{\mathcal{B}} |(\nabla\mathbf{v})_s|^2\,d\mathbf{x} + \int_{\mathcal{B}} |\mathbf{v}|^2\,d\mathbf{x} \ge k\,\|\mathbf{v}\|^2_{1,2,\mathcal{B}} \qquad \textit{for all}\ \ \mathbf{v} \in W^{1,2}(\mathcal{B};\mathbb{R}^2) \tag{2.1}$$

and, consequently,

$$\int_{\mathcal{B}} |(\nabla\mathbf{v})_s|^2\,d\mathbf{x} \ge \hat{k}\,\|\mathbf{v}\|^2_{1,2,\mathcal{B}} \tag{2.2}$$

for all $\mathbf{v} \in W^{1,2}(\mathcal{B};\mathbb{R}^2)$ that satisfy $\mathbf{v} = \mathbf{0}$ on $\mathcal{D}$ (in the sense of trace), where $\mathcal{D} \subset \partial\mathcal{B}$ is any nonempty relatively open set and $\hat{k} = \hat{k}(\mathcal{B},\mathcal{D}) > 0$ is a constant. Moreover, if $\mathcal{B} = \mathcal{R} = (-R,R) \times (0,L)$ then (2.2) is satisfied by all

$$\mathbf{v} \in \mathrm{Var} = \{\mathbf{v} \in W^{1,2}(\mathcal{R};\mathbb{R}^2) : \int_{\mathcal{R}} v_1\,d\mathbf{x} = 0 \ \ \textit{and}\ \ v_2(\cdot,0) = v_2(\cdot,L) = 0\}. \tag{2.3}$$

For a proof of (2.1) see, e.g., Fichera [28] or Nitsche [43]. Inequality (2.2) then follows from (2.1) in a standard manner, see, e.g., Ciarlet [22, pp. 292–294].

3. The Constitutive Relation

We consider a body that, for convenience, we identify with the region $\overline{\mathcal{B}}$ that it occupies in a fixed homogeneous reference configuration in $\mathbb{R}^2$. A *deformation* $\mathbf{f}$ of the body is a member of the space

$$\mathrm{Def} = \left\{ \mathbf{f} \in C^1(\overline{\mathcal{B}};\mathbb{R}^2) : \det \nabla\mathbf{f} > 0 \text{ on } \overline{\mathcal{B}} \right\}.$$

We assume that the body is composed of a homogeneous, compressible, hyperelastic material with continuous response function $W : \mathrm{Lin}^+ \to [0,\infty)$. W gives the *stored energy* $W(\nabla\mathbf{f}(\mathbf{x}))$ at each point $\mathbf{x} \in \overline{\mathcal{B}}$ when the body is deformed by $\mathbf{f} \in \mathrm{Def}$. We will assume that:

(H1) *$W(\mathbf{F}) = \Phi(\nu_1,\nu_2)$ for all $\mathbf{F} \in \mathrm{Lin}^+$, where $\Phi \in C^2(\mathbb{R}^+ \times \mathbb{R}^+;[0,\infty))$ and ν_1 and ν_2 are the principal stretches, i.e., the eigenvalues of $\sqrt{\mathbf{F}\mathbf{F}^{\mathrm{T}}}$.*

Consequently, W satisfies the *axiom of frame-indifference* and the material which occupies $\overline{\mathcal{B}}$ is *isotropic*; thus

$$W(\mathbf{QF}) = W(\mathbf{F}), \qquad W(\mathbf{FQ}) = W(\mathbf{F}) \tag{3.1}$$

for all[11] $\mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{Q} \in \mathrm{Orth}^+$.

The following result is then due to Ball [9, Thms. 6.4 and 6.9] and Sylvester [59]. See also Chadwick and Ogden [19, 18] and Šilhavý [52].

Proposition 3.1. *Let (H1) be satisfied. Then $W \in C^2(\mathrm{Lin}^+;[0,\infty))$. Moreover, the derivatives*

$$\mathbf{S}(\mathbf{F}) := \frac{\mathrm{d}}{\mathrm{d}\mathbf{F}}W(\mathbf{F}), \qquad \mathsf{C}(\mathbf{F}) := \frac{\mathrm{d}^2}{\mathrm{d}\mathbf{F}^2}W(\mathbf{F}) \tag{3.2}$$

satisfy[12]

$$\mathbf{S}(\mathbf{D}_{\mu,\lambda}) = \begin{bmatrix} \Phi,_1 & 0 \\ 0 & \Phi,_2 \end{bmatrix}, \tag{3.3}$$

$$\begin{aligned} \mathbf{H} : \mathsf{C}(\mathbf{D}_{\mu,\lambda})[\mathbf{H}] = {} & \left[\frac{\mu\Phi,_1 - \lambda\Phi,_2}{\mu^2 - \lambda^2}\right]\left[(H_{12})^2 + (H_{21})^2\right] + 2\left[\frac{\lambda\Phi,_1 - \mu\Phi,_2}{\mu^2-\lambda^2}\right]H_{12}H_{21} \\ & + \Phi,_{11}(H_{11})^2 + \Phi,_{22}(H_{22})^2 + 2\Phi,_{12}H_{11}H_{22}, \end{aligned} \tag{3.4}$$

[11] Equation (3.1) implies that $W(\mathbf{QFQ}) = W(\mathbf{F})$ for all $\mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{Q} \in \mathrm{Orth}^-$. See footnote 46.

[12] When $\mu = \lambda$ one interprets the difference quotients as derivatives.

where

$$\mathbf{D}_{\mu,\lambda} = \mathrm{diag}\{\mu, \lambda\} := \begin{bmatrix} \mu & 0 \\ 0 & \lambda \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

for every diagonal matrix $\mathbf{D}_{\mu,\lambda} \in \mathrm{Lin}^+$ *and every* $\mathbf{H} \in \mathrm{Lin}$, *respectively. Here, e.g.,* $\Phi,_{ij} = \Phi,_{ij}(\mu, \lambda)$. *Moreover, for any* $m \in \mathbb{Z}^+$ *with* $m \geq 3$

$$W \in C^m(\mathrm{Lin}^+, [0,\infty)) \ \ \textit{if and only if} \ \ \Phi \in C^m(\mathbb{R}^+ \times \mathbb{R}^+, [0,\infty)). \tag{3.5}$$

In addition, there exists $\sigma \in C^1(\mathbb{R}^+ \times \mathbb{R}^+; [0,\infty))$ *such that*

$$W(\mathbf{F}) = \sigma\left(\frac{1}{2}\mathbf{F}:\mathbf{F},\ \det \mathbf{F}\right) \ \textit{for all} \ \ \mathbf{F} \in \mathrm{Lin}^+$$

and hence

$$\mathbf{S}(\mathbf{F}) = \begin{bmatrix} (\sigma,_1)F_{11} + (\sigma,_2)F_{22} & (\sigma,_1)F_{12} - (\sigma,_2)F_{21} \\ (\sigma,_1)F_{21} - (\sigma,_2)F_{12} & (\sigma,_1)F_{22} + (\sigma,_2)F_{11} \end{bmatrix}. \tag{3.6}$$

The derivatives given in (3.2) are called, respectively, the (Piola-Kirchhoff) *stress* and *elasticity* tensors; for each $\mathbf{F} \in \mathrm{Lin}^+$, $\mathsf{C}(\mathbf{F}) \in \mathrm{LinLin}$ is a linear mapping that is *symmetric*:

$$\mathbf{K}:\mathsf{C}(\mathbf{F})[\mathbf{H}] = \mathbf{H}:\mathsf{C}(\mathbf{F})[\mathbf{K}] \tag{3.7}$$

for all $\mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{H}, \mathbf{K} \in \mathrm{Lin}$. If W is sufficiently smooth we also define the tensors[13]

$$\mathbb{D}(\mathbf{F}) := \frac{\mathrm{d}^3}{\mathrm{d}\mathbf{F}^3} W(\mathbf{F}), \qquad \mathbb{E}(\mathbf{F}) := \frac{\mathrm{d}^4}{\mathrm{d}\mathbf{F}^4} W(\mathbf{F}), \tag{3.8}$$

and note the symmetry property

$$\mathbf{K}:\mathbb{D}(\mathbf{F})[\mathbf{H}, \mathbf{L}] = \mathbf{K}:\mathbb{D}(\mathbf{F})[\mathbf{L}, \mathbf{H}] = \mathbf{H}:\mathbb{D}(\mathbf{F})[\mathbf{K}, \mathbf{L}] \tag{3.9}$$

for all $\mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{H}, \mathbf{K}, \mathbf{L} \in \mathrm{Lin}$.

We call the scalars $s_i := \Phi,_i$ the *principal stresses.*[14] The *tension-extension inequality* is the requirement that each principal stress is an increasing function of the corresponding principal stretch. Slightly stronger than this is the requirement

$$\Phi,_{11} = \frac{\partial s_1}{\partial \nu_1} > 0, \qquad \Phi,_{22} = \frac{\partial s_2}{\partial \nu_2} > 0. \tag{3.10}$$

[13] We view $\mathbb{D}(\mathbf{F})$ and $\mathbb{E}(\mathbf{F})$ as a bilinear $\mathbb{D}(\mathbf{F}) : \mathrm{Lin} \times \mathrm{Lin} \to \mathrm{Lin}$ and a trilinear $\mathbb{E}(\mathbf{F}) : \mathrm{Lin} \times \mathrm{Lin} \times \mathrm{Lin} \to \mathrm{Lin}$ mapping, respectively.

[14] The eigenvalues t_i of the (Cauchy) stress $\mathbf{T} = \mathbf{S}\mathbf{F}^{\mathrm{T}} / \det \mathbf{F}$ are usually referred to as the principal stresses. However, since $t_1 = s_1/\nu_2$ and $t_2 = s_2/\nu_1$ the tension-extension inequality does not depend on this choice.

If the body in its reference configuration is *natural*, i.e., if $\mathbf{S}(\mathbf{I}) = \mathbf{0}$, then[15] $(3.1)_1$ implies that for all $\mathbf{H} \in \mathrm{Lin}$,

$$\mathbf{H} : \mathsf{C}(\mathbf{I})[\mathbf{H}] = \mathbf{H}_s : \mathsf{C}(\mathbf{I})[\mathbf{H}_s],$$

where, as in Korn's inequality, $\mathbf{H}_s := \frac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}})$. In this case we say that $\mathsf{C}(\mathbf{I})$ is *positive definite* (on its restriction to symmetric tensors) if there is a constant $k > 0$ such that

$$\mathbf{E} : \mathsf{C}(\mathbf{I})[\mathbf{E}] \geq k|\mathbf{E}|^2$$

for all $\mathbf{E} \in \mathrm{Lin}$ that satisfy $\mathbf{E} = \mathbf{E}^{\mathrm{T}}$. Given $\mathbf{f} \in \mathrm{Def}$ and $\mathbf{x}_0 \in \overline{\mathcal{B}}$ we write $\mathsf{C}_0 := \mathsf{C}(\nabla \mathbf{f}(\mathbf{x}_0))$ and say that C_0 satisfies the *strong-ellipticity condition* provided there is a constant $k > 0$ such that

$$\mathbf{a} \otimes \mathbf{b} : \mathsf{C}_0[\mathbf{a} \otimes \mathbf{b}] \geq k|\mathbf{a}|^2|\mathbf{b}|^2 \quad \text{for all} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^2.$$

For $\mathbf{x}_0 \in \partial\mathcal{B}$ we write $\mathbf{n}_0 = \mathbf{n}(\mathbf{x}_0)$ for the outward unit normal to $\partial\mathcal{B}$ at $\mathbf{x}_0$, we let $\mathcal{H}$ denote the half-space

$$\mathcal{H} := \left\{ \mathbf{x} \in \mathbb{R}^2 \, : \, (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}_0 < 0 \right\},$$

and we consider the problem: find $\mathbf{w} : \overline{\mathcal{H}} \to \mathbb{R}^2$ that satisfies

$$\begin{aligned} \operatorname{div} \mathsf{C}_0[\nabla \mathbf{w}] &= \tau \mathbf{w} \quad \text{in } \mathcal{H}, \\ \mathsf{C}_0[\nabla \mathbf{w}]\mathbf{n}_0 &= \mathbf{0} \quad \text{on } \partial\mathcal{H}, \end{aligned} \tag{3.11}$$

where $\tau \geq 0$. We seek solutions of (3.11) that are *bounded exponentials*, i.e.,

$$\mathbf{w}(\mathbf{x}) = \mathbf{z}(-(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}_0) \exp(\mathrm{i}(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{t})$$

for some unit vector $\mathbf{t} \in \mathbb{R}^2$ that is tangent to $\partial\mathcal{H}$ (i.e., $\mathbf{t} \cdot \mathbf{n}_0 = 0$) and some $\mathbf{z} \in C^2([0, \infty); \mathbb{C}^2)$ that satisfies $\sup_s |\mathbf{z}(s)| < \infty$. We say that the pair $(\mathsf{C}_0, \mathbf{n}_0)$ satisfies *Agmon's condition* provided[16] that for every real $\tau > 0$ the only bounded exponential solution of (3.11) is $\mathbf{w} = \mathbf{0}$. Furthermore we say that $(\mathsf{C}_0, \mathbf{n}_0)$ satisfies the *complementing condition*[17] if (3.11) with $\tau = 0$ has no nontrivial bounded exponential solution.[18] We note that the existence of exponential solutions of (3.11) is determined solely by $\mathbf{n}_0$ and the components of C_0 and as such, Agmon's condition and the complementing condition are *algebraic conditions*[19] on

[15] See, e.g., Gurtin [33, §29].
[16] Cf. Agmon [3], de Figueiredo [29], Friedman [30].
[17] Cf. Agmon, Douglis, and Nirenberg [5], Friedman [30].
[18] For a physical interpretation of these conditions in terms of Rayleigh waves in a half-space see [56] or Thompson [60]. In particular, Thompson notes that Agmon's condition is equivalent to the requirement that no Rayleigh wave $\mathbf{w}(\mathbf{x}) \exp(\imath\nu t)$ in $\mathcal{H}$ can propagate with pure imaginary speed ν; and similarly the complementing condition requires $\nu \neq 0$. Thus these conditions have an interpretation in terms of dynamic stability.
[19] See also Mielke and Sprenger [41]. See [57] for the relation with the concept of quasiconvexity at the boundary due to Ball and Marsden [10]. For linear elasticity see [55].

the pair $(\mathsf{C}_0, \mathbf{n}_0)$. Finally, it is clear that if $(\mathsf{C}_0, \mathbf{n}_0)$ satisfies one of these conditions then $(\mathsf{C}_0, -\mathbf{n}_0)$ satisfies the same condition, a property that will be useful for the rectangular body used in the analysis in this paper.

In the sequel we will be interested in deformations whose gradients are constant diagonal matrices. Thus if we take the derivative of (3.4) with respect to $\mathbf{H}$ we find that

$$\begin{aligned} \mathsf{C}(\mathbf{D}_{\mu,\lambda})\,[\mathbf{H}] = H_{11} \begin{bmatrix} K & 0 \\ 0 & N \end{bmatrix} + H_{12} \begin{bmatrix} 0 & P \\ M-N & 0 \end{bmatrix} \\ + H_{21} \begin{bmatrix} 0 & M-N \\ P & 0 \end{bmatrix} + H_{22} \begin{bmatrix} N & 0 \\ 0 & T \end{bmatrix}, \end{aligned} \tag{3.12}$$

where

$$K := \Phi,_{11}\ , \qquad T := \Phi,_{22}\ , \qquad N := \Phi,_{12}\ , \tag{3.13a}$$

$$P := \left[\frac{\mu\Phi,_1 - \lambda\Phi,_2}{\mu^2 - \lambda^2}\right], \quad M := N + \left[\frac{\lambda\Phi,_1 - \mu\Phi,_2}{\mu^2 - \lambda^2}\right]. \tag{3.13b}$$

Proposition 3.2. *Let (H1) be satisfied.*

(i) $\mathsf{C}(\mathbf{D}_{\mu,\lambda})$ *satisfies the strong-ellipticity condition if and only if*

$$K > 0, \quad P > 0, \quad T > 0, \quad P + \sqrt{KT} > |M|\,.$$

(ii) Assume $\mathsf{C}(\mathbf{D}_{\mu,\lambda})$ *satisfies the strong-ellipticity condition. Then the pair* $(\mathsf{C}(\mathbf{D}_{\mu,\lambda}), \mathbf{e}_1)$ *satisfies the complementing condition if and only if*

$$A = A(\mu,\lambda) := P[KT - N^2] + \sqrt{KT}\,[P^2 - (N-M)^2] \neq 0. \tag{3.14}$$

(iii) Assume $\mathsf{C}(\mathbf{D}_{\mu,\lambda})$ *satisfies the strong-ellipticity condition. Then the pair* $(\mathsf{C}(\mathbf{D}_{\mu,\lambda}), \mathbf{e}_1)$ *satisfies Agmon's condition if and only if*

$$A(\mu,\lambda) = P[KT - N^2] + \sqrt{KT}\,[P^2 - (N-M)^2] \geq 0.$$

(iv) Assume the reference configuration is natural. Then $\mathsf{C}(\mathbf{I})$ *is positive definite if and only if* $P > 0$ *and* $M > 0$*. Furthermore, in this case* $(\mathsf{C}(\mathbf{I}), \mathbf{n})$ *satisfies the complementing and Agmon's conditions for all unit vectors* $\mathbf{n} \in \mathbb{R}^2$*. In particular with the notation of parts (ii) and (iii)*

$$A = A(1,1) = 4MP^2 > 0.$$

Remarks. 1. Part (i) is due to Knowles and Sternberg [37]. Part (ii) is due to Davies [24, Theorem 8.1]. The strengthened tension-extension inequalities, (3.10), reduce to $K > 0$ and $T > 0$. The inequality $P = \sigma,_1 > 0$ is the (strict) Baker-Ericksen inequality (see [53] and Truesdell and Noll [62, §51]).

2. If the reference configuration is natural and $\mathsf{C}(\mathbf{I})$ is positive definite it follows from (iv) and the continuity of C (i.e., (H1)) that there is a neighborhood of the reference configuration in Def where $\mathsf{C}(\nabla\mathbf{f}(\mathbf{x}))$ satisfies the strong-ellipticity condition for every $\mathbf{x} \in \overline{\mathcal{B}}$ and $(\mathsf{C}(\nabla\mathbf{f}(\mathbf{x})), \mathbf{n})$ satisfies Agmon's condition and the complementing condition for every $\mathbf{x} \in \partial\mathcal{B}$ and all unit vectors $\mathbf{n} \in \mathbb{R}^2$.

Proof of Proposition 3.2. Part (i) follows as in, e.g., [24, 37, 53, 54]. To prove (ii) and (iii) we substitute $\mathbf{H} = \nabla\mathbf{w}$ into (3.12) and the result into (3.11) to arrive at

$$K\,\partial_{xx}w_1 + P\,\partial_{yy}w_1 + M\,\partial_{xy}w_2 = \tau w_1 \quad \text{in } \mathcal{H}, \tag{3.15a}$$

$$P\,\partial_{xx}w_2 + T\,\partial_{yy}w_2 + M\,\partial_{xy}w_1 = \tau w_2 \quad \text{in } \mathcal{H}, \tag{3.15b}$$

$$K\,\partial_x w_1 + N\,\partial_y w_2 = 0 \quad \text{on } \partial\mathcal{H}, \tag{3.15c}$$

$$P\,\partial_x w_2 + (M - N)\,\partial_y w_1 = 0 \quad \text{on } \partial\mathcal{H}, \tag{3.15d}$$

where $\mathcal{H} = \{(x, y) : x < 0\}$. To analyze the solutions of this system we note the characteristic equation of (3.15a)–(3.15b) is

$$KP\,r^4 + (M^2 - KT_\tau - PP_\tau)\,r^2 + P_\tau T_\tau = 0, \tag{3.16}$$

where

$$P_\tau := P + \tau \geq P, \qquad T_\tau := T + \tau \geq T, \tag{3.17}$$

and the roots $r = r^{(\tau)}$ come in pairs $\pm r_1$ and $\pm r_2$ with[20] $\mathrm{Re}\{r_1\} > 0$ and $\mathrm{Re}\{r_2\} > 0$.

If $r_1 \neq r_2$ and $M \neq 0$ the bounded exponential solutions of (3.15a)–(3.15b), as $x \to -\infty$, are given by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_j = \begin{bmatrix} -Mr_j \exp(r_j x + \mathrm{i}y) \\ \mathrm{i}(P_\tau - Kr_j^2)\exp(r_j x + \mathrm{i}y) \end{bmatrix} \tag{3.18}$$

for $j = 1, 2$. The substitution of a linear combination of these solutions into (3.15c)–(3.15d) implies that the vectors

$$\left[-KMr^2 + N(Kr^2 - P_\tau), -\mathrm{i}P(Kr^2 - P_\tau)r - \mathrm{i}(M - N)Mr\right],$$

at $r = r_1^{(\tau)}$ and $r = r_2^{(\tau)}$ are linearly independent in $\mathbb{C}^2$ if and only if Agmon's condition ($\tau > 0$) or the complementing condition ($\tau = 0$) is satisfied. This is equivalent to $\det \mathbf{C}_\tau \neq 0$, where

$$\mathbf{C}_\tau := \begin{bmatrix} p_1^{(\tau)}(r_1^{(\tau)}) & p_1^{(\tau)}(r_2^{(\tau)}) \\ p_2^{(\tau)}(r_1^{(\tau)}) & p_2^{(\tau)}(r_2^{(\tau)}) \end{bmatrix}$$

[20] The strong ellipticity condition implies $\mathrm{Re}\{r_1^{(\tau)}\} \neq 0$ and $\mathrm{Re}\{r_2^{(\tau)}\} \neq 0$ for all $\tau \geq 0$.

and

$$p_1^{(\tau)}(r) := -KM\,r^2 + N(Kr^2 - P_\tau), \tag{3.19a}$$
$$p_2^{(\tau)}(r) := P(Kr^2 - P_\tau)r + (M-N)Mr. \tag{3.19b}$$

If $r_1 = r_2$ and $M \neq 0$ the bounded exponential solutions of (3.15a)–(3.15b) are given by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_1 \text{ as in (3.18),}$$

and

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_2 = \frac{\partial}{\partial r}\begin{bmatrix} -Mr\exp(rx+\mathrm{i}y) \\ \mathrm{i}(P_\tau - Kr^2)\exp(rx+\mathrm{i}y) \end{bmatrix}\Bigg|_{r=r_1}.$$

A computation similar to the previous shows that the matrix

$$\mathbf{C}_\tau^* := \begin{bmatrix} p_1^{(\tau)}(r_1^{(\tau)}) & \left(\frac{d}{dr}p_1^{(\tau)}(r)\right)\Big|_{r=r_1^{(\tau)}} \\ p_2^{(\tau)}(r_1^{(\tau)}) & \left(\frac{d}{dr}p_2^{(\tau)}(r)\right)\Big|_{r=r_1^{(\tau)}} \end{bmatrix}$$

is nonsingular if and only if Agmon's condition ($\tau > 0$) or the complementing condition ($\tau = 0$) is satisfied, where $p_1^{(\tau)}$ and $p_2^{(\tau)}$ are given by (3.19).

If we take the determinant of $\mathbf{C}_\tau$ and $\mathbf{C}_\tau^*$, respectively, and simplify we find, with the aid of (3.16), that

$$\det \mathbf{C}_\tau = -M\hat{A}(\tau)\sqrt{\frac{P_\tau}{P}}\left(r_2^{(\tau)} - r_1^{(\tau)}\right), \qquad \det \mathbf{C}_\tau^* = -M\hat{A}(\tau)\sqrt{\frac{P_\tau}{P}}, \tag{3.20}$$

where

$$\hat{A}(\tau) := \sqrt{PP_\tau}\left[KT_\tau - N^2\right] + \sqrt{KT_\tau}\left[PP_\tau - (N-M)^2\right]. \tag{3.21}$$

(ii)($M \neq 0$). We now let $\tau = 0$ in (3.20) and (3.21) to conclude, with the aid of (3.17) that $\det \mathbf{C}_0 = -\hat{A}(0)M(r_2^{(0)} - r_1^{(0)})$ and $\det \mathbf{C}_0^* = -\hat{A}(0)M$. Therefore, if $M \neq 0$ it follows that $\hat{A}(0) \neq 0$ if and only if the complementing condition is satisfied.

(iii)($M \neq 0$). We note that, by (3.17) and (3.21),

$$\frac{\hat{A}(\tau)}{\sqrt{PP_\tau KT_\tau}} = \left[\sqrt{K(T+\tau)} - \frac{N^2}{\sqrt{K(T+\tau)}}\right] + \left[\sqrt{P(P+\tau)} - \frac{(N-M)^2}{\sqrt{P(P+\tau)}}\right]$$

is an increasing function of $\tau \in [0,\infty)$ that becomes infinite as $\tau \to +\infty$. Therefore $\hat{A}(\tau)$ is nonzero for all $\tau > 0$ if and only if $A = \hat{A}(0) \geq 0$, which, in view of (3.14) and (3.20) completes the proof of (iii) when $M \neq 0$.

If $M = 0$ the bounded exponential solutions of (3.15a)–(3.15b), as $x \to -\infty$, are given by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_1 = \begin{bmatrix} \exp(x\sqrt{\frac{P_\tau}{K}} + \mathrm{i}y) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_2 = \begin{bmatrix} 0 \\ \exp(x\sqrt{\frac{T_\tau}{P}} + \mathrm{i}y) \end{bmatrix},$$

where P_τ and T_τ are given by (3.17). The substitution of a linear combination into (3.15c)–(3.15d) implies that the matrix

$$\mathbf{C}_\tau^\dagger := \begin{bmatrix} \sqrt{P_\tau K} & N \\ N & \sqrt{T_\tau P} \end{bmatrix}$$

is nonsingular if and only if Agmon's condition ($\tau > 0$) or the complementing condition ($\tau = 0$) is satisfied.

(ii)($M = 0$). We take the determinant of $\mathbf{C}_0^\dagger$ to conclude, with the aid of (3.14), that

$$\hat{A}(0) = \left(P + \sqrt{TK}\right) \det[\,\mathbf{C}_0^\dagger\,] \neq 0, \tag{3.22}$$

which completes the proof of (ii).

(iii)($M = 0$). We note that

$$\det \mathbf{C}_\tau^\dagger = \sqrt{PK(T+\tau)(P+\tau)} - N^2 \geq P\sqrt{KT} - N^2 = \det \mathbf{C}_0^\dagger$$

and hence $\det \mathbf{C}_\tau^\dagger$ is nonzero for all $\tau > 0$ if and only if $\det \mathbf{C}_0^\dagger \geq 0$, which together with (3.22) completes the proof of (iii).

(iv) Since $\mathbf{S}(\mathbf{I}) = \mathbf{0}$ it follows from (3.6) that $\sigma,_1(1,1) + \sigma,_2(1,1) = 0$ and hence, in view of (3.13a) and (3.13b) with $\mu = \lambda = 1$, $K = T = M + P$ and $N = M - P$. Therefore, by (3.12),

$$\begin{aligned} \mathbf{H} : \mathsf{C}(\mathbf{I})\,[\mathbf{H}] &= (M+P)\left(H_{11}^2 + H_{22}^2\right) + 2(M-P)H_{11}H_{22} + P\left(H_{12} + H_{21}\right)^2 \\ &= M\left(H_{11} + H_{22}\right)^2 + P\left(H_{11} - H_{22}\right)^2 + P\left(H_{12} + H_{21}\right)^2, \end{aligned}$$

which is positive for all symmetric $\mathbf{H}_s = \frac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}}) \neq \mathbf{0}$ if and only if $M > 0$ and $P > 0$.

Next, let $\mathbf{n}$ be a unit vector in $\mathbb{R}^2$ and consider the half-space and half-disk, respectively,

$$\mathcal{H} = \left\{\mathbf{x} \in \mathbb{R}^2 \,:\, \mathbf{x}\cdot\mathbf{n} < 0\right\}, \qquad \mathcal{HD} = \{\mathbf{x} \in \mathcal{H} \,:\, |\mathbf{x}| < 1\}.$$

Then $\mathsf{C}(\mathbf{I})$ positive definite implies, for some $c > 0$,

$$\int_{\mathcal{HD}} \nabla\mathbf{w} : \mathsf{C}(\mathbf{I})\,[\nabla\mathbf{w}]\; d\mathbf{x} \geq c\,\|(\nabla\mathbf{w})_s\|_{0,2,\mathcal{HD}}^2 \tag{3.23}$$

for all $\mathbf{w} \in W^{1,2}(\mathcal{HD};\mathbb{R}^2)$ such that $\mathbf{w} = \mathbf{0}$ on $\mathcal{H} \cap \partial(\mathcal{HD})$. Korn's inequality (2.2) shows we may replace the right side of (3.23) with $c_1 \|\mathbf{w}\|^2_{1,2,\mathcal{HD}}$ ($c_1 > 0$). The desired results are now well known (see, e.g., [56, Theorems 1 and 3]).

Finally, by (3.14) and since $K = T = M + P$ and $N = M - P$,

$$A(1,1) = P[(M+P)^2 - (M-P)^2] = 4MP^2 > 0.$$

□

4. The Nonlinear Problem

We assume that our two-dimensional body occupies the *rectangular region*

$$\mathcal{R} = \{(x,y) : -R < x < R,\ 0 < y < L\}$$

in a fixed homogeneous reference configuration. We label the *top, bottom*, and *sides* as $\mathcal{R}_T$, $\mathcal{R}_B$, and $\mathcal{S}$, respectively, i.e.,

$$\begin{aligned} \mathcal{R}_T &= \{(x,y) : -R < x < R,\ y = L\}, \\ \mathcal{R}_B &= \{(x,y) : -R < x < R,\ y = 0\}, \\ \mathcal{S} &= \{(x,y) : x = \pm R,\ 0 < y < L\}. \end{aligned}$$

We suppose that hard loading is applied at the top $\mathcal{R}_T$ with compressive displacement supplied by pressing on perfectly lubricated plates and that the body rests on a lubricated flat surface fixed at $\mathcal{R}_B$. The sides $\mathcal{S}$ are free. Let $\lambda \in (0,1]$, the *load modulus*, denote the ratio of height (in the y-direction) after compression to that prior to loading.

According to the principle of minimum energy the body tends to a deformation $\mathbf{f}$ that minimizes the *total energy*

$$E(\mathbf{f}) = \int_{\mathcal{R}} W(\nabla \mathbf{f}(\mathbf{x}))\, d\mathbf{x}, \tag{4.1}$$

where

$$\mathbf{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}, \quad (x,y) \in \mathcal{R}$$

is allowed to vary over a class of functions smooth[21] enough to be included in Def and that satisfy the boundary conditions $f_2(x,L) = \lambda L$ on $\mathcal{R}_T$ and $f_2(x,0) = 0$ on $\mathcal{R}_B$. If we suppose

[21] The C^1 smoothness of functions in Def assumes that their determinant is positive and the deformations preserve orientation (see, e.g., Ciarlet [22]).

that $\mathbf{f}$ minimizes E, then the *equations of equilibrium* are satisfied:

$$\operatorname{div}\mathbf{S}(\nabla\mathbf{f}) = \mathbf{0} \quad \text{in } \mathcal{R}, \tag{4.2a}$$
$$f_2 = \lambda L \quad \text{on } \mathcal{R}_T, \tag{4.2b}$$
$$f_2 = 0 \quad \text{on } \mathcal{R}_B, \tag{4.2c}$$
$$(\mathbf{S}(\nabla\mathbf{f})\mathbf{n})_1 = 0 \quad \text{on } \mathcal{R}_T \cup \mathcal{R}_B, \tag{4.2d}$$
$$\mathbf{S}(\nabla\mathbf{f})\mathbf{n} = \mathbf{0} \quad \text{on } \mathcal{S}, \tag{4.2e}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\mathcal{R}$. We note that if a deformation $\mathbf{f}$ satisfies (4.2), then so does $\mathbf{g}\circ\mathbf{f}$ where $\mathbf{g}$ is any translation in the x-direction. In order to eliminate this trivial nonuniqueness of solutions we impose the additional constraint

$$\int_{\mathcal{R}} f_1 \, d\mathbf{x} = 0. \tag{4.3}$$

A deformation that satisfies (4.2)–(4.3) is called a solution of the nonlinear problem or simply a *nonlinear solution.*

It is known that, under reasonable constitutive hypothesis on W and for each $\lambda \in (0,\infty)$, equations (4.2)–(4.3) have a (unique) *homogeneous solution* $\mathbf{f}_\lambda$ and for some critical value $\lambda_c \in (0,1)$ this solution loses stability[22], i.e., ceases to be a weak relative minimizer of the total energy E. In the remainder of this paper we prove that a branch of solutions of (4.2)–(4.3) bifurcates (locally) from $\mathbf{f}_\lambda$ at the critical value λ_c and that the stability of the branch is determined by the direction of bifurcation. Furthermore we show[23] that bifurcations also occur at a discrete set of values of λ that decreases to a limit $\lambda_\infty \in (0,1)$. These bifurcations are of either barrelling or buckling type, i.e., the rectangular column bulges on both sides of its axis symmetrically or else deforms toward one side of the axis, respectively. We show that the bifurcations are also of pitchfork type, i.e., occur in one side of the critical value of λ and we determine conditions that fix the direction of bifurcation in λ. We also estimate the total stored energy along each branch. Finally we consider the case of a special, compressible neo-Hookean material.

Remark. The condition

$$(\mathbf{S}(\nabla\mathbf{f})\mathbf{n})_2 \le 0 \quad \text{on } \mathcal{R}_T \cup \mathcal{R}_B$$

is necessary for the rectangular rod to remain in contact with the lubricated plates that are compressing it. If the inequality is strict at some point on a one-parameter family of solutions then, by continuity, any branch of solutions that bifurcates from the given solution branch at that point will also remain in contact, at least in a small neighborhood of the bifurcation

[22] This is under the hypotheses (H1)–(H11) which we list for convenience later in the paper.
[23] See also Davies [24] for a rigorous analysis of the linearized problem.

point. See Proposition 6.6 for conditions under which one can obtain this inequality for the homogeneous solutions $\mathbf{f}_\lambda$, $\lambda \in (0, 1]$ (see Proposition 4.1 below).

In order to prove the existence and uniqueness of a homogeneous solution to (4.2)–(4.3) we assume that:

(H2) $W(\mathbf{F}) \to +\infty$ *as*[24] $\det \mathbf{F} \to 0^+$ *and also as* $|\mathbf{F}| \to \infty$*;*

(H3) *The reference configuration is natural:* $\mathbf{S}(\mathbf{I}) = \mathbf{0}$;

(H4) *The strengthened tension-extension inequalities:*

$$K = K(\mu, \lambda) := \Phi,_{11}(\mu, \lambda) > 0 \quad \textit{and} \quad T = T(\mu, \lambda) := \Phi,_{22}(\mu, \lambda) > 0$$

are satisfied for each $\lambda > 0$ *and each* $\mu > 0$.

Proposition 4.1 (Davies)**.** *Assume (H1)–(H2). Then for each* $\lambda > 0$*, there is a constant* $\mu > 0$ *such that* $(\mu x, \lambda y)^{\mathrm{T}}$ *is a solution of (4.2)–(4.3). Moreover, if (H3) and (H4) are satisfied then, for each* $\lambda > 0$*,* $\mu = \mu(\lambda)$ *is unique,* $\mu(1) = 1$*, and* $\lambda \mapsto \mu(\lambda) \in C^1(\mathbb{R}^+; \mathbb{R}^+)$*. In this case we write*

$$\mathbf{f}_\lambda(\mathbf{x}) := \begin{bmatrix} \mu(\lambda)x \\ \lambda y \end{bmatrix}, \qquad \mathbf{F}_\lambda := \nabla \mathbf{f}_\lambda(\mathbf{x}) \equiv \begin{bmatrix} \mu(\lambda) & 0 \\ 0 & \lambda \end{bmatrix}. \tag{4.4}$$

In addition, if $W \in C^{k+1}(\mathrm{Lin}^+; \mathbb{R})$ *for some integer* $k \geq 2$ *then* $\lambda \mapsto \mu(\lambda) \in C^k(\mathbb{R}^+; \mathbb{R}^+)$*.*

Proof.[25] Clearly $\mathbf{f}_\lambda$ satisfies (4.2a)–(4.2d) and (4.3) for any constants μ and λ (see (3.3)). By (4.2e), μ is determined by

$$\Phi,_1(\mu, \lambda) = 0. \tag{4.5}$$

To show that (4.5) has a solution we note that for fixed λ, $\det \mathbf{D}_{\mu,\lambda} = \mu\lambda \to 0^+$ as $\mu \to 0^+$ and $|\mathbf{D}_{\mu,\lambda}|^2 = \mu^2 + \lambda^2 \to +\infty$ as $\mu \to +\infty$. Therefore (H1) and (H2) imply that $\mu \mapsto \Phi(\mu, \lambda)$ has a minimum that satisfies (4.5). Next, (H4) implies that $\mu \mapsto \Phi(\mu, \lambda)$ is strictly convex: $K(\mu, \lambda) = \Phi,_{11}(\mu, \lambda) > 0$. The implicit function theorem then yields the smoothness of $\lambda \mapsto \mu(\lambda)$. Uniqueness also follows from the strict convexity of $\mu \mapsto \Phi(\mu, \lambda)$. Since, by (H3), the reference configuration is natural it is clear that $\mu(1) = 1$. Finally, we note that the additional smoothness of the mapping $\lambda \mapsto \mu(\lambda)$ follows from (3.5) and the implicit function theorem. □

Before we proceed to the bifurcation analysis we note some simplifications of the tangential traction boundary condition (4.2d) at the top and bottom, $\mathcal{R}_T \cup \mathcal{R}_B$, that will be useful later.

[24] See Antman [6] and Ball [8] for discussions of assumptions of this type.
[25] For completeness we include the proof of Davies [24].

Proposition 4.2. *Let* $\mathbf{f} \in \mathrm{Def}$. *Assume that (H1) is satisfied and that* f_2 *satisfies* (4.2b) *and* (4.2c). *Then*

(i) $\partial_y f_1 = 0$ *on* $\mathcal{R}_T \cup \mathcal{R}_B$

implies

(ii) $(\mathbf{S}(\nabla \mathbf{f})\mathbf{n})_1 = 0$ *on* $\mathcal{R}_T \cup \mathcal{R}_B$.

Moreover, if $\mathsf{C}(\nabla \mathbf{f}(\mathbf{x}))$ *satisfies the strong ellipticity condition at each* $\mathbf{x} \in \mathcal{R}_T \cup \mathcal{R}_B$ *then (i) and (ii) are equivalent.*

Proof. We first note that (3.6) yields $(\mathbf{S}(\nabla \mathbf{f})\mathbf{e}_2)_1 = (\sigma,_1\,)\partial_y f_1 - (\sigma,_2\,)\partial_x f_2$. However, f_2 is constant on $\mathcal{R}_T \cup \mathcal{R}_B$ and hence $\partial_x f_2 = 0$; thus,

$$(\mathbf{S}(\nabla \mathbf{f})\mathbf{e}_2)_1 = (\sigma,_1\,)\partial_y f_1 \quad \text{on } \mathcal{R}_T \cup \mathcal{R}_B.$$

The implication (i) $\Rightarrow$ (ii) is now clear.

Conversely, let $\mathsf{C}(\nabla \mathbf{f}(\mathbf{x}))$ satisfy the strong ellipticity condition at $\mathbf{x} \in \mathcal{R}_T \cup \mathcal{R}_B$ then a standard result (see, e.g., [62, p. 168], [53, p. 63]) in the elasticity literature is that the strict Baker-Ericksen inequality, $\sigma,_1 > 0$, is satisfied at this $\mathbf{x}$. The implication (ii) $\Rightarrow$ (i) now follows. □

To analyze the bifurcation of solutions of (4.2) from the homogeneous branch $\mathbf{f}_\lambda$ we recast the equilibrium equations, (4.2) and (4.3), in the form $\mathfrak{F} = \mathbf{0}$ where $\mathfrak{F}$ is a nonlinear map between function spaces $\mathcal{X}_{m,p}$ and $\mathcal{Y}_{m,p}$; the latter incorporating the boundary conditions (4.2b)–(4.2d) and (4.3). To simplify matters we extend the region $\mathcal{R}$ to an infinite strip

$$\Omega := \{(x, y) \,:\, -R < x < R\,,\;\; -\infty < y < \infty\}$$

and extend $\mathbf{f}$ to a $2L$-periodic function on Ω. Let $\mathbf{u} := \mathbf{f} - \mathbf{f}_\lambda$ denote the *displacement* from the homogeneous deformation $\mathbf{f}_\lambda$. Motivated by Proposition 4.2 and equations (4.2)(b,c,d) we require the component u_2 (u_1) to be odd (even) about the lines $y = 0, \pm L, \pm 2L, \ldots$. Similarly, since the left side of (4.2) preserves these parity characteristics we extend the values of those operators periodically.

Fix $p \in (1, \infty)$. Then for each $m \in \mathbb{N}$ we write $\mathcal{X}_{m,p}$ for the set of $\mathbf{u} \in W^{m,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ that satisfy

$\mathrm{X}_{(i)}:\;\; \int_{\mathcal{R}} u_1 \, d\mathbf{x} = 0$,

$\mathrm{X}_{(ii)}:\;\; \mathbf{u}(x, y + 2L) = \mathbf{u}(x, y)$ for all $(x, y) \in \Omega$, and

$$\mathrm{X}_{(iii)}: \quad \begin{array}{rcl} u_1(x,-y) & = & u_1(x,y) \\ u_2(x,-y) & = & -u_2(x,y) \end{array} \quad \text{for all } (x,y) \in \Omega$$

with norm

$$\|\mathbf{u}\|_{\mathcal{X}_{m,p}} := \|\mathbf{u}\|_{m,p,\mathcal{R}_e}\,,$$

where

$$\mathcal{R}_e := (-R,R)\times(-L,2L). \tag{4.6}$$

Clearly, conditions $\mathrm{X}_{(i)-(iii)}$ and the norm we have chosen yield the following standard result.

Lemma 4.3. *Let $p\in(1,\infty)$ and $m\in\mathbb{N}$ with $m\geq 2$. Then we have the following isomorphisms:*

$$\begin{aligned}
\mathcal{X}_{0,p} &\cong L^p(\mathcal{R};\mathbb{R}^2)/\mathbb{R} = \left\{\mathbf{u}\in L^p(\mathcal{R};\mathbb{R}^2)\ :\ \int_{\mathcal{R}} u_1\,d\mathbf{x}=0\right\},\\
\mathcal{X}_{1,p} &\cong \left\{\mathbf{u}\in W^{1,p}(\mathcal{R};\mathbb{R}^2)\ :\ \int_{\mathcal{R}} u_1\,d\mathbf{x}=0,\quad u_2=0\ \ \text{on}\ \ \mathcal{R}_T\cup\mathcal{R}_B\right\},\\
\mathcal{X}_{m,p} &\cong \left\{\mathbf{u}\in W^{m,p}(\mathcal{R};\mathbb{R}^2)\ :\ \int_{\mathcal{R}} u_1\,d\mathbf{x}=0,\quad \partial_y^{2l}u_2=\partial_y^{2k-1}u_1=0\ \ \text{on}\ \ \mathcal{R}_T\cup\mathcal{R}_B\right\}
\end{aligned}$$

for $l=0,1,\ldots,[\![\frac{m-1}{2}]\!]$ and $k=1,2,\ldots,[\![\frac{m}{2}]\!]$. Here $[\![r]\!]$ denotes the greatest integer less than or equal to $r>0$ and $\partial_y^0u_2:=u_2$.

Fix $p\in(1,\infty)$. Then for integers $m\geq 2$ we define $\mathcal{Y}_{m,p}$ to be the set of pairs of vector-valued functions $(\mathbf{h},\mathbf{g})\in W^{m-2,p}_{\text{loc}}(\Omega;\mathbb{R}^2)\times W^{m-1-1/p,p}_{\text{loc}}(\partial\Omega;\mathbb{R}^2)$ that satisfy

$\mathrm{Y}_{(i)}$: $\int_{\mathcal{R}} h_1\,d\mathbf{x}=\int_{\mathcal{S}} g_1\,dS_{\mathbf{x}}$,

$$\mathrm{Y}_{(ii)}:\quad \begin{array}{ll}\mathbf{h}(x,y+2L)=\mathbf{h}(x,y) & \text{for all}\quad (x,y)\in\Omega\\ \mathbf{g}(x,y+2L)=\mathbf{g}(x,y) & \text{for all}\quad (x,y)\in\partial\Omega,\end{array}$$

$$\mathrm{Y}_{(iii)}:\quad \begin{array}{rcl}h_1(x,-y)&=&h_1(x,y)\\ h_2(x,-y)&=&-h_2(x,y)\end{array}\quad \text{for all } (x,y)\in\Omega, \text{ and}$$

$$\mathrm{Y}_{(iv)}:\quad \begin{array}{rcl}g_1(x,-y)&=&g_1(x,y)\\ g_2(x,-y)&=&-g_2(x,y)\end{array}\quad \text{for all } (x,y)\in\partial\Omega$$

with norm[26]

$$\|(\mathbf{h},\mathbf{g})\|_{\mathcal{Y}_{m,p}} := \|\mathbf{h}\|_{m-2,p,\mathcal{R}_e}+\|\mathbf{g}\|_{m-1-1/p,p,\mathcal{S}_e}\,,$$

where

$$\mathcal{S}_e := \partial\Omega\cap\partial\mathcal{R}_e. \tag{4.7}$$

[26] To simplify the technical details we use a larger rectangle than $\mathcal{R}$ in this norm and the norm on $\mathcal{X}_{m,p}$. It is clear that the Sobolev norms on the extended sets, $\mathcal{R}_e$ and $\mathcal{S}_e$, are equivalent to the corresponding norm on the original sets $\mathcal{R}$ and $\mathcal{S}$. However, there are technical difficulties with the existence of periodic extensions for the Sobolev space $W^{s,p}$, when s is not an integer, that we avoid by using the larger sets.

For $p \in (1,\infty)$ and integers $m > 1+2/p$ we define the map $\mathfrak{F} : \mathrm{Dom}_{m,p}(\mathfrak{F}) \subset \mathbb{R}^+ \times \mathcal{X}_{m,p} \to \mathcal{Y}_{m,p}$ by[27]

$$\mathfrak{F}(\lambda, \mathbf{u}) = \big(\operatorname{div} \mathbf{S}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u}),\ \mathbf{S}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u})\mathbf{n}\big), \tag{4.8}$$

where

$$\mathrm{Dom}_{m,p}(\mathfrak{F}) := \big\{(\lambda, \mathbf{u}) \in (0,\infty) \times \mathcal{X}_{m,p} : \det(\mathbf{F}_\lambda + \nabla \mathbf{u}(\mathbf{x})) > 0 \text{ for every } \mathbf{x} \in \overline{\Omega}\big\} \tag{4.9}$$

and $\mathbf{n} = \mathbf{n}(\pm R, y) = \pm \mathbf{e}_1$. In view of Proposition 4.2 it is clear that any solution pair $(\lambda, \mathbf{u})$ of $\mathfrak{F}(\lambda, \mathbf{u}) = \mathbf{0}$ will, formally, satisfy (4.2)–(4.3) with $\mathbf{f} := \mathbf{f}_\lambda + \mathbf{u}$. To verify that $\mathfrak{F}$ maps into $\mathcal{Y}_{m,p}$ and establish its smoothness we have:

Proposition 4.4. *Assume that (H1)–(H4) are satisfied. Fix $p \in (1,\infty)$ and an integer $m > 1 + 2/p$. Then $\mathrm{Dom}_{m,p}(\mathfrak{F})$ is an open subset of $(0,\infty) \times \mathcal{X}_{m,p}$. Suppose further that $(\lambda_0, \mathbf{u}_0) \in \mathrm{Dom}_{m,p}(\mathfrak{F})$, $\mathbf{f}_{\lambda_0}$ is given by Proposition 4.1,*

$$\mathbf{g}_0 := \mathbf{f}_{\lambda_0} + \mathbf{u}_0 \in W^{m,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^2),$$

and W is of class C^{m+2} in a neighborhood of the set

$$\mathcal{M}_0 := \{\nabla \mathbf{g}_0(\mathbf{x}) : \mathbf{x} \in \overline{\Omega}\} \subset \mathrm{Lin}^+.$$

Then there exists a neighborhood $\mathcal{N} \subset (0,\infty) \times \mathcal{X}_{m,p}$ of $(\lambda_0, \mathbf{u}_0)$ on which $\mathfrak{F}$ is of class C^2 and assumes its values in $\mathcal{Y}_{m,p}$.

Proof. Define $\mathcal{R}_e$ and $\mathcal{S}_e$ by (4.6) and (4.7), respectively. We then first note that, in view of Proposition 4.1 and the continuity of the imbedding $W^{m-1,p}(\mathcal{R}_e) \to C^0(\overline{\mathcal{R}}_e)$, the map $(\lambda, \mathbf{u}) \mapsto \det(\mathbf{F}_\lambda + \nabla \mathbf{u}) : (0,\infty) \times \mathcal{X}_{m,p} \to C^0(\overline{\mathcal{R}}_e)$ is continuous and, consequently, $\mathrm{Dom}_{m,p}(\mathfrak{F})$ is open in $(0,\infty) \times \mathcal{X}_{m,p}$.

Next, by hypothesis $\mathbf{S} : \mathrm{Lin}^+ \to \mathrm{Lin}$ is of class C^{m+1} in a neighborhood $\mathcal{V} \subset \mathrm{Lin}^+$ of $\mathcal{M}_0$. Define the map $\tilde{\mathbf{S}} : W^{m-1,p}(\mathcal{R}_e; \mathrm{Lin}^+) \to W^{m-1,p}(\mathcal{R}_e; \mathrm{Lin})$ by $\tilde{\mathbf{S}}(\mathbf{F})(\mathbf{x}) := \mathbf{S}(\mathbf{F}(\mathbf{x}))$ for all $\mathbf{F} \in W^{m-1,p}(\mathcal{R}_e; \mathrm{Lin}^+)$ and $\mathbf{x} \in \mathcal{R}_e$. Then Theorems 3.1 and 4.3 in Valent [63, Chapter II] show that $\tilde{\mathbf{S}}$ maps into $W^{m-1,p}(\mathcal{R}_e; \mathrm{Lin})$ and is C^2 on a sufficiently small neighborhood $\mathcal{W}$ of $\nabla \mathbf{g}_0$ in $W^{m-1,p}(\mathcal{R}_e; \mathrm{Lin}^+)$ (thus $\mathbf{F} \in \mathcal{W}$ implies $\mathbf{F}(\mathbf{x}) \in \mathcal{V}$ for each $\mathbf{x} \in \overline{\mathcal{R}}_e$; this makes use of the imbedding of $W^{m-1,p}(\mathcal{R}_e; \mathrm{Lin}^+)$ into $C^0(\overline{\mathcal{R}}_e; \mathrm{Lin}^+)$.) Since $\nabla : W^{m,p}(\mathcal{R}_e; \mathbb{R}^2) \to W^{m-1,p}(\mathcal{R}_e; \mathrm{Lin})$ is a linear mapping, $\tilde{\mathbf{S}} \circ \nabla : \mathcal{U} \to W^{m-1,p}(\mathcal{R}_e; \mathrm{Lin})$ is C^2 on a neighborhood $\mathcal{U}$ of $\mathbf{f}_{\lambda_0} + \mathbf{u}_0$ in $W^{m,p}(\mathcal{R}_e; \mathbb{R}^2)$. The maps trace : $W^{m-1,p}(\mathcal{R}_e; \mathbb{R}^2) \to W^{m-1-1/p,p}(\mathcal{S}_e; \mathbb{R}^2)$ and div : $W^{m-1,p}(\mathcal{R}_e; \mathrm{Lin}) \to W^{m-2,p}(\mathcal{R}_e; \mathbb{R}^2)$ are linear and, by Proposition 4.1, $\lambda \mapsto \nabla \mathbf{f}_\lambda$ is C^2 in a neighborhood of λ_0.

[27] The second component of $\mathfrak{F}$, $\mathbf{S}(\nabla \mathbf{f}_\lambda + \nabla \mathbf{u})\mathbf{n}$, is to be interpreted in the sense of trace.

Now let $\mathfrak{F}$ be defined by (4.8). Then by periodicity (see (4.10) below) the previous paragraph shows that $\mathfrak{F}$, when restricted to an appropriate neighborhood $\mathcal{N} \subset \mathrm{Dom}_{m,p}(\mathfrak{F})$ of the point $(\lambda_0, \mathbf{u}_0)$, has values in $W^{m-2,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^2) \times W^{m-1-1/p,p}_{\mathrm{loc}}(\partial\Omega;\mathbb{R}^2)$.

To show $\mathfrak{F}$ maps into $\mathcal{Y}_{m,p}$ it suffices, by continuity, to show $\mathfrak{F}$ maps $\mathcal{N} \cap ((0,\infty) \times C^\infty(\overline{\Omega};\mathbb{R}^2))$ into $\mathcal{Y}_{m,p}$, since $\mathcal{X}_{m,p} \cap C^\infty(\overline{\Omega};\mathbb{R}^2)$ is dense in $\mathcal{X}_{m,p}$. To prove $\mathrm{Y}_{(ii)-(iv)}$ we recall (3.6), i.e.,

$$\mathbf{S}(\nabla\mathbf{f}) = \begin{bmatrix} (\sigma,_1)\partial_x f_1 + (\sigma,_2)\partial_y f_2 & (\sigma,_1)\partial_y f_1 - (\sigma,_2)\partial_x f_2 \\ (\sigma,_1)\partial_x f_2 - (\sigma,_2)\partial_y f_1 & (\sigma,_1)\partial_y f_2 + (\sigma,_2)\partial_x f_1 \end{bmatrix}, \tag{4.10}$$

which together with $\mathrm{X}_{(ii)}$ show that the stress is $2L$-periodic in y. Next recall $\mathbf{u} = \mathbf{f} - \mathbf{f}_\lambda$ and let $p \in \mathbb{Z}$; then under the transformation $y \mapsto 2pL - y$ we find that $u_1 \mapsto u_1$, $u_2 \mapsto -u_2$, the components of $\mathbf{S} = \mathbf{S}(\nabla\mathbf{f})$ transform according to $S_{11} \mapsto S_{11}$, $S_{12} \mapsto -S_{12}$, $S_{21} \mapsto -S_{21}$, and $S_{22} \mapsto S_{22}$. Properties $\mathrm{Y}_{(ii)-(iv)}$ now follow. To verify $\mathrm{Y}_{(i)}$ integrate $\int_{\mathcal{R}} (\operatorname{div}\mathbf{S}(\nabla\mathbf{f}))_1\, d\mathbf{x}$ by parts and make use of Proposition 4.2.

Finally, $\mathrm{X}_{(ii)}$ and $\mathrm{X}_{(iii)}$ together with the argument in the first paragraph applied to the Frechét derivatives of $\mathfrak{F}$ shows that $\mathfrak{F}$ is of class C^2 on $\mathcal{N}$. This completes the proof. □

5. The Linearized Operator I: Sobolev Estimates and Spectral Theory

In this section we establish some properties of the operators given by the linearization of (4.2)–(4.3), i.e., the linearization of $\mathfrak{F}$ with respect to $\mathbf{u}$. Given $\mathbf{f} \in \mathrm{Def}$ that satisfies (4.2)(b,c) and (4.3) we call $\mathbf{f}$ a *weak relative minimizer* of the total energy (4.1) provided there is an $\epsilon > 0$ such that

$$E(\mathbf{g}) \geq E(\mathbf{f}) \;\text{ for all }\; \mathbf{g} \in \mathrm{Kin}_{\mathbf{f}} \;\text{ that satisfy }\; \|\nabla\mathbf{f} - \nabla\mathbf{g}\|_{C^0(\overline{\mathcal{R}};\mathrm{Lin})} < \epsilon,$$

where

$$\mathrm{Kin}_{\mathbf{f}} := \left\{ \mathbf{g} \in \mathrm{Def} : \int_{\mathcal{R}} (g_1 - f_1)\, d\mathbf{x} = 0, \quad g_2 = f_2 \text{ on } \mathcal{R}_T \cup \mathcal{R}_B \right\}$$

are the kinematically admissible deformations. The sign of the *quadratic form*

$$\mathrm{Q}_{\mathbf{f}}[\mathbf{v}] := \int_{\mathcal{R}} \nabla\mathbf{v}(\mathbf{x}) : \mathsf{C}(\nabla\mathbf{f}(\mathbf{x}))\,[\nabla\mathbf{v}(\mathbf{x})]\, d\mathbf{x} \tag{5.1}$$

for *variations* (cf. (2.3))

$$\mathbf{v} \in \mathrm{Var} := \left\{ \mathbf{v} \in W^{1,2}(\mathcal{R};\mathbb{R}^2) \;:\; \int_{\mathcal{R}} v_1\, d\mathbf{x} = 0, \quad v_2 = 0 \text{ on } \mathcal{R}_T \cup \mathcal{R}_B \right\}$$

may determine if $\mathbf{f}$ is a weak relative minimizer (see, e.g., van Hove [64]):

Proposition 5.1. *Assume (H1) and that* $\mathbf{f} \in \mathrm{Def}$ *satisfies* (4.2)–(4.3). *Then a necessary condition and a sufficient condition for* $\mathbf{f}$ *to be a weak relative minimizer of* (4.1) *are, respectively,*

$$\mathrm{Q}_{\mathbf{f}}[\mathbf{v}] \geq 0 \quad \textit{and} \quad \mathrm{Q}_{\mathbf{f}}[\mathbf{v}] \geq k \,\|\mathbf{v}\|^2_{1,2,\mathcal{R}} \tag{5.2}$$

for some $k > 0$ *and all* $\mathbf{v} \in \mathrm{Var}$.

We say that the quadratic functional $\mathrm{Q}_{\mathbf{f}}$ defined by (5.1) is *uniformly positive* if $(5.2)_2$ is satisfied for all $\mathbf{v} \in \mathrm{Var}$ and we call $\mathrm{Q}_{\mathbf{f}}$ *coercive* if there exists $k > 0$ and $k_1 \in \mathbb{R}$ such that

$$\mathrm{Q}_{\mathbf{f}}[\mathbf{v}] \geq k \,\|\mathbf{v}\|^2_{1,2,\mathcal{R}} - k_1 \,\|\mathbf{v}\|^2_{0,2,\mathcal{R}} \quad \text{for all} \quad \mathbf{v} \in \mathrm{Var}. \tag{5.3}$$

Fix $p \in (1,\infty)$, an integer[28] $m > 1+2/p$, and assume (H1)–(H4). If $(\lambda, \mathbf{u}) \in \mathrm{Dom}_{m,p}(\mathfrak{F})$, we write $\mathbf{f} := \mathbf{f}_\lambda + \mathbf{u}$. If in addition W is of class C^{m+2} (near the values of $\nabla\mathbf{f}$), then as in Proposition 4.4 and Theorem 4.3 in Valent [63, Chapter II], the Frechét derivative $\mathfrak{L}_{\mathbf{f}} := (\partial_{\mathbf{u}}\mathfrak{F})(\lambda, \mathbf{u}) \in \mathrm{BL}(\mathcal{X}_{m,p}; \mathcal{Y}_{m,p})$ exists and the mapping $(\lambda, \mathbf{u}) \mapsto (\partial_{\mathbf{u}}\mathfrak{F})(\lambda, \mathbf{u})$ is continuous at each $(\lambda, \mathbf{u}) \in \mathrm{Dom}_{m,p}(\mathfrak{F})$ ($\mathsf{C}_{\mathbf{f}} \in W^{m-1,p}_{\mathrm{loc}}(\Omega; \mathrm{LinLin}) \subset C^0(\overline{\Omega}; \mathrm{LinLin})$ where $\mathsf{C}_{\mathbf{f}}(\mathbf{x}) := \mathsf{C}(\nabla\mathbf{f}(\mathbf{x}))$ for $\mathbf{x} \in \overline{\Omega}$). We note that $\mathfrak{L}_{\mathbf{f}} = (-\mathfrak{D}_{\mathbf{f}}, \mathfrak{B}_{\mathbf{f}})$, where the second-order *differential* operator, $\mathfrak{D}_{\mathbf{f}}$, together with the corresponding (natural) *boundary* operator, $\mathfrak{B}_{\mathbf{f}}$ are given by

$$\begin{aligned} \mathfrak{D}_{\mathbf{f}}[\mathbf{v}] &:= -\mathrm{div}\,\mathsf{C}_{\mathbf{f}}[\nabla\mathbf{v}] \quad &&\text{on } \Omega, \\ \mathfrak{B}_{\mathbf{f}}[\mathbf{v}] &:= \mathsf{C}_{\mathbf{f}}[\nabla\mathbf{v}]\,\mathbf{n} &&\text{on } \partial\Omega. \end{aligned} \tag{5.4}$$

The linearization of (4.2)–(4.3) about the deformation $\mathbf{f} = \mathbf{f}_\lambda + \mathbf{u}$ is then

$$\mathrm{div}\,\mathsf{C}_{\mathbf{f}}[\nabla\mathbf{v}] = \mathbf{0} \ \text{ in } \mathcal{R}, \tag{5.5a}$$
$$v_2 = 0 \ \text{ on } \mathcal{R}_T \cup \mathcal{R}_B, \tag{5.5b}$$
$$\left(\mathsf{C}_{\mathbf{f}}[\nabla\mathbf{v}]\,\mathbf{n}\right)_1 = 0 \ \text{ on } \mathcal{R}_T \cup \mathcal{R}_B, \tag{5.5c}$$
$$\mathsf{C}_{\mathbf{f}}[\nabla\mathbf{v}]\,\mathbf{n} = \mathbf{0} \ \text{ on } \mathcal{S}, \tag{5.5d}$$
$$\int_{\mathcal{R}} v_1\,d\mathbf{x} = 0. \tag{5.5e}$$

We note that if $\mathbf{v} \in \mathcal{X}_{m,p}$ satisfies $\mathfrak{L}_{\mathbf{f}}[\mathbf{v}] = \mathbf{0}$ then $\mathbf{v}$ will also satisfy (5.5). This follows from the fact that $\mathbf{v} \in \mathcal{X}_{m,p}$ and Lemma 4.3 imply $v_2 = \partial_x v_2 = \partial_y v_1 = 0$ on $\mathcal{R}_T \cup \mathcal{R}_B$ and hence $\nabla\mathbf{v}$ is diagonal on the top and bottom of the rectangle; similarly for $\nabla\mathbf{f}$. The proof of Proposition 4.2 then gives us (5.5c). We also remark that (5.5a)–(5.5d) are the Euler-Lagrange equations of the quadratic functional $\mathrm{Q}_{\mathbf{f}}$ defined in (5.1).

We will make use of the following result whose proof we postpone until the Appendix.

[28]The assumption $m > 1 + 2/p$ together with a standard imbedding theorem implies that each $\mathbf{v} \in \mathcal{X}_{m,p}$ has a C^1 representative.

Proposition 5.2. *Assume that (H1)–(H4) are satisfied. Let* $(\lambda, \mathbf{u}) \in \mathrm{Dom}_{m,p}(\mathfrak{F})$ *and define* $\mathbf{f} := \mathbf{f}_\lambda + \mathbf{u}$. *Suppose* $\mathsf{C}_\mathbf{f}(\mathbf{x})$ *satisfies the strong-ellipticity condition at every* $\mathbf{x} \in \overline{\mathcal{R}}$ *and that* $(\mathsf{C}_\mathbf{f}(\mathbf{x}), \mathbf{e}_1)$ *satisfies the complementing condition at every* $\mathbf{x} \in \overline{\mathcal{S}}$. *Then a necessary and sufficient condition for the quadratic functional* $\mathrm{Q}_\mathbf{f}$ *to be coercive is that the pair* $(\mathsf{C}_\mathbf{f}(\mathbf{x}), \mathbf{e}_1)$ *satisfies Agmon's condition at every* $\mathbf{x} \in \overline{\mathcal{S}}$.

For the remainder of this section we assume that (H1)–(H4) *are satisfied and we fix* $p \in (1, \infty)$, *an integer* $m \geq 2$, $\lambda \in (0, 1]$, *and* $\mathbf{f} := \mathbf{f}_\lambda$, *as defined in* (4.4), *where* $\mu = \mu(\lambda)$ *is given by Proposition 4.1.* We write

$$\mathsf{C}_\lambda := \mathsf{C}_{\mathbf{f}_\lambda}, \quad \mathfrak{D}_\lambda := \mathfrak{D}_{\mathbf{f}_\lambda}, \quad \mathfrak{B}_\lambda := \mathfrak{B}_{\mathbf{f}_\lambda}, \quad \mathrm{Q}_\lambda := \mathrm{Q}_{\mathbf{f}_\lambda}, \tag{5.6}$$

and note that our linearized system, (5.5), now has constant coefficients ($\mathsf{C}_\lambda \in \mathrm{LinLin}$ for each λ). We write

$$\mathfrak{L}_\lambda := (-\mathfrak{D}_\lambda, \mathfrak{B}_\lambda) \in \mathrm{BL}(\mathcal{X}_{m,p}; \mathcal{Y}_{m,p}). \tag{5.7}$$

We also consider $\mathfrak{D}_\lambda : \mathrm{Dom}_{m,p}(\mathfrak{D}_\lambda) \subset \mathcal{X}_{m-2,p} \to \mathcal{X}_{m-2,p}$ as an (unbounded) operator with domain

$$\mathrm{Dom}_{m,p}(\mathfrak{D}_\lambda) := \{\mathbf{v} \in \mathcal{X}_{m,p}, : \mathfrak{B}_\lambda[\mathbf{v}] = \mathbf{0} \text{ on } \partial\Omega\}.$$

Note that $\mathfrak{D}_\lambda$ maps into $\mathcal{X}_{m-2,p}$ since, by (5.5)(c,d),

$$\int_{\mathcal{R}} (\mathrm{div}\, \mathsf{C}_\lambda[\nabla \mathbf{v}])_1 \, d\mathbf{x} = 0$$

for every $\mathbf{v} \in \mathrm{Dom}_{m,p}(\mathfrak{D}_\lambda)$. Thus, in view of (3.12) (or the left-hand sides of (3.15a) and (3.15b)), $\mathfrak{D}_\lambda[\mathbf{v}]$ satisfies $X_{(i)-(iii)}$.

Since our ultimate goal is to prove that a second solution branch emanates from the trivial homogeneous solution branch we will need to show that an eigenvalue of the one-parameter family of linear operators $\mathfrak{D}_\lambda$ crosses through zero, at a nontrivial rate, as λ passes through some critical value. In order to accomplish this we will require detailed information on the entire spectrum of each of the operators $\mathfrak{D}_\lambda$. If $p = 2$ then $\mathrm{Dom}_{2,2}(\mathfrak{D}_\lambda) \subset \mathcal{X}_{0,2}$, which is a Hilbert space and in view of the symmetry of C_λ, (3.7), it can be shown that each operator $\mathfrak{D}_\lambda$ is self-adjoint. Standard results then imply that the spectrum consists solely of real eigenvalues that have no finite accumulation point. When $p \neq 2$ the spaces $\mathcal{X}_{m,p}$ and $\mathcal{Y}_{m,p}$ are Banach spaces; the standard[29] approach to the spectral theory for linear operators on a Banach space over the real numbers requires the consideration of the complexification[30] of these spaces, e.g.,

$$\mathcal{X}^c_{m,p} := \left\{\mathbf{w} \in W^{m,p}_{\mathrm{loc}}(\Omega; \mathbb{C}^2) : \mathbf{w} \text{ satisfies } \mathrm{X}_{(i)}\text{–}\mathrm{X}_{(iii)} \text{ in } \S 4\right\}.$$

[29]See, for example, Chapters II-V in Kato [36] for spectral theory of unbounded operators in general and, in particular, Fredholm and semi-Fredholm operators.

[30]Equivalently one can instead define the complexification via ordered pairs, each of which resides in the original space. See, e.g., [51, pp. 147–148]

The *spectrum* of $\mathfrak{D}_\lambda : \mathrm{Dom}^c_{m,p}(\mathfrak{D}_\lambda) \subset \mathcal{X}^c_{m-2,p} \to \mathcal{X}^c_{m-2,p}$ is defined to be the set of complex numbers μ such that $\mathfrak{D}_\lambda - \mu\mathfrak{I}$ is not a bijection from $\mathrm{Dom}^c_{m,p}(\mathfrak{D}_\lambda)$ onto $\mathcal{X}^c_{m-2,p}$, where $\mathfrak{I} : \mathcal{X}^c_{m-2,p} \to \mathcal{X}^c_{m-2,p}$ is the identity map. The spectrum is said to be *bounded below* if there is a real number c such that the real part of every μ in the spectrum is larger than c. An *eigenvalue* $\mu \in \mathbb{C}$ satisfies $\mathfrak{D}_\lambda[\mathbf{v}] = \mu\mathbf{v}$ and $\mathfrak{B}_\lambda[\mathbf{v}] = 0$ for some *eigenfunction* $\mathbf{v} \in \mathcal{X}^c_{m,p}$ with $\mathbf{v} \neq \mathbf{0}$. We note, by the Sobolev imbedding theorems, $\mathcal{X}^c_{m,p} \subset \mathcal{X}^c_{1,2} \cap C^0(\overline{\Omega}; \mathbb{C}^2)$. We also note that for any $\mathbf{v}, \mathbf{w} \in \mathcal{X}^c_{m,p}$ the divergence theorem together with the identities $w_2 = 0$ and[31] $(\mathsf{C}_\mathbf{f}[\nabla\mathbf{v}]\,\mathbf{n})_1 = 0$ on $\mathcal{R}_T \cup \mathcal{R}_B$ yield

$$\int_{\mathcal{R}} \overline{\nabla\mathbf{w}} : \mathsf{C}_\lambda[\nabla\mathbf{v}]\; d\mathbf{x} = \int_{\mathcal{R}} \overline{\mathbf{w}} \cdot \mathfrak{D}_\lambda[\mathbf{v}]\; d\mathbf{x} + \int_{\mathcal{S}} \overline{\mathbf{w}} \cdot \mathfrak{B}_\lambda[\mathbf{v}]\; dS_\mathbf{x}, \tag{5.8}$$

where $\overline{\mathbf{w}}$ denotes the complex conjugate of $\mathbf{w}$. Moreover, if $\mathbf{v}, \mathbf{w} \in \mathcal{X}_{m,p}$ then (5.8) is also valid, but the complex conjugate is then, of course, unnecessary.

In order to proceed with our analysis we will require the following technical results, which we prove in the Appendix.

Proposition 5.3. *Fix $\lambda \in (0,1]$ and suppose C_λ satisfies the strong-ellipticity condition and $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies the complementing condition. Then:*

(i) There exists a constant $\omega > 0$, which depends only on m, p, $\mathcal{R}$, and C_λ, such that

$$\|\mathbf{v}\|_{\mathcal{X}_{m,p}} \le \omega \left(\|\mathfrak{L}_\lambda[\mathbf{v}]\|_{\mathcal{Y}_{m,p}} + \|\mathbf{v}\|_{0,p,\mathcal{R}_e} \right) \quad \textit{for all} \;\; \mathbf{v} \in \mathcal{X}_{m,p}. \tag{5.9}$$

Thus $\mathfrak{L}_\lambda \in \mathrm{BL}(\mathcal{X}_{m,p}; \mathcal{Y}_{m,p})$ has finite dimensional null space and closed range in $\mathcal{Y}_{m,p}$. Moreover, $\mathfrak{D}_\lambda$ (with domain $\mathrm{Dom}_{m,p}(\mathfrak{D}_\lambda)$) is a closed operator in $\mathcal{X}_{m-2,p}$ with finite dimensional null space and closed range in $\mathcal{X}_{m-2,p}$.

In addition, if $m \ge 3$ (or $m = 2$ and $p \ge 2$) and the spectrum of $\mathfrak{D}_\lambda$ is real, then:

(ii) $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies Agmon's condition if and only if the spectrum is bounded below.

Proposition 5.4. *Suppose that $\mathbf{S}(\mathbf{I}) = \mathbf{0}$ and $\mathsf{C}(\mathbf{I})$ is positive definite (on symmetric tensors). Then the operator $\mathfrak{D}_1$ is a bijection from $\mathrm{Dom}_{m,p}(\mathfrak{D}_1)$ onto $\mathcal{X}_{m-2,p}$ and consequently a Fredholm operator with index zero.*

We will henceforth make the following additional assumptions:

(H5) $\mathsf{C}_\lambda := \mathsf{C}(\nabla\mathbf{f}_\lambda)$ *satisfies the strong-ellipticity condition at each $\lambda \in (0,1]$. Here, and in the remaining hypotheses, $\mathbf{f}_\lambda$ is defined in* (4.4) *and $\mu = \mu(\lambda)$ is given by Prop. 4.1;*

(H6) $\mathsf{C}(\mathbf{I})$ *is positive definite (on symmetric tensors);*

[31] This identity is a consequence of $\partial_x v_2 = \partial_y v_1 = 0$ on $\mathcal{R}_T \cup \mathcal{R}_B$, see Lemma 4.3.

The main result of this section is (cf., e.g., [56, Proposition 4.1]):

Theorem 5.5. *Assume (H1)–(H6). Let $m \geq 2$, $1 < p < \infty$, and $\lambda_0 \in (0,1)$. Suppose that $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies the complementing condition for all $\lambda \in (\lambda_0, 1]$. Then for each $\lambda \in (\lambda_0, 1]$:*

(i) The operators $\mathfrak{L}_\lambda$ and $\mathfrak{D}_\lambda$ are each Fredholm operators with index zero;

(ii) The spectrum of $\mathfrak{D}_\lambda$ consists solely of countably many real eigenvalues whose only accumulation point is $+\infty$; in particular it is bounded below;

(iii) $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies Agmon's condition and Q_λ is coercive;

(iv) Q_λ is uniformly positive if and only if the smallest eigenvalue, $\sigma(\lambda)$, of $\mathfrak{D}_\lambda$ is positive;

(v) $t \mapsto \sigma(t)$ is continuous at $t = \lambda$;

(vi) If $m = p = 2$ then $\mathfrak{D}_\lambda$ is self-adjoint and has a complete orthonormal sequence of eigenfunctions in $\mathcal{X}_{0,2}$;

(vii) The spectrum of $\mathfrak{D}_\lambda$ is independent of m and p as is the null space N_μ of $\mathfrak{D}_\lambda - \mu\mathfrak{T}$, for each $\mu \in \mathbb{C}$, where $\mathfrak{T} : \mathcal{X}^c_{m-2,p} \to \mathcal{X}^c_{m-2,p}$ is the identity map. Moreover, $N_\mu \subset C^\infty(\overline{\Omega}; \mathbb{C}^2)$.

Proof of Theorem 5.5. (iii). To prove that $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies Agmon's condition for all $\lambda \in (\lambda_0, 1]$ we first note that this condition is valid at $\lambda = 1$ by Proposition 3.2(iv) and (H6). Moreover, we have assumed that the complementing condition is satisfied at each $\lambda \in (\lambda_0, 1]$ and thus Proposition 3.2(ii)–(iii) together with the continuity of the map $\lambda \mapsto A(\mu(\lambda), \lambda)$ imply that Agmon's condition cannot fail at any $\lambda \in (\lambda_0, 1]$ (see also [35, Theorem 2.2]). Consequently, by Proposition 5.2, Q_λ is coercive for all $\lambda \in (\lambda_0, 1]$, which proves (iii).

(i). For each $\lambda \in (\lambda_0, 1]$ Proposition 5.3(i) implies that $\mathfrak{L}_\lambda \in \mathrm{BL}(\mathcal{X}_{m,p}; \mathcal{Y}_{m,p})$ is a semi-Fredholm operator. Due to the continuity of the map $\lambda \mapsto \mathfrak{L}_\lambda$, the index of $\mathfrak{L}_\lambda$ is independent[32] of λ. Thus, by Proposition 5.4, the index of $\mathfrak{L}_\lambda$ is zero for all $\lambda \in (\lambda_0, 1]$. Similarly, by the proof of Proposition 5.3(i), $(\mathfrak{D}_\lambda - \mu\mathfrak{T}, \mathfrak{B}_\lambda) \in \mathrm{BL}(\mathcal{X}^c_{m,p}; \mathcal{Y}^c_{m,p})$ is a Fredholm operator with index zero for all $\mu \in \mathbb{C}$ and $\lambda \in (\lambda_0, 1]$ (the index is independent of μ). In addition, $\mathfrak{D}_\lambda - \mu\mathfrak{T}$ is semi-Fredholm for all such λ and μ with index independent of μ (cf. (A.7)).

Now fix $\lambda \in (\lambda_0, 1]$ and note that Q_λ is coercive by part (iii). Let μ be a real eigenvalue of $\mathfrak{D}_\lambda$, i.e., suppose there exists $\mathbf{v} \in \mathcal{X}_{m,p} \setminus \{\mathbf{0}\}$ that satisfies $\mathfrak{D}_\lambda[\mathbf{v}] = \mu\mathbf{v}$ and $\mathfrak{B}_\lambda[\mathbf{v}] = \mathbf{0}$. Then, by (5.3) and (5.8)

$$-k_1 \|\mathbf{v}\|^2_{0,2,\mathcal{R}} \leq \mathrm{Q}_\lambda[\mathbf{v}] = \int_{\mathcal{R}} \mathbf{v} \cdot \mathfrak{D}_\lambda[\mathbf{v}]\, d\mathbf{x} = \mu \|\mathbf{v}\|^2_{0,2,\mathcal{R}}\,. \tag{5.10}$$

Thus each real eigenvalue μ of $\mathfrak{D}_\lambda$ must satisfy $\mu \geq -k_1$. Now choose $\mu_0 < -k_1$. Then by the above argument $(\mathfrak{D}_\lambda - \mu_0\mathfrak{T}, \mathfrak{B}_\lambda) \in \mathrm{BL}(\mathcal{X}_{m,p}; \mathcal{Y}_{m,p})$ is injective. Since the operator

[32]See, e.g., Kato [36, Chapter IV, Theorems 2.14 and 5.17].

$(\mathfrak{D}_\lambda - \mu_0\mathfrak{T}, \mathfrak{B}_\lambda)$ is Fredholm with index zero it must therefore be a bijection.[33] Consequently, $\mathfrak{D}_\lambda - \mu_0\mathfrak{T}$ is also a bijection and so has index zero. Therefore $\mathfrak{D}_\lambda - \zeta\mathfrak{T}$ has index zero for all $\zeta \in \mathbb{C}$, which proves (i).

To prove (ii), define $\mathcal{K} : \mathcal{X}^c_{m-2,p} \to \mathrm{Dom}^c_{m,p}(\mathfrak{D}_\lambda) \subset \mathcal{X}^c_{m-2,p}$ by $\mathcal{K} = \mathcal{I} \circ (\mathfrak{D}_\lambda - \mu_0\mathfrak{T})^{-1}$, where $\mathcal{I} \in \mathrm{BL}(\mathcal{X}^c_{m,p}; \mathcal{X}^c_{m-2,p})$ is the (compact) imbedding mapping $\mathcal{I}(\mathbf{u}) := \mathbf{u}$. Proposition 5.3(i) implies that $\mathcal{K}$ is closed and so it is bounded by the closed graph theorem. Moreover, since $\mathcal{I}$ is compact so is $\mathcal{K}$. Therefore (see, e.g., the proof of [56, Proposition 4.1]), the spectrum of $\mathfrak{D}_\lambda$ consists solely of eigenvalues of finite multiplicity with no finite accumulation point in the complex plane; the eigenfunctions of $\mathcal{K}$ are also those of $\mathfrak{D}_\lambda = \mathcal{K}^{-1} + \mu_0\mathfrak{T}$. To see that these eigenvalues are real let $\mathfrak{D}_\lambda[\mathbf{v}] = \zeta\mathbf{v}$ and $\mathfrak{B}_\lambda[\mathbf{v}] = \mathbf{0}$ with $\mathbf{v} \in \mathcal{X}^c_{m,p} \setminus \{\mathbf{0}\}$ and $\zeta \in \mathbb{C}$. Then by (3.7) (the symmetry of C_λ) and (5.8)

$$\zeta \|\mathbf{v}\|^2_{0,2,\mathcal{R}} = \int_{\mathcal{R}} \overline{\mathbf{v}} \cdot \mathfrak{D}_\lambda[\mathbf{v}]\ d\mathbf{x} = \int_{\mathcal{R}} \nabla\overline{\mathbf{v}} : \mathsf{C}_\lambda[\nabla\mathbf{v}]\ d\mathbf{x}$$
$$= \int_{\mathcal{R}} \nabla\mathbf{v} : \mathsf{C}_\lambda[\nabla\overline{\mathbf{v}}]\ d\mathbf{x} = \int_{\mathcal{R}} \mathbf{v} \cdot \mathfrak{D}_\lambda[\overline{\mathbf{v}}]\ d\mathbf{x} = \overline{\zeta}\,\|\mathbf{v}\|^2_{0,2,\mathcal{R}}$$

and hence $\zeta = \overline{\zeta}$. Finally, in view of (5.10) the spectrum is bounded below; in particular, each eigenvalue μ satisfies $\mu \geq -k_1$. This proves (ii).

(vi). For $p = m = 2$ fix $\lambda \in (\lambda_0, 1]$ and define $\widehat{\mathcal{D}}_\lambda := \mathfrak{D}_\lambda : \mathrm{Dom}_{2,2}(\mathfrak{D}_\lambda) \subset \mathcal{X}_{0,2} \to \mathcal{X}_{0,2}$. By the proof of Proposition 5.3(ii), $\widehat{\mathcal{D}}_\lambda$ is self-adjoint. Its normalized eigenfunctions form a complete orthonormal set in $\mathcal{X}_{0,2}$ (see, e.g., [36, Chapter V, §3.8]).

(iv). Fix $\lambda \in (\lambda_0, 1]$. Let $\Sigma = \Sigma(\lambda) \subset \mathbb{C}$ and $\widehat{\Sigma} = \widehat{\Sigma}(\lambda) \subset \mathbb{C}$ denote the spectra of $\mathfrak{D}_\lambda$ and $\widehat{\mathcal{D}}_\lambda$, respectively while $\sigma = \sigma(\lambda)$ and $\widehat{\sigma} = \widehat{\sigma}(\lambda)$ denote the smallest eigenvalues of $\mathfrak{D}_\lambda$ and $\widehat{\mathcal{D}}_\lambda$, respectively. Then if $m = 2$ and $p \geq 2$, or if $m \geq 3$, it follows as in the proof of Proposition 5.3(ii) that $\widehat{\Sigma} \subset \Sigma$ and hence $\sigma \leq \widehat{\sigma}$. To see that $\sigma \leq \widehat{\sigma}$ when $m = 2$ and $1 < p < 2$ also, let $\widehat{\mu} \in \widehat{\Sigma}$; then $\widehat{\mu}$ is an eigenvalue of $\widehat{\mathcal{D}}_\lambda$. However, $\mathcal{X}_{2,2} \subset \mathcal{X}_{m,p}$ and hence $\widehat{\mu}$ is also an eigenvalue of $\mathfrak{D}_\lambda$. Therefore, $\sigma \leq \widehat{\sigma}$.

We claim that $\sigma = \widehat{\sigma}$. To see this we first note that, by (5.8) and the fact that $\mathrm{D} := \mathrm{Dom}_{2,2}(\mathfrak{D}_\lambda)$ is dense in Var,

$$\widehat{\sigma} = \inf_{\substack{\mathbf{v}\in\mathrm{D} \\ \|\mathbf{v}\|_{0,2,\mathcal{R}}=1}} \int_{\mathcal{R}} \mathbf{v} \cdot \widehat{\mathcal{D}}_\lambda[\mathbf{v}]\, d\mathbf{x} = \inf_{\substack{\mathbf{v}\in\mathrm{Var} \\ \|\mathbf{v}\|_{0,2,\mathcal{R}}=1}} \mathrm{Q}_\lambda[\mathbf{v}]. \tag{5.11}$$

Next, since σ is an eigenvalue of $\mathfrak{D}_\lambda$ there exists an eigenfunction $\mathbf{w} \in \mathrm{Dom}_{m,p}(\mathfrak{D}_\lambda)\setminus\{\mathbf{0}\}$, i.e., $\mathfrak{D}_\lambda[\mathbf{w}] = \sigma\mathbf{w}$. Consequently, by (5.8)

$$\mathrm{Q}_\lambda[\mathbf{w}] = \int_{\mathcal{R}} \mathbf{w} \cdot \mathfrak{D}_\lambda[\mathbf{w}]\, d\mathbf{x} = \sigma\,\|\mathbf{w}\|^2_{0,2,\mathcal{R}}\,,$$

[33] In particular, $\mathfrak{B}_\lambda$ is surjective.

which together with (5.11) yields $\sigma \geq \widehat{\sigma}$, which proves the claim.

Finally, if Q_λ is uniformly positive then $(5.2)_2$ and (5.11) imply $\sigma(\lambda) > 0$. Conversely, if $\sigma(\lambda) > 0$ then Q_λ is strictly positive on $\mathrm{Var}\setminus\{\mathbf{0}\}$. Then arguing as in the proof of Lemma 6.5 we see Q_λ is uniformly positive (this uses the coercivity of Q_λ). See also [56, Theorem 2].

(v). By the proof of (iv), $\sigma(\lambda) = \widehat{\sigma}(\lambda)$ and hence it suffices to prove $\lambda \mapsto \widehat{\sigma}(\lambda)$ is continuous. To prove upper semicontinuity of $\widehat{\sigma}$ fix $\lambda_* \in (\lambda_0, 1]$. Let $\mathbf{v} \in \mathrm{Dom}_{2,2}(\widehat{\mathcal{D}}_{\lambda_*})$ with $\|\mathbf{v}\|_{0,2,\mathcal{R}} = 1$ satisfy $\widehat{\mathcal{D}}_{\lambda_*}[\mathbf{v}] = \widehat{\sigma}(\lambda_*)\mathbf{v}$. Then by (5.8), (5.11), and the continuity of $\lambda \mapsto \mathsf{C}_\lambda$

$$\limsup_{\lambda\to\lambda_*} \widehat{\sigma}(\lambda) \leq \limsup_{\lambda\to\lambda_*} \mathrm{Q}_\lambda[\mathbf{v}] = \mathrm{Q}_{\lambda_*}[\mathbf{v}] = \int_{\mathcal{R}} \mathbf{v}\cdot\widehat{\mathcal{D}}_{\lambda_*}[\mathbf{v}]\,d\mathbf{x} = \widehat{\sigma}(\lambda_*).$$

To prove lower semicontinuity let $\varepsilon \in (0,1)$. Then by (5.11),

$$\mathrm{Q}_{\lambda_*}[\mathbf{v}] = \varepsilon\mathrm{Q}_{\lambda_*}[\mathbf{v}] + (1-\varepsilon)\mathrm{Q}_{\lambda_*}[\mathbf{v}] \geq \varepsilon\mathrm{Q}_{\lambda_*}[\mathbf{v}] + \widehat{\sigma}(\lambda_*)(1-\varepsilon)\,\|\mathbf{v}\|^2_{0,2,\mathcal{R}} \tag{5.12}$$

for all $\mathbf{v} \in \mathrm{Var}$. Next, (5.12) and the coercivity of Q_{λ_*} (see (iii) and (5.3)) imply

$$\mathrm{Q}_{\lambda_*}[\mathbf{v}] \geq \varepsilon k\,\|\mathbf{v}\|^2_{1,2,\mathcal{R}} + \left[\widehat{\sigma}(\lambda_*)(1-\varepsilon) - \varepsilon k_1\right]\|\mathbf{v}\|^2_{0,2,\mathcal{R}} \tag{5.13}$$

for all $\mathbf{v} \in \mathrm{Var}$. However the continuity of the map $\lambda \mapsto \mathsf{C}_\lambda$ yields a $\delta > 0$ such that

$$\left|\mathrm{Q}_\lambda[\mathbf{v}] - \mathrm{Q}_{\lambda_*}[\mathbf{v}]\right| \leq \frac{\varepsilon k}{2}\,\|\mathbf{v}\|^2_{1,2,\mathcal{R}} \tag{5.14}$$

whenever $|\lambda - \lambda_*| < \delta$. Therefore by (5.13) and (5.14) we find that, for all $\mathbf{v} \in \mathrm{Var}$ and $\lambda \in (\lambda_* - \delta, \lambda_* + \delta)$,

$$\mathrm{Q}_\lambda[\mathbf{v}] \geq \left[\widehat{\sigma}(\lambda_*)(1-\varepsilon) - \varepsilon k_1\right]\|\mathbf{v}\|^2_{0,2,\mathcal{R}}$$

and consequently, in view of (5.11),

$$\widehat{\sigma}(\lambda) \geq \widehat{\sigma}(\lambda_*)(1-\varepsilon) - \varepsilon k_1. \tag{5.15}$$

Therefore, since ε is arbitrary,

$$\liminf_{\lambda\to\lambda_*} \widehat{\sigma}(\lambda) \geq \widehat{\sigma}(\lambda_*).$$

(vii). Fix $\lambda \in (\lambda_0, 1]$ and $p \in (1,\infty)$. We denote by $\Sigma^{(m)}$ the spectrum of the operator $\mathfrak{D}_\lambda$ in $\mathcal{X}^c_{m-2,p}$ with domain $\mathrm{Dom}^c_{m,p}(\mathfrak{D}_\lambda)$ for $m = 2, 3, \ldots$. For each $\mu \in \mathbb{C}$ let $N^{(m)}_\mu$ denote the null space of $\mathfrak{D}_\lambda - \mu\mathfrak{T}$ in $\mathrm{Dom}^c_{m,p}(\mathfrak{D}_\lambda)$ and let $R^{(m)}_\mu$ denote the range of $\mathfrak{D}_\lambda - \mu\mathfrak{T}$ in $\mathcal{X}^c_{m-2,p}$, where $\mathfrak{T} : \mathcal{X}^c_{m-2,p} \to \mathcal{X}^c_{m-2,p}$ is the identity operator. Then $\mathfrak{D}_\lambda - \mu\mathfrak{T}$ is Fredholm with index zero and so each $N^{(m)}_\mu$ is finite dimensional and each $R^{(m)}_\mu$ is closed. Since $\mathcal{X}^c_{m+1,p} \subset \mathcal{X}^c_{m,p}$ (and hence $\mathrm{Dom}^c_{m+1,p}(\mathfrak{D}_\lambda) \subset \mathrm{Dom}^c_{m,p}(\mathfrak{D}_\lambda)$) it follows that $N^{(m+1)}_\mu \subset N^{(m)}_\mu$.

The reverse inclusion will follow from the inequality $\dim(N_\mu^{(m)}) \le \dim(N_\mu^{(m+1)})$, which is an immediate consequence of the inequality we will next prove:

$$\operatorname{codim}(R_\mu^{(m)}) \le \operatorname{codim}(R_\mu^{(m+1)}). \tag{5.16}$$

Define $n = \operatorname{codim}(R_\mu^{(m)})$; we may suppose $n \ne 0$ since otherwise (5.16) is clear. Suppose that

$$\operatorname{span}\{\mathbf{w}_1, ..., \mathbf{w}_n\} \cap R_\mu^{(m)} = \{\mathbf{0}\}, \quad \text{where } \mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_n \in \mathcal{X}^c_{m-2,p}$$

are linearly independent. Then, by the density of C^∞ in our spaces and since $R_\mu^{(m)}$ is closed, we claim that we may assume that each $\mathbf{w}_i \in C^\infty(\overline{\Omega}; \mathbb{C}^2) \cap \mathcal{X}^c_{m-2,p}$. To see this suppose otherwise, then by the density of C^∞ there are sequences $\mathbf{w}_i^{(k)} \in C^\infty(\overline{\Omega}; \mathbb{C}^2) \cap \mathcal{X}^c_{m-2,p}$ that converge to $\mathbf{w}_i$ in $\mathcal{X}^c_{m-2,p}$ and satisfy

$$\sum_{i=1}^n c_i^{(k)} \mathbf{w}_i^{(k)} \in R_\mu^{(m)}, \qquad \mathbf{c}^{(k)} = (c_1^{(k)}, ..., c_n^{(k)}) \in \mathbb{C}^n \setminus \{\mathbf{0}\}.$$

We can then rescale, if necessary, and assume $|\mathbf{c}^{(k)}| = 1$. If we now take the limit as $k \to \infty$ we find that

$$\sum_{i=1}^n c_i \mathbf{w}_i \in R_\mu^{(m)} \text{ with } |\mathbf{c}| = 1,$$

which contradicts the fact that the vectors are linearly independent.

Now let $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and consider

$$\mathbf{w} = \sum_{i=1}^n c_i \mathbf{w}_i.$$

If $\mathbf{w} \in R_\mu^{(m+1)}$ then $\mathbf{w} = (\mathfrak{D}_\lambda - \mu\mathfrak{I})[\mathbf{v}]$ for some $\mathbf{v} \in \mathrm{Dom}^c_{m+1,p}(\mathfrak{D}_\lambda)$ and hence $\mathbf{v} \in \mathrm{Dom}^c_{m,p}(\mathfrak{D}_\lambda)$. Thus $\mathbf{w} \in R_\mu^{(m)}$, which is not possible. Consequently, $\mathbf{w} \notin R_\mu^{(m+1)}$ for any such $\mathbf{c}$ and hence $n \le \operatorname{codim}(R_\mu^{(m+1)})$, which proves (5.16).

We have therefore shown that $N_\mu^{(2)} = N_\mu^{(3)} = \ldots$ for each $\mu \in \mathbb{C}$, which implies $\Sigma^{(2)} = \Sigma^{(3)} = \ldots$. Finally, let $\mathbf{v} \in N_\mu^{(m)}$. Then $\mathbf{v} \in N_\mu^{(k)} \cap \mathcal{X}^c_{k,p}$ for any $k \in \mathbb{Z}^+$. By the Sobolev imbedding theorems, $\mathbf{v} \in C^\infty(\overline{\Omega}; \mathbb{C}^2)$. The proof for fixed m and any $p \in (1, \infty)$ is similar. □

Remarks. 1. The upper semicontinuity of $\sigma(\lambda)$ (Theorem 5.5(v)) is also proved in Kato [36, Chapter IV, Theorem 3.16]. In fact, for any circle Γ of radius ρ centered at $\sigma(\lambda_*)$ in the complex plane, for $|\lambda - \lambda_*|$ sufficiently small, $\mathfrak{D}_\lambda$ has an eigenvalue inside Γ and hence $\sigma(\lambda) \le \sigma(\lambda_*) + \rho$. However the (above quoted) theorem in [36] requires that the gap $\hat{\delta}(\mathfrak{D}_\lambda, \mathfrak{D}_{\lambda_*}) \to 0$ as $|\lambda - \lambda_*| \to 0$. In this regard we refer to Healey and Simpson [35, Proposition A.2] (which requires that $\mathfrak{B}_{\lambda_*}$ be surjective, cf. footnote 33).

2. We note that a crucial hypothesis in Theorem 5.5, namely Agmon's condition on $\overline{\mathcal{S}}$ (which holds for all $\lambda \in (\lambda_0, 1]$) implies that the spectrum of $\mathfrak{D}_\lambda$ does not suddenly expand, as λ varies, by allowing an eigenvalue to enter from $-\infty$ on the real axis. This is seen in (5.15) where the coercivity of the quadratic form Q_λ plays an essential role via Proposition 5.2. In this way the least eigenvalue $\sigma(\lambda)$ of $\mathfrak{D}_\lambda$ varies continuously with λ as long as Agmon's condition holds. If it crosses zero then the possibility of bifurcation of a branch of nontrivial (nonhomogeneous) solutions of (4.2) arises; this is the subject of Section 7.

3. If the Sobolev spaces $\mathcal{X}_{m,p}$ are replaced by Hölder spaces most of the above results are still valid. For $m \in \{2, 3, ...\}$ and $0 < \alpha < 1$ let $\hat{\mathcal{X}}_{m,\alpha}$ denote the set of $\mathbf{u} \in C^{m,\alpha}(\overline{\Omega}; \mathbb{R}^2)$ that satisfy $X_{(i)-(iii)}$ and let $\hat{\mathcal{Y}}_{m,\alpha}$ denote the set of pairs $(\mathbf{h}, \mathbf{g}) \in C^{m-2,\alpha}(\overline{\Omega}; \mathbb{R}^2) \times C^{m-1,\alpha}(\partial\Omega; \mathbb{R}^2)$ that satisfy $Y_{(i)-(iv)}$, with corresponding Hölder norms $\|\cdot\|_{m,\alpha,\overline{\mathcal{R}}_e}$ and $\|\cdot\|_{m-1,\alpha,\overline{\mathcal{S}}_e}$ on $\overline{\mathcal{R}}_e$ and $\overline{\mathcal{S}}_e$, respectively; then $\hat{\mathcal{X}}_{m,\alpha} \subset \mathcal{X}_{1,2}$. With the same operators and domain as above (and $\hat{\mathcal{X}}_{m,\alpha}$, and $\hat{\mathcal{Y}}_{m,\alpha}$ in place of $\mathcal{X}_{m,p}$ and $\mathcal{Y}_{m,p}$, respectively), Proposition 5.3(i) holds with Schauder estimates in (5.9); Proposition 5.4 and Theorem 5.5(i)–(v) and (vii) also are valid. In fact, in the proof of Proposition 5.3(i) one uses $\|\mathbf{v}\|_{m,\alpha,\overline{\mathcal{R}}_e} \le c\,\|\mathbf{v}\|_{m,\alpha,\overline{\mathcal{R}}}$ and the proof is essentially the same. In Proposition 5.4 the solvability of the Poisson problem may be established by a technique in Fourier series similar to the one we use in the Appendix. In Theorem 5.5 again the lower bound on the spectrum of $\mathfrak{D}_\lambda$ is proved directly by coercivity of Q_λ; in part (iv), $\widehat{\Sigma}(\lambda) \subset \Sigma(\lambda)$ follows as in the proof of Proposition 5.3(ii) using $\hat{\mathcal{X}}_{m,\alpha} \subset \mathcal{X}_{2,2}$ and the fact that $\hat{\mathcal{X}}_{m-2,\alpha}$ is dense in $\mathcal{X}_{0,2}$. In proving part (vii) one can establish that the spectrum of $\mathfrak{D}_\lambda$ in the space $\hat{\mathcal{X}}_{m,\alpha}$ equals its spectrum in the space $\mathcal{X}_{m,p}$ for any $m \ge 2$, $1 < p < \infty$, and $0 < \alpha < 1$.

4. Theorem 5.5(i) is also valid under weaker hypotheses. In particular one need not have a continuous one-parameter family of deformations that connects $\mathbf{f}_\lambda$ to a positive natural state. Instead, if C_λ satisfies the strong-ellipticity condition and $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies both the complementing and Agmon's conditions then, by Proposition 5.2, the quadratic form Q_λ is coercive and consequently the operator $(\mathfrak{D}_\lambda + k_1\mathfrak{T}, \mathfrak{B}_\lambda)$ is injective. By a generalization of a theorem of Schechter [50] (see, e.g., [56, Proposition 9.2]) and the fact that $\mathfrak{B}_\lambda$ is surjective (footnote 33), it follows that $(\mathfrak{D}_\lambda + k_1\mathfrak{T}, \mathfrak{B}_\lambda)$ is bijective and hence has index zero. Consequently, $\mathfrak{D}_\lambda + k_1\mathfrak{T}$ also has index zero. A continuity argument (in k_1) then shows that both $\mathfrak{L}_\lambda$ and $\mathfrak{D}_\lambda$ have index zero (Proposition 5.3(i)).

6. The Linearized Operator II: Separation of Variables

In this section we continue our analysis of the linearized operator to determine conditions under which it has a nontrivial null space along the trivial branch $\mathbf{f}_\lambda$. We take $p = 2$, $m \ge 2$,

$\lambda \in (0,1]$, and $\mathbf{f} = \mathbf{f}_\lambda$, as defined in (4.4), where $\mu = \mu(\lambda)$ is given by Proposition 4.1. As in the previous section we let $\mathsf{C}_\lambda, \mathfrak{D}_\lambda, \mathfrak{B}_\lambda$, and Q_λ be defined by (5.6).

In this section we will assume, in addition, that

(H7) $\dfrac{d\mu(\lambda)}{d\lambda} \le 0$ *for* $\lambda \in (0,1]$.

We first note that the constants (in x and y) K, M, N, P, and T defined by (3.13a) and (3.13b) now depend on λ alone. By Proposition 3.2 we have, in particular, assumed in (H5) and (H6) that $M(1) > 0$ and $K(\lambda) > 0$, $P(\lambda) > 0$, and $T(\lambda) > 0$ on $(0,1]$. We note for future reference.

Lemma 6.1. *Assume (H1)–(H5) and (H7). Then* $N(\lambda) \ge 0$ *and* $M(\lambda) > 0$ *for* $\lambda \in (0,1]$.

Proof. We differentiate (4.5) with respect to λ to conclude, with the aid of (3.13a), that

$$K\frac{\mathrm{d}\mu(\lambda)}{\mathrm{d}\lambda} + N = 0. \tag{6.1}$$

Consequently, $N \ge 0$ now follows from (H4), (H7), and (6.1). Finally, by (3.13b) and (4.5)

$$M = N - \sigma,_2 = N + \frac{\mu(\lambda)}{\lambda}P,$$

which is strictly positive by (H5) and Proposition 3.2(i). □

We next substitute $\mathbf{H} = \nabla\mathbf{v}$ into (3.12) and the result into (5.5), with $\mathbf{f} = \mathbf{f}_\lambda$, to conclude

$$K\,\partial_{xx}v_1 + P\,\partial_{yy}v_1 + M\,\partial_{xy}v_2 = 0 \quad \text{in } \mathcal{R}, \tag{6.2a}$$

$$P\,\partial_{xx}v_2 + T\,\partial_{yy}v_2 + M\,\partial_{xy}v_1 = 0 \quad \text{in } \mathcal{R}, \tag{6.2b}$$

$$v_2 = \partial_y v_1 = 0 \quad \text{on } \mathcal{R}_T \cup \mathcal{R}_B, \tag{6.2c}$$

$$K\,\partial_x v_1 + N\,\partial_y v_2 = 0 \quad \text{on } \mathcal{S}, \tag{6.2d}$$

$$P\,\partial_x v_2 + (M-N)\,\partial_y v_1 = 0 \quad \text{on } \mathcal{S}, \tag{6.2e}$$

$$\int_{-R}^{R}\int_0^L v_1(x,y)\,dydx = 0. \tag{6.2f}$$

The characteristic equation of (6.2a)–(6.2b) is (3.16) with $\tau = 0$, where the roots r come in pairs $\pm r_1$ and $\pm r_2$. If $r_1 \ne r_2$, four linearly independent solutions of (6.2a)–(6.2c) and (6.2f) are[34]

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_i = \begin{bmatrix} -Mr_i\sinh(r_i\Lambda_n x)\cos(\Lambda_n y) \\ (Kr_i^2 - P)\cosh(r_i\Lambda_n x)\sin(\Lambda_n y) \end{bmatrix}, \tag{6.3a}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_i = \begin{bmatrix} -Mr_i\cosh(r_i\Lambda_n x)\cos(\Lambda_n y) \\ (Kr_i^2 - P)\sinh(r_i\Lambda_n x)\sin(\Lambda_n y) \end{bmatrix} \tag{6.3b}$$

[34]Cf. Proposition 3.2. Note, however, that here $M > 0$ by Lemma 6.1.

with $i = 1, 2$, $\Lambda_n := n\pi/L$, and $n \in \mathbb{Z}^+$.

If $r_1 = r_2$ the corresponding solutions are

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_1 \text{ as in (6.3a),} \tag{6.4a}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_2 = \frac{\partial}{\partial r} \begin{bmatrix} -Mr \sinh(r\Lambda_n x)\cos(\Lambda_n y) \\ (Kr^2 - P)\cosh(r\Lambda_n x)\sin(\Lambda_n y) \end{bmatrix}\Bigg|_{r=r_1}, \tag{6.4b}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_1 \text{ as in (6.3b),} \tag{6.4c}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_2 = \frac{\partial}{\partial r} \begin{bmatrix} -Mr \cosh(r\Lambda_n x)\cos(\Lambda_n y) \\ (Kr^2 - P)\sinh(r\Lambda_n x)\sin(\Lambda_n y) \end{bmatrix}\Bigg|_{r=r_1}. \tag{6.4d}$$

The real and imaginary parts of the solutions (6.3a), (6.4a), and (6.4b) represent infinitesimal *barrelling* modes (superimposed on $\mathbf{f}_\lambda$) since v_1 is odd about $x = 0$. Those in (6.3b), (6.4c), and (6.4d) represent infinitesimal *buckling* modes since v_1 is even about $x = 0$. We call (6.3a), (6.4a), (6.4b), as well as any linear combinations of these solution *barrelling modes of mode* n with similar terminology in the case of buckling modes.

Let

$$p_1(r) = -KM\,r^2 + N(Kr^2 - P),$$
$$p_2(r) = P(Kr^2 - P)r + (M - N)Mr.$$

If $r_1 \neq r_2$ define the matrices, $n \in \mathbb{Z}^+$,

$$\mathbf{C}^{(0)} = \begin{bmatrix} p_1(r_1) & p_1(r_2) \\ p_2(r_1) & p_2(r_2) \end{bmatrix}, \tag{6.5a}$$

$$\mathbf{C}_n^{(1)} = \begin{bmatrix} p_1(r_1)\cosh(r_1\Lambda_n R) & p_1(r_2)\cosh(r_2\Lambda_n R) \\ p_2(r_1)\sinh(r_1\Lambda_n R) & p_2(r_2)\sinh(r_2\Lambda_n R) \end{bmatrix}, \tag{6.5b}$$

$$\mathbf{C}_n^{(2)} = \begin{bmatrix} p_1(r_1)\sinh(r_1\Lambda_n R) & p_1(r_2)\sinh(r_2\Lambda_n R) \\ p_2(r_1)\cosh(r_1\Lambda_n R) & p_2(r_2)\cosh(r_2\Lambda_n R) \end{bmatrix}, \tag{6.5c}$$

and, if $r_1 = r_2$, define analogous matrices with the same first column and the second column replaced with the derivatives of the first column with respect to r_1. We note that $\mathbf{C}_n^{(1)}$ and $\mathbf{C}_n^{(2)}$ are obtained by the substitution of (6.3a)–(6.4d) into (6.2d)–(6.2e). Define

$$\overline{\mathbf{C}}_n^{(i)} := \begin{cases} \dfrac{1}{\sqrt{r_2 - r_1}}\,\mathbf{C}_n^{(i)}, & r_1 \neq r_2, \\[2ex] \mathbf{C}_n^{(i)}, & r_1 = r_2, \end{cases} \tag{6.6}$$

for $i = 0, 1, 2$; then it is straightforward to show that $\det \overline{\mathbf{C}}_n^{(i)}$ are continuous in λ.

Proposition 6.2. *Assume (H1)–(H4) and (H7). Let* $\lambda \in (0, 1]$. *Suppose that* C_λ *satisfies the strong-ellipticity condition and that* $(\mathsf{C}_\lambda, \mathbf{e}_1)$ *satisfies the complementing condition. Then every solution of* (6.2) *of class* $W^{m,p}(\mathcal{R}; \mathbb{R}^2)$, $m \geq 2$, $1 \leq p \leq \infty$, *is a finite linear combination of solutions of the form* (6.3a) *(or* (6.4a), (6.4b)*) for those* $n \in \mathbb{Z}^+$ *that satisfy* $\det \mathbf{C}_n^{(1)} = 0$ *added to a finite linear combination of solutions of* (6.3b) *(or* (6.4c), (6.4d)*) for* $n \in \mathbb{Z}^+$ *that satisfy* $\det \mathbf{C}_n^{(2)} = 0$.

Proof. This follows as in Davies [24] by substitution of linear combinations of (6.3a) or (6.3b) into (6.2d)–(6.2e) (similarly if $r_1 = r_2$). See, also, Ogden [44, §6.3.1]. □

We will now assume:

(H8) *There exists* $\lambda_\infty \in (0, 1)$ *such that* $A(\lambda_\infty) = 0$ *and* $A(\lambda) := A(\mu(\lambda), \lambda) > 0$ *for* $\lambda \in (\lambda_\infty, 1]$.

Remark. It follows from the proof of Proposition 3.2(ii) that the complementing condition is equivalent to $\det \mathbf{C}^{(0)} \neq 0$. $A(\lambda)$ (see (H8) and (3.14)) is equal to P/λ times the numerator of $\psi(\lambda)$ in Davies [24]. We also note that $\det \overline{\mathbf{C}}^{(0)} = -MA(\lambda)$.

The determination of those λ and n such that $\det \mathbf{C}_n^{(1)} = 0$ or $\det \mathbf{C}_n^{(2)} = 0$ is due to Davies [24] and here we quote some of her results. Assuming[35] (H1)–(H7) Davies proves that, for each mode number $k \in \mathbb{Z}^+$, there exists $\lambda_k \in (\lambda_\infty, 1)$ such that (6.2) has a nontrivial solution at $\lambda = \lambda_k$ of either barrelling or buckling type. In particular there exists a largest $\lambda_c \in (\lambda_\infty, 1)$ such that (6.2) has a nontrivial solution at $\lambda = \lambda_c$. Furthermore, the quadratic form Q_λ is uniformly positive, $(5.2)_2$, for all $\lambda \in (\lambda_c, 1]$. Thus the possibility of bifurcation occurs at $\lambda = \lambda_c$. More information on the nontrivial solutions of (6.2) can be obtained if the roots r_1 and r_2 are real, i.e., if

$$\Delta(\lambda) := \sqrt{KT} - M - P \geq 0.$$

In this case Davies shows that $\lambda_k > \lambda_{k+1}$ and each corresponds to a buckling mode.

We now prove some of the results stated above by a different method. By a *nontrivial solution* of (6.2) we mean a function in $W^{m,p}(\mathcal{R}; \mathbb{R}^2) \setminus \{\mathbf{0}\}$, $m \geq 2$, $1 < p < \infty$, that satisfies (6.2).

Theorem 6.3. *Assume (H1)–(H8). Then:*

(i) There exists $\lambda_c \in (\lambda_\infty, 1)$ *such that* $\mathbf{f}_\lambda$ *is a weak relative minimizer of* E *for* $\lambda \in (\lambda_c, 1]$ *and* (6.2) *has a nontrivial solution at* $\lambda = \lambda_c$*;*

[35] Davies also requires that $\Phi(\lambda, \mu(\lambda)) \leq c\lambda^{-k}$ near $\lambda = 0$ and $|\mu'(\lambda)| \to \infty$ as $\lambda \to 0$, rather than our hypothesis (H8).

(ii) There is a sequence $\lambda_k \in (\lambda_\infty, 1)$ that tends to λ_∞ such that (6.2) *has a nontrivial solution at $\lambda = \lambda_k$ $(k = 1, 2, 3, \ldots)$ that is a linear combination of either barrelling or buckling modes* (6.3a)–(6.4d) *(or both) with mode number k.*

(iii) If $\Delta(\lambda) \geq 0$ for all $\lambda \in (\lambda_\infty, 1]$ then the kernel of $\mathfrak{D}_{\lambda_k}$ is one dimensional for each $k \in \mathbb{Z}^+$. Moreover, the solution $\mathbf{v}_k$ of the linear problem is a buckling solution.

Proof. (i) Since the elasticity tensor is positive definite at $\lambda = 1$ we find that for some $k_1 > 0$

$$\mathrm{Q}_1[\mathbf{u}] \geq k_1 \, \|(\nabla \mathbf{u})_s\|^2_{0,2,\mathcal{R}} \quad \text{for all } \mathbf{u} \in W^{1,2}(\mathcal{R}; \mathbb{R}^2).$$

Now $\mathbf{u} \in \mathrm{Var}$ satisfies (5.5b) and (5.5e) and hence Korn's inequality (2.2) implies that Q_1 is uniformly positive.

Lemma 6.4.[36] *Assume (H1)–(H5), (H7), and (H8). Then there exists $\mathbf{v} \in \mathrm{Var}$ and $\lambda \in (\lambda_\infty, 1)$ such that $\mathrm{Q}_\lambda[\mathbf{v}] < 0$.*

In view of (H3), (H5), (H6), and (H8), Proposition 3.2(ii)–(iv) imply that the pair $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies both the complementing condition and Agmon's condition at each $\lambda \in (\lambda_\infty, 1]$. Therefore, by (5.11) and Theorem 5.5(iv)–(v), the least eigenvalue $\sigma(\lambda)$ of $\mathfrak{D}_\lambda$ varies from positive at $\lambda = 1$ to negative and by continuity equals zero at some $\lambda \in (\lambda_\infty, 1]$. Define

$$\lambda_c := \sup \{\lambda \in (\lambda_\infty, 1] \, : \, \sigma(\lambda) = 0\} \, .$$

Thus for $\lambda \in (\lambda_c, 1]$, $\sigma(\lambda) > 0$ and by Proposition 5.1 and Theorem 5.5(iv), $\mathbf{f}_\lambda$ is a weak relative minimizer of E. Moreover, (6.2) has a nontrivial solution at $\lambda = \lambda_c$ since $\sigma(\lambda_c) = 0$.

(ii) An examination of the proof shows that Lemma 6.4 remains valid if $\mathbf{v}$ is restricted to the subspace, $k \in \mathbb{Z}^+$,

$$\mathrm{Var}_k = \left\{ \mathbf{v} \in \mathrm{Var} \, : \, \mathbf{v} = \begin{bmatrix} v_1(x) \cos(\frac{k\pi}{L} y) \\ v_2(x) \sin(\frac{k\pi}{L} y) \end{bmatrix}, \quad v_i \in W^{1,2}(-R, R), \; i = 1, 2 \right\}.$$

Define

$$\lambda_k := \sup \{\lambda \in (\lambda_\infty, 1] \, : \, \mathrm{Q}_\lambda[\mathbf{v}] \leq 0 \text{ for some } \mathbf{v} \in \mathrm{Var}_k \setminus \{\mathbf{0}\}\} \, ; \tag{6.7}$$

again the proof of Lemma 6.4 together with (H6) imply $\lambda_k \in (\lambda_\infty, \lambda_c]$.

Lemma 6.5. *Assume (H1)–(H6) and (H8). Then for each $k \in \mathbb{Z}^+$ there exists $\mathbf{z} \in \mathrm{Dom}_{m,p}(\mathfrak{D}_{\lambda_k}) \cap \mathrm{Var}_k$, such that $\mathrm{Q}_{\lambda_k}[\mathbf{z}] = 0$ and $\mathbf{z}$ is a nontrivial solution of* (6.2) *at $\lambda = \lambda_k$.*

[36] For a proof of this lemma and Lemma 6.5 see the Appendix.

Then Proposition 6.2 implies $\mathbf{z}$ is a linear combination of barrelling and/or buckling modes with mode number k.

To prove $\lim_{k\to\infty} \lambda_k = \lambda_\infty$ note that

$$\lim_{k\to\infty} 4\mathrm{e}^{-(r_1+r_2)\Lambda_k R} \det \overline{\mathbf{C}}_k^{(i)} = \det \overline{\mathbf{C}}^{(0)},$$

for $i = 1, 2$ (see (6.6)). Thus $\det \mathbf{C}^{(0)}$ vanishes at any accumulation point λ^* of the sequence $\lambda_1, \lambda_2, \ldots$ and so Proposition 3.2(ii), (H8), and the remark following (H8) (i.e., $\det \overline{\mathbf{C}}^{(0)} = -MA(\lambda)$) imply $\lambda^* = \lambda_\infty$.

(iii) If the dimension of the null space of $\mathfrak{D}_{\lambda_k}$ is greater than one, then Proposition 6.2 and the fact that the null space consists only of buckling modes with mode number k (Davies [24]) imply the dimension equals two, i.e., $\mathbf{C}_k^{(2)}$ is the zero matrix. Consequently, $\mathbf{C}^{(0)} = \mathbf{0}$ and hence by Proposition 3.2(ii) and (H8) we conclude that $A(\lambda_k) = 0$ which is impossible since $\lambda_k > \lambda_\infty$. □

Finally we examine the total energy of the homogeneous deformation $\mathbf{f}_\lambda$:

$$\mathcal{E}(\lambda) := E(\mathbf{f}_\lambda) = \int_{\mathcal{R}} W(\nabla \mathbf{f}_\lambda)\, d\mathbf{x}.$$

Proposition 6.6. *Assume (H1)–(H5) and (H7). Then*

$$\frac{d}{d\lambda}\mathcal{E}(\lambda) < 0, \qquad S_{22}(\lambda) := S_{22}(\nabla \mathbf{f}_\lambda) < 0 \quad \textit{for all } \lambda \in (0,1).$$

Moreover, if in addition (H6) and (H8) are satisfied then

$$\frac{d^2}{d\lambda^2}\mathcal{E}(\lambda) > 0 \quad \textit{for all } \lambda \in (\lambda_\infty, 1].$$

Proof. We first note that by (H1)–(H4) (Proposition 4.1), $\mu(1) = 1$ and hence by (H7)

$$1 \le \mu(\lambda) \quad \text{for all } \lambda \in (0,1]. \tag{6.8}$$

Next, we differentiate $\mathcal{E}$ with respect to λ to conclude, with the aid of $(3.2)_1$, (4.2e) (with $\mathbf{f} = \mathbf{f}_\lambda$), (4.4), and an integration by parts, that

$$\mathcal{E}'(\lambda) = \int_{\mathcal{R}} \nabla \mathbf{f}'_\lambda : \mathbf{S}(\nabla \mathbf{f}_\lambda)\, d\mathbf{x} = |\mathcal{R}|\, S_{22}(\lambda), \tag{6.9}$$

where $|\mathcal{R}|$ denotes the area of $\mathcal{R}$ and $\mathbf{f}'_\lambda := \mathrm{d}\mathbf{f}_\lambda/\mathrm{d}\lambda$. By (3.3), (4.5), $(3.13\mathrm{b})_1$, (H5), Proposition 3.2(i), and (6.8)

$$S_{22}(\lambda) = P\lambda \left[1 - \left(\frac{\mu(\lambda)}{\lambda}\right)^2\right] < 0$$

for all $\lambda \in (0,1)$, which together with (6.9) yields the first pair of inequalities.

We now differentiate (6.9) with respect to λ and make use of $(3.2)_2$, (4.2e) (with $\mathbf{f} = \mathbf{f}_\lambda$), (4.4), (5.5d) (with $\mathbf{v} = \mathbf{f}_\lambda$), (3.12), and (6.1) to conclude

$$\mathcal{E}''(\lambda) = \int_{\mathcal{R}} \nabla \mathbf{f}'_\lambda : \mathsf{C}_\lambda \big[\nabla \mathbf{f}'_\lambda\big] \, d\mathbf{x} = |\mathcal{R}| \left(\mathsf{C}_\lambda\big[\nabla \mathbf{f}'_\lambda\big]\right)_{22} = |\mathcal{R}| \left(T - \frac{N^2}{K}\right). \tag{6.10}$$

Next, (3.13b) and (4.5) imply $N - M = -(\mu/\lambda)P$, which together with (3.14), (6.9), and (6.10) yield

$$A(\lambda) = |\mathcal{R}|^{-1} \, P \left[\mathcal{E}''(\lambda) K + \mathcal{E}'(\lambda)\lambda^{-1}\sqrt{KT}\right]. \tag{6.11}$$

Finally, we note that $P > 0$, $T > 0$, and $K > 0$ by (H5) and Proposition 3.2(i), $A(\lambda) > 0$ for $\lambda \in (\lambda_\infty, 1]$ by (H6) and (H8), and $\mathcal{E}'(\lambda) < 0$ for $\lambda \in (0,1)$ by the first part of this proof. Thus the desired convexity of $\mathcal{E}$ on $(\lambda_\infty, 1]$ follows from (6.11). □

7. Local Existence and Stability of Bifurcated Solution Branches

In this section we apply a theorem of Crandall and Rabinowitz [23] to prove that bifurcation occurs from the homogeneous branch in the interval $(0,1)$ (see Theorem 6.3). Throughout this section we will fix $p \in (1,\infty)$ and an integer $m > 1 + 2/p$ (so that each element of $W^{m,p}$ has a C^1 representative) and impose the following additional hypotheses:

(H9) *For some $\lambda_0 \in (\lambda_\infty, 1)$, some $p \in (1,\infty)$, and some integer $m > 1 + 2/p$ the stored energy W is of class C^{m+2} in a neighborhood of*

$$\mathbf{F}_{\lambda_0} := \nabla \mathbf{f}_{\lambda_0}(\mathbf{x}) \ \ \textit{for all} \ \ \mathbf{x} \in \Omega.$$

(H10) *For λ_0 as in (H9) the linear equations* (5.5)*, with $\mathbf{f} = \mathbf{f}_\lambda$ and $\lambda = \lambda_0$, possess a one-dimensional solution set spanned by $\mathbf{u}_0 \in \mathcal{X}_{m,p}$. We write $\mathcal{V}_{m,p}$ for the L^2-orthogonal complement*[37] *of $\mathbf{u}_0$ in $\mathcal{X}_{m,p}$:*

$$\mathcal{V}_{m,p} := \left\{ \mathbf{v} \in \mathcal{X}_{m,p} : \int_{\mathcal{R}} \mathbf{v} \cdot \mathbf{u}_0 \, d\mathbf{x} = 0 \right\}. \tag{7.1}$$

(H11) *At $\lambda = \lambda_0$ as given in (H9) and (H10)*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_{\mathcal{R}} \nabla \mathbf{u}_0 : \mathsf{C}(\nabla \mathbf{f}_\lambda) \left[\nabla \mathbf{u}_0\right] \, d\mathbf{x} \neq 0. \tag{7.2}$$

[37] By Proposition 6.2, $\mathbf{u}_0 \in C^\infty(\overline{\mathcal{R}}; \mathbb{R}^2)$ and so the integral in (7.1) exists. Note that $\mathcal{V}_{m,p}$ is closed in $\mathcal{X}_{m,p}$.

Remark. Given (H9) and (H10) condition (7.2) is equivalent to (cf. Theorem 5.5(iv))

$$\sigma'(\lambda_0) := \frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}(\lambda_0) \neq 0$$

(see, e.g., Chow and Hale [21, pp. 473–474]) since $\sigma(\lambda)$ is a simple eigenvalue. If λ_0 is the largest value of $\lambda \in (0,1]$ such that $\sigma(\lambda_0) = 0$, i.e., $\lambda_0 = \lambda_c$ in Theorem 6.3, then under the additional hypotheses (H1)–(H8), condition (7.2) is equivalent to $\sigma'(\lambda_0) > 0$.

Recall (see (4.8)–(4.9)) that

$$\mathfrak{F} : \mathrm{Dom}(\mathfrak{F}) \subset (0,\infty) \times \mathcal{X} \to \mathcal{Y},$$

where

$$\mathcal{X} := \mathcal{X}_{m,p}, \text{ with norm } \|\cdot\|_{m,p,\mathcal{R}_e},$$

is the subspace of all $\mathbf{u} \in W^{m,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ that satisfy $\mathrm{X}_{(i)-(iii)}$ in §4;

$$\mathcal{Y} := \mathcal{Y}_{m,p}, \text{ with norm } \|(\mathbf{g},\mathbf{h})\|_{\mathcal{Y}} := \|\mathbf{g}\|_{m-2,p,\mathcal{R}_e} + \|\mathbf{h}\|_{m-1-1/p,p,\mathcal{S}_e},$$

is the subspace of all $(\mathbf{g},\mathbf{h}) \in W^{m-2,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^2) \times W^{m-1-1/p,p}_{\mathrm{loc}}(\partial\Omega;\mathbb{R}^2)$ that satisfy $\mathrm{Y}_{(i)-(iv)}$ in §4; and

$$\mathrm{Dom}(\mathfrak{F}) := \left\{(\lambda,\mathbf{u}) \in (0,\infty) \times \mathcal{X} : \det\left(\mathbf{F}_\lambda + \nabla\mathbf{u}(\mathbf{x})\right) > 0 \text{ for every } \mathbf{x} \in \overline{\Omega}\right\}.$$

Theorem 7.1. *Assume (H1)–(H6) and (H8)–(H11). Then there exists a neighborhood* $\mathcal{U}$ *of* $(\lambda_0,\mathbf{0})$ *in* $\mathrm{Dom}(\mathfrak{F})$ *such that a smooth branch of solutions* $(\lambda,\mathbf{f}_\lambda+\mathbf{u})$ *of* (4.2) *with* $(\lambda,\mathbf{u}) \in \mathcal{U}$ *and* $\mathbf{f}_\lambda + \mathbf{u} \in \mathrm{Def}$ *bifurcates from* $\mathbf{f}_{\lambda_0}$*; more specifically, there exists an* $\varepsilon > 0$ *and* C^1 *functions* $\Lambda : (-\varepsilon,\varepsilon) \to (0,1]$ *and* $\mathbf{g} : (-\varepsilon,\varepsilon) \to \mathcal{V} := \mathcal{V}_{m,p}$ *such that* $\Lambda(0) = \lambda_0$*,* $\mathbf{g}(0) = \mathbf{0}$*, and the solution set of* (4.2)*,* $(\lambda,\mathbf{f}_\lambda+\mathbf{u})$ *with* $(\lambda,\mathbf{u}) \in \mathcal{U}$*, consists of the distinct branches* $\{(\lambda,\mathbf{f}_\lambda) : \lambda \in (0,1]$ *and* $(\lambda,\mathbf{0}) \in \mathcal{U}\}$ *and*

$$\left\{(\Lambda(s),\ \mathbf{f}_{\Lambda(s)} + s\mathbf{u}_0 + s\mathbf{g}(s))\ :\ |s| < \varepsilon \text{ and } (\Lambda(s), s\mathbf{u}_0 + s\mathbf{g}(s)) \in \mathcal{U}\right\}. \tag{7.3}$$

Moreover, there are no other solutions $(\lambda,\mathbf{f}_\lambda+\mathbf{v})$ *of* (4.2) *with* $(\lambda,\mathbf{v}) \in \mathcal{U}$*. Furthermore,* $\Lambda(s) = \Lambda(-s)$ *for all* $s \in (-\varepsilon,\varepsilon)$*.*

Remark. Since $\Lambda(s)$ is even the bifurcation is of *pitchfork type.*

Proof. We first note that $\mathfrak{F}(\lambda,\mathbf{0}) = \mathbf{0}$ for $\lambda \in (0,1]$. By Proposition 4.4 and (H9), $\mathfrak{F}$ is C^2 in a neighborhood of the bifurcation point $(\lambda_0,\mathbf{0})$. Moreover, by (H5), (H8), and Proposition 3.2(ii) the linearized operator, $\mathfrak{L}_{\lambda_0} = (\partial_{\mathbf{u}}\mathfrak{F})(\lambda_0,\mathbf{0}) \in \mathrm{BL}(\mathcal{X};\mathcal{Y})$ satisfies the hypotheses of Theorem 5.5 and consequently, by Theorem 5.5(i), $\mathfrak{L}_{\lambda_0}$ is a Fredholm operator with index zero. In addition, in view of (H10), $\mathfrak{L}_{\lambda_0}$ has a one-dimensional kernel and cokernel.

We next note that it follows from Proposition 4.4 that $(\partial_\lambda \partial_{\mathbf{u}} \mathfrak{F})(\lambda_0, \mathbf{0}) \in \mathrm{BL}(\mathcal{X}, \mathcal{Y})$. Thus, in order to apply the desired theorem in [23] all we now need to show is that

$$(\partial_\lambda \partial_{\mathbf{u}} \mathfrak{F})(\lambda_0, \mathbf{0})[\mathbf{u}_0] \notin \mathrm{Range}\left((\partial_{\mathbf{u}} \mathfrak{F})(\lambda_0, \mathbf{0})\right). \tag{7.4}$$

We will show that this containment constraint is equivalent to our bifurcation condition (7.2). Consider the nontrivial bounded[38] linear functional $\psi \in \mathcal{Y}^*$ defined by

$$\psi(\mathbf{h}, \mathbf{g}) := -\int_{\mathcal{R}} \mathbf{u}_0 \cdot \mathbf{h}\, d\mathbf{x} + \int_{\mathcal{S}} \mathbf{u}_0 \cdot \mathbf{g}\, dS_{\mathbf{x}}, \quad (\mathbf{h}, \mathbf{g}) \in \mathcal{Y}.$$

If $(\mathbf{h}, \mathbf{g}) \in \mathrm{Range}\left((\partial_{\mathbf{u}} \mathfrak{F})(\lambda_0, \mathbf{0})\right)$ let $\mathbf{v} \in \mathcal{X}$ satisfy $(\partial_{\mathbf{u}} \mathfrak{F})(\lambda_0, \mathbf{0})[\mathbf{v}] = (\mathbf{h}, \mathbf{g})$. Then, by (3.7), (5.8), and the paragraph following (5.5)

$$\begin{aligned}
\psi(\mathbf{h}, \mathbf{g}) &= \int_{\mathcal{R}} \mathbf{u}_0 \cdot \mathfrak{D}_{\lambda_0}[\mathbf{v}]\ d\mathbf{x} + \int_{\mathcal{S}} \mathbf{u}_0 \cdot \mathfrak{B}_{\lambda_0}[\mathbf{v}]\ dS_{\mathbf{x}} \\
&= \int_{\mathcal{R}} \nabla \mathbf{u}_0 : \mathsf{C}_{\lambda_0}[\nabla \mathbf{v}]\ d\mathbf{x} = \int_{\mathcal{R}} \nabla \mathbf{v} : \mathsf{C}_{\lambda_0}[\nabla \mathbf{u}_0]\ d\mathbf{x} \\
&= \int_{\mathcal{R}} \mathbf{v} \cdot \mathfrak{D}_{\lambda_0}[\mathbf{u}_0]\ d\mathbf{x} + \int_{\mathcal{S}} \mathbf{v} \cdot \mathfrak{B}_{\lambda_0}[\mathbf{u}_0]\ dS_{\mathbf{x}} = 0.
\end{aligned}$$

Thus the kernel of ψ coincides with Range$((\partial_{\mathbf{u}} \mathfrak{F})(\lambda_0, \mathbf{0}))$ and hence (7.4) is satisfied if and only if $\psi\left((\partial_\lambda \partial_{\mathbf{u}} \mathfrak{F})(\lambda_0, \mathbf{0})[\mathbf{u}_0]\right) \neq 0$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\int_{\mathcal{R}} \mathbf{u}_0 \cdot \mathfrak{D}_\lambda[\mathbf{u}_0]\ d\mathbf{x} + \int_{\mathcal{S}} \mathbf{u}_0 \cdot \mathfrak{B}_\lambda[\mathbf{u}_0]\ dS_{\mathbf{x}}\right]\Bigg|_{\lambda=\lambda_0} \neq 0,$$

which is (7.2) after integration by parts.

We have now verified all of the hypotheses of Theorem 1.7 in Crandall and Rabinowitz [23] and so can conclude the existence of $\varepsilon > 0$, $\mathcal{U}$, Λ, and $\mathbf{g}$ such that (7.3) is a solution branch with $(\Lambda(s), s\mathbf{u}_0 + s\mathbf{g}(s)) \in \mathcal{U}$ for $s \in (-\varepsilon, \varepsilon)$, as well as the nonexistence of any other solutions $(\lambda, \mathbf{f}_\lambda + \mathbf{v})$ of (4.2) with $(\lambda, \mathbf{v}) \in \mathcal{U}$.

In order to prove that $s \mapsto \Lambda(s)$ is even, suppose $\mathbf{u}_0$ is a (buckled or barrelled) solution of (6.2) with mode number $n \geq 1$. Now replace the rectangle $\mathcal{R}$ of height L with the rectangle $\mathcal{R}_n := \mathcal{R} \cap \{0 < y < L/n\}$ with lateral boundary $\mathcal{S}_n := \mathcal{S} \cap \{0 < y < L/n\}$. The spaces $\mathcal{X}^n_{m,p}$ and $\mathcal{Y}^n_{m,p}$ are then defined analogously with $\mathcal{R}$ replaced by $\mathcal{R}_n$, $\mathcal{S}$ replaced by $\mathcal{S}_n$, and the same operator $\mathfrak{F}$ given by (4.8). Then $\mathbf{u}_0 \in \mathcal{X}^n_{m,p}$ and hypotheses (H1)–(H6) and (H8)–(H11) are still valid (note the integral in (7.2) is equal to n times the integral over $\mathcal{R}_n$). Thus the first part of the theorem implies that, for $|s| \leq \varepsilon_1 \leq \varepsilon$, $\mathbf{u}(s) := s\mathbf{u}_0 + s\mathbf{g}(s)$ lies in $\mathcal{X}^n_{m,p}$

[38] The continuity of ψ is a consequence of $\mathbf{u}_0 \in C^\infty(\overline{\mathcal{R}}; \mathbb{R}^2)$, see footnote 37.

and $\mathbf{g}(s) \in \mathcal{V}^n_{m,p}$ (with $\mathcal{X}_{m,p}$ and $\mathcal{R}$ replaced by $\mathcal{X}^n_{m,p}$ and $\mathcal{R}_n$, respectively, in (7.1)). Choose $\varepsilon_1 > 0$ so that, in addition,

$$\|\mathbf{u}(s) - s\mathbf{u}_0\| < |s|\,\|\mathbf{u}_0\| \quad \text{for} \quad |s| < \varepsilon_1, \tag{7.5}$$

where $\|\cdot\|$ is the norm on $\mathcal{X}^n_{m,p}$.

Define the norm-preserving isomorphism $\mathfrak{R} : \mathcal{X}^n_{m,p} \to \mathcal{X}^n_{m,p}$ by

$$(\mathfrak{R}\mathbf{v})(x,y) := \begin{bmatrix} v_1(x, L/n - y) \\ -v_2(x, L/n - y) \end{bmatrix} \text{ for } \mathbf{v} \in \mathcal{X}^n_{m,p}$$

and note that $\mathfrak{R}$ is self-adjoint under the $L^2(\mathcal{R}_n)$-inner product. Then the nonlinear operator $\mathfrak{F}$ satisfies $\mathfrak{F}\circ\mathfrak{R} = \mathfrak{R}\circ\mathfrak{F}$ and hence $\mathfrak{F}(\lambda, \mathfrak{R}\mathbf{u}) = \mathbf{0}$ whenever $\mathfrak{F}(\lambda, \mathbf{u}) = \mathbf{0}$. However, $\mathfrak{R}\mathbf{u}_0 = -\mathbf{u}_0$ and so $\mathfrak{R}\mathbf{u}(s) = -s\mathbf{u}_0 + s\mathfrak{R}\mathbf{g}(s)$. It then follows that $\mathfrak{R}\mathbf{u}(s)$ must be nonzero for small $s \neq 0$ since $\mathfrak{R}\mathbf{u}(s) = \mathbf{0}$ together with (7.5) yield

$$\|s\mathbf{u}_0\| = \|s\mathfrak{R}\mathbf{g}(s)\| = \|s\mathbf{g}(s)\| = \|\mathbf{u}(s) - s\mathbf{u}_0\| < |s|\,\|\mathbf{u}_0\|\,,$$

which is a contradiction. Since the only other solution that lies in the neighborhood $\mathcal{U}$ is the solution branch (7.3) we must have $(\Lambda(s), \mathfrak{R}\mathbf{u}(s)) = (\Lambda(r(s)), \mathbf{u}(r(s)))$ for some $r(s) \in (-\varepsilon_1, \varepsilon_1)$. If we take the dot product of $\mathfrak{R}\mathbf{u}(s) = \mathbf{u}(r(s))$ with $\mathbf{u}_0$ and integrate over the region $\mathcal{R}_n$ we conclude $r(s) = -s$ since $\mathbf{g}(s) \in \mathcal{V}^n_{m,p} = \mathfrak{R}(\mathcal{V}^n_{m,p})$. Finally, $\Lambda(s) = \Lambda(r(s)) = \Lambda(-s)$ concludes the proof. □

Remarks. 1. The proof shows that

$$\mathbf{u}(-s) = \mathfrak{R}\mathbf{u}(s), \quad -\varepsilon_1 < s < \varepsilon_1$$

and hence each branch of the (pitchfork) bifurcation is a reflection of the other branch under $\mathfrak{R}$. This result may also be obtained directly from a theorem in Golubitsky and Schaeffer [31, p. 306] that shows the reduced equations maintain the reflection symmetry of the original problem.

2. The conclusions of Theorem 7.1 may be obtained by replacing $\mathcal{X}_{m,p}$ and $\mathcal{Y}_{m,p}$ with the Hölder spaces $C^{m,\alpha}(\overline{\Omega};\mathbb{R}^2)$ and $C^{m-2,\alpha}(\overline{\Omega};\mathbb{R}^2) \times C^{m-1,\alpha}(\partial\Omega;\mathbb{R}^2)$, respectively (with the reflection, periodicity and normalization conditions $X_{(i)-(iii)}$, $Y_{(i)-(iv)}$, $m \geq 2$, and $0 < \alpha < 1$); see Remark 3 following Theorem 5.5. Assuming (H1)–(H6) and (H8)–(H11) with W of class C^{m+3} near $\mathbf{F}_{\lambda_0}$ in (H9), instead, and the Hölder space replacing $\mathcal{X}_{m,p}$ in Dom($\mathfrak{F}$), then Proposition 4.4 (see Valent [63, Chapter II, Theorem 4.4]), Theorem 5.5 and Theorem 7.1 remain valid. Therefore we conclude that if $W \in C^{m+3}$ near $\mathbf{F}_{\lambda_0}$ then each element of the solution branch in (7.3) is of class $C^{m,\alpha}$ $(0 < \alpha < 1)$: $\mathbf{g}(s) \in C^{m,\alpha}(\overline{\Omega};\mathbb{R}^2)$ for each s sufficiently small.

The next proposition determines the direction of bifurcation in Theorem 7.1 in the sense that $\Lambda(s) - \lambda_0$ is positive or negative.[39] Define

$$\rho := \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_{\mathcal{R}} \nabla \mathbf{u}_0 : \mathsf{C}_\lambda[\nabla \mathbf{u}_0] \, d\mathbf{x} \right) \bigg|_{\lambda=\lambda_0} \tag{7.6}$$

and write $\mathbf{u}(s) := s\mathbf{u}_0 + s\mathbf{g}(s)$ as in (7.3).

Proposition 7.2. *Assume (H1)–(H6), (H8)–(H11), and that W is of class C^{m+3} in a neighborhood of $\mathbf{F}_{\lambda_0}$. Then*

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}s}(0) = 0, \tag{7.7a}$$

$$\frac{\mathrm{d}^2\Lambda}{\mathrm{d}s^2}(0) = (3\rho)^{-1} \left\{ 3 \int_{\mathcal{R}} \nabla \mathbf{z} : \mathsf{C}_{\lambda_0}[\nabla \mathbf{z}] \, d\mathbf{x} - \int_{\mathcal{R}} \nabla \mathbf{u}_0 : \mathbb{E}_0[\nabla \mathbf{u}_0, \nabla \mathbf{u}_0, \nabla \mathbf{u}_0] \, d\mathbf{x} \right\}, \tag{7.7b}$$

where $\mathbf{z} \in \mathcal{X}$ is the unique solution of the boundary value problem

$$\operatorname{div} \mathsf{C}_{\lambda_0}[\nabla \mathbf{z}] = -\operatorname{div} \mathbb{D}_0[\nabla \mathbf{u}_0, \nabla \mathbf{u}_0] \quad \text{in } \mathcal{R}, \tag{7.8a}$$

$$\mathsf{C}_{\lambda_0}[\nabla \mathbf{z}] \, \mathbf{n} = -\mathbb{D}_0[\nabla \mathbf{u}_0, \nabla \mathbf{u}_0] \, \mathbf{n} \quad \text{on } \mathcal{S}, \tag{7.8b}$$

and $\mathbf{z}$ is $L^2(\mathcal{R})$-orthogonal to $\mathbf{u}_0$. Here $\mathbb{D}_0 := \mathbb{D}(\nabla \mathbf{f}_{\lambda_0})$ and $\mathbb{E}_0 := \mathbb{E}(\nabla \mathbf{f}_{\lambda_0})$ are defined by (3.8).

Proof. By Theorem 7.1 the function Λ is even and hence (7.7a) is satisfied. Since $W \in C^{m+3}$, Proposition 4.1 and (an extension of) Proposition 4.4 imply that $\lambda \mapsto \mu(\lambda)$ is of class C^2 and, in some neighborhood of $(\lambda_0, \mathbf{0})$, $\mathfrak{F}$ is a C^2 mapping and $\partial^3_{\mathbf{u}}\mathfrak{F}$ exists and is continuous. Thus by Theorem 1.18 in Crandall and Rabinowitz [23], Λ and $\mathbf{g}$ are of class C^2 in the variable s.

We will next differentiate $\mathfrak{F}(\Lambda(s), \mathbf{u}(s)) = \mathbf{0}$ with respect to s. We recall that the homogeneous solution branch is given by $\mathfrak{F}(\lambda, \mathbf{0}) = \mathbf{0}$ for $\lambda \in (0, 1]$ and to simplify the computation we note that

$$\Lambda(0) = \lambda_0, \qquad \mathbf{u}(0) = \mathbf{0}, \qquad \dot{\mathbf{u}}(0) = \mathbf{u}_0, \tag{7.9a}$$

$$(\partial_\lambda \mathfrak{F})(\lambda_0, \mathbf{0}) = (\partial^2_\lambda \mathfrak{F})(\lambda_0, \mathbf{0}) = (\partial^3_\lambda \mathfrak{F})(\lambda_0, \mathbf{0}) = \mathbf{0}, \tag{7.9b}$$

where the dot denotes the derivative with respect to s. Thus, if we differentiate $\mathfrak{F}(\Lambda(s), \mathbf{u}(s)) = \mathbf{0}$ three times with respect to s and let $s = 0$ in the result of each differentiation we conclude that

$$(\partial_{\mathbf{u}}\mathfrak{F})^0 \, [\dot{\mathbf{u}}(0)] = \mathbf{0}, \qquad (\partial_{\mathbf{u}}\mathfrak{F})^0 \, [\ddot{\mathbf{u}}(0)] + (\partial^2_{\mathbf{u}}\mathfrak{F})^0 \, [\mathbf{u}_0, \mathbf{u}_0] = \mathbf{0}, \tag{7.10a}$$

$$(\partial_{\mathbf{u}}\mathfrak{F})^0 \, [\dddot{\mathbf{u}}(0)] + 3(\partial^2_{\mathbf{u}}\mathfrak{F})^0 \, [\ddot{\mathbf{u}}(0), \mathbf{u}_0] + 3\ddot{\lambda}(0)(\partial_{\mathbf{u}}\partial_\lambda \mathfrak{F})^0 \, [\mathbf{u}_0] + (\partial^3_{\mathbf{u}}\mathfrak{F})^0 \, [\mathbf{u}_0, \mathbf{u}_0, \mathbf{u}_0] = \mathbf{0}, \tag{7.10b}$$

[39] Triantafyllidis, Scherzinger, and Huang [61] have previously obtained similar results for the same problem.

where the superscript zero denotes evaluation of a derivative of $\mathfrak{F}$ at $(\lambda, \mathbf{u}) = (\lambda_0, \mathbf{0})$. Furthermore, it is clear from (3.8) and (4.8) that

$$(\partial^2_{\mathbf{u}}\mathfrak{F})^0\,[\mathbf{u}_0, \mathbf{u}_0] = (\operatorname{div}\mathbb{D}_0[\nabla\mathbf{u}_0, \nabla\mathbf{u}_0]\,, \mathbb{D}_0[\nabla\mathbf{u}_0, \nabla\mathbf{u}_0]\,\mathbf{n})\,.$$

However, $\ddot{\mathbf{u}}(0) = 2\dot{\mathbf{g}}(0) \in \mathcal{V}$ and hence it is $L^2(\mathcal{R})$-orthogonal to $\mathbf{u}_0$. Since equations (7.8) and $(7.10\text{a})_2$ are equivalent we conclude that $\mathbf{z} = \ddot{\mathbf{u}}(0)$ and that (7.8a) and (7.8b) are satisfied.

Next, $(7.10\text{a})_2$ and (7.10b) are each equivalent to identities in both the interior and on the lateral surface of the rectangle $\mathcal{R}$. In order to proceed we will need similar identities on the top and bottom surfaces of the rectangle. We first note that if we differentiate (4.2d) with respect to λ and evaluate at $\lambda = \lambda_0$ we find that

$$\big(\mathsf{C}_{\lambda_0}\big[\,\nabla\mathbf{f}_0'\,\big]\mathbf{n}\big)_1 = 0 \quad \text{on } \mathcal{R}_T \cup \mathcal{R}_B, \qquad \mathbf{f}_0' := \frac{d}{d\lambda}\mathbf{f}_\lambda\bigg|_{\lambda=\lambda_0}. \tag{7.11}$$

Similarly, $\mathbf{f}(s) := \mathbf{f}_{\Lambda(s)} + \mathbf{u}(s)$ given by Theorem 7.1 also satisfies (4.2d), that is,

$$\big[\mathbf{S}(\nabla\mathbf{f}(s))\mathbf{n}\big]_1 = 0 \quad \text{on } \mathcal{R}_T \cup \mathcal{R}_B.$$

We differentiate this equation three times with respect to s and let $s = 0$ in the result of each differentiation to conclude, with the aid of (7.7a), (7.9a), $(7.11)_1$, and $\mathbf{z} = \ddot{\mathbf{u}}(0)$, that the following equations are satisfied on $\mathcal{R}_T \cup \mathcal{R}_B$:

$$\big(\mathsf{C}_{\lambda_0}[\nabla\mathbf{u}_0]\mathbf{n}\big)_1 = 0, \qquad \big[\big(\mathsf{C}_{\lambda_0}[\nabla\mathbf{z}] + \mathbb{D}_0[\nabla\mathbf{u}_0, \nabla\mathbf{u}_0]\big)\mathbf{n}\big]_1 = 0, \tag{7.12a}$$

$$\big[\big(\mathsf{C}_{\lambda_0}[\nabla\dddot{\mathbf{u}}(0)] + 3\mathbb{D}_0[\nabla\mathbf{z} + \ddot{\lambda}(0)\nabla\mathbf{f}_0', \nabla\mathbf{u}_0] + \mathbb{E}_0[\nabla\mathbf{u}_0, \nabla\mathbf{u}_0, \nabla\mathbf{u}_0]\big)\mathbf{n}\big]_1 = 0. \tag{7.12b}$$

We next take the inner product of (7.8a) with $\mathbf{z}$, integrate over the region $\mathcal{R}$, and make use of the divergence theorem, (5.8), (7.8b), $(7.12\text{a})_2$, and the fact[40] that $z_2 = 0$ on $\mathcal{R}_T \cup \mathcal{R}_B$ to conclude

$$0 = \int_{\mathcal{R}} \nabla\mathbf{z} : \big(\mathsf{C}_{\lambda_0}[\nabla\mathbf{z}] + \mathbb{D}_0[\nabla\mathbf{u}_0, \nabla\mathbf{u}_0]\big)\,d\mathbf{x}. \tag{7.13}$$

In order to simplify a similar computation that starts with (7.10b) we note that

$$\begin{aligned}
(\partial_{\mathbf{u}}\partial_\lambda\mathfrak{F})^0\,[\mathbf{u}_0] &= \big(\operatorname{div}\mathbb{D}_0\big[\nabla\mathbf{f}_0', \nabla\mathbf{u}_0\big]\,, \mathbb{D}_0\big[\nabla\mathbf{f}_0', \nabla\mathbf{u}_0\big]\mathbf{n}\big)\,,\\
(\partial^3_{\mathbf{u}}\mathfrak{F})^0\,[\mathbf{u}_0, \mathbf{u}_0, \mathbf{u}_0] &= (\operatorname{div}\mathbb{E}_0[\nabla\mathbf{u}_0, \nabla\mathbf{u}_0, \nabla\mathbf{u}_0]\,, \mathbb{E}_0[\nabla\mathbf{u}_0, \nabla\mathbf{u}_0, \nabla\mathbf{u}_0]\mathbf{n})\,,
\end{aligned} \tag{7.14}$$

where $\mathbf{f}_0'$ is defined by $(7.11)_2$. We take the inner product of the first component of (7.10b) with $\mathbf{u}_0$, integrate over the region $\mathcal{R}$, and make use of the divergence theorem, (5.8), (7.12b),

[40] Recall $\mathbf{z} = \ddot{\mathbf{u}}(0)$ and $\mathbf{u}(s) \in \mathcal{X}$ for each s.

(7.14), the second component of (7.10b), and the fact that $\dot{u}_2(0) = 0$ on $\mathcal{R}_T \cup \mathcal{R}_B$ to arrive at

$$0 = \int_{\mathcal{R}} \nabla \mathbf{u}_0 : \Big(\mathsf{C}_{\lambda_0}[\nabla \ddot{\mathbf{u}}(0)] + 3\mathbb{D}_0\Big[\nabla \mathbf{z} + \ddot{\lambda}(0)\nabla \mathbf{f}_0', \nabla \mathbf{u}_0\Big] + \mathbb{E}_0[\nabla \mathbf{u}_0, \nabla \mathbf{u}_0, \nabla \mathbf{u}_0]\Big)\, d\mathbf{x}. \quad (7.15)$$

Finally, $(7.10\text{a})_1$, (5.8), $(7.12\text{a})_1$, and the fact that $\ddot{u}_2(0) = 0$ on $\mathcal{R}_T \cup \mathcal{R}_B$ imply

$$\begin{aligned} \int_{\mathcal{R}} \nabla \mathbf{u}_0 : \mathsf{C}_{\lambda_0}[\nabla \ddot{\mathbf{u}}(0)]\, d\mathbf{x} &= \int_{\mathcal{R}} \nabla \ddot{\mathbf{u}}(0) : \mathsf{C}_{\lambda_0}[\nabla \mathbf{u}_0]\, d\mathbf{x} \\ &= \int_{\mathcal{R}} \ddot{\mathbf{u}}(0) \cdot \mathfrak{D}_{\lambda_0}[\mathbf{u}_0]\, d\mathbf{x} = 0 \end{aligned}$$

and, consequently, (7.7b) now follows from (7.6), (7.13), (7.15), (3.9) (the symmetry of $\mathbb{D}_0$, viz., $\nabla \mathbf{z} : \mathbb{D}_0[\nabla \mathbf{u}_0, \nabla \mathbf{u}_0] = \nabla \mathbf{u}_0 : \mathbb{D}_0[\nabla \mathbf{z}, \nabla \mathbf{u}_0]$), and the fact that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big(\nabla \mathbf{u}_0 : \mathsf{C}_\lambda[\nabla \mathbf{u}_0]\Big)\Big|_{\lambda=\lambda_0} = \nabla \mathbf{u}_0 : \mathbb{D}_0\big[\nabla \mathbf{f}_0', \nabla \mathbf{u}_0\big]\,.$$

$\square$

Remarks. 1. If $\dot{\Lambda}(s) \neq 0$ for $0 < |s| < s_0 < \varepsilon$, where ε is given by Theorem 7.1, a further result of Crandall and Rabinowitz [23, Theorem 1.17] implies that $\mathfrak{L}_s := -(\partial_{\mathbf{u}}\mathfrak{F})(\Lambda(s), \mathbf{u}(s))$, $\mathfrak{L}_s \in \mathrm{BL}(\mathcal{X}, \mathcal{Y})$, is a bijection for $s \in (-s_0, 0) \cup (0, s_0)$ when s_0 is sufficiently small. In particular, if $\ddot{\Lambda}(0) \neq 0$ then since $\dot{\Lambda}(0) = 0$ it will follow that $\dot{\Lambda}(s) \neq 0$ on some such interval and hence $\mathfrak{L}_s$ will be a bijection on that interval.

2. If $\ddot{\Lambda}(0) < 0$ and $\lambda_0 = \lambda_c$ (see Theorem 6.3) then, since $\sigma(\lambda) < 0$ for $\lambda \in (\lambda_c - \varepsilon, \lambda_c)$, the invariance of the topological degree of Healey and Simpson [35, Equation 4.19], continuity of the lowest eigenvalue (Theorem 5.5(v)), and the fact that $\mathfrak{L}_s$ is a bijection show that the lowest eigenvalue of $\mathfrak{L}_s$ is positive.[41] Therefore the bifurcation solution $\mathbf{f}(s) = \mathbf{f}_{\Lambda(s)} + \mathbf{u}(s)$ is a relative minimizer of the energy E as long as $\dot{\Lambda}(s) \neq 0$ and s is sufficiently small. Note the operator $\mathfrak{L}_s$ has all the properties of Theorem 5.5 for such s (see [58]).

Next we examine the energy difference between the homogeneous and bifurcated branches. This is given by

$$(\Delta E)(s) = \int_{\mathcal{R}} \big[W(\nabla \mathbf{f}(s)) - W(\nabla \mathbf{f}_{\Lambda(s)})\big]\, d\mathbf{x}, \quad (7.16)$$

where $\mathbf{f}(s) = \mathbf{f}_{\Lambda(s)} + \mathbf{u}(\mathbf{s})$. We show that, if $\rho > 0$ (see (7.6)), then $(\Delta E)(s)$ is positive or negative with $\Lambda(s) - \lambda_0$.

[41] In equation (4.19) in [35], ν will count the number of negative eigenvalues of $\mathfrak{L}_{\Lambda(s)}$ and $\mathfrak{L}_s$ with multiplicity.

Proposition 7.3. *If the hypotheses of Proposition 7.2 are assumed and $\rho > 0$, then $\Lambda(s)$ nonincreasing (nondecreasing) for $s \in (0, \varepsilon)$ implies $(\Delta E)(s)$ is nonincreasing (nondecreasing) for ε sufficiently small.*

Proof. It follows from Proposition 7.2 that

$$\mathbf{u}(s) = \mathbf{f}(s) - \mathbf{f}_{\Lambda(s)} = s\mathbf{u}_0 + \frac{1}{2}s^2\mathbf{z} + o(s^2), \quad \text{as} \quad s \to 0. \tag{7.17}$$

If we differentiate (7.16) we find that

$$\frac{\mathrm{d}(\Delta E)(s)}{\mathrm{d}s} = \frac{\mathrm{d}\Lambda}{\mathrm{d}s} \int_{\mathcal{R}} \nabla \mathbf{f}'_{\Lambda(s)} : \left[\mathbf{S}(\nabla \mathbf{f}(s)) - \mathbf{S}(\nabla \mathbf{f}_{\Lambda(s)})\right] d\mathbf{x}, \tag{7.18}$$

where $\mathbf{f}'_\lambda := \mathrm{d}\mathbf{f}_\lambda/\mathrm{d}\lambda$. Let $I(s)$ denote the integral on the right-hand side of (7.18) and use Taylor series to expand the bracketed quantity about $\mathbf{u} = \mathbf{0}$ to arrive at

$$\begin{aligned} I(s) = &\int_{\mathcal{R}} \nabla \mathbf{f}'_{\Lambda(s)} : \mathsf{C}(\nabla \mathbf{f}_{\Lambda(s)}) \left[\nabla \mathbf{u}(s)\right] \, d\mathbf{x} \\ &+ \frac{1}{2} \int_{\mathcal{R}} \nabla \mathbf{f}'_{\Lambda(s)} : \mathbb{D}(\nabla \mathbf{f}_{\Lambda(s)}) \left[\nabla \mathbf{u}(s), \nabla \mathbf{u}(s)\right] \, d\mathbf{x} + o(\|\mathbf{u}(s)\|^2_{1,\mathcal{R}}). \end{aligned}$$

Next if one differentiates (4.2) with respect to λ at $\mathbf{f} = \mathbf{f}_\lambda$, makes use of (3.7) (the symmetry of C), and the divergence theorem, it follows that the first integral in the right-hand side of $I(s)$ is zero. Thus by (7.17) and (3.9)

$$\begin{aligned} 2I(s) &= s^2 \int_{\mathcal{R}} \nabla \mathbf{f}'_{\lambda_0} : \mathbb{D}_0 \left[\nabla \mathbf{u}_0, \nabla \mathbf{u}_0\right] \, d\mathbf{x} + o(s^2) \\ &= s^2 \int_{\mathcal{R}} \nabla \mathbf{u}_0 : \mathbb{D}_0 \left[\nabla \mathbf{f}'_{\lambda_0}, \nabla \mathbf{u}_0\right] d\mathbf{x} + o(s^2) \\ &= s^2 \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_{\mathcal{R}} \nabla \mathbf{u}_0 : \mathsf{C}(\nabla \mathbf{f}_\lambda) \left[\nabla \mathbf{u}_0\right] \, d\mathbf{x}\right)\Bigg|_{\lambda=\lambda_0} + o(s^2) \\ &= \rho s^2 + o(s^2). \end{aligned}$$

□

8. An Example: A compressible neo-Hookean material.

In this section we consider a model stored energy function and verify the bifurcation of buckled configurations from the homogeneous branch. We let $k > 0$ and assume the stored energy is of the form[42]

$$W(\mathbf{F}) = \frac{1}{2}\,\mathbf{F} : \mathbf{F} + \frac{1}{k}\,(\det \mathbf{F})^{-k}. \tag{8.1}$$

[42] Blatz and Ko [14] have shown this function can be matched to the experimental data of Bridgman [15] with $k = 13.3$. See also Burgess and Levinson [17].

This material is a compressible neo-Hookean material and also a special form of the Hadamard-Green material; it is both isotropic and compressible. The stress tensor is

$$\mathbf{S}(\mathbf{F}) = \mathbf{F} - (\det \mathbf{F})^{-k-1} \operatorname{adj} \mathbf{F}.$$

The strong-ellipticity condition is satisfied for all $\mathbf{F} \in \mathrm{Lin}^+$ and it is easy to verify that hypotheses (H1)–(H6) and (H9) are satisfied. By (4.5) the homogeneous solution $\mathbf{f}_\lambda$ has

$$\mu(\lambda) = \lambda^{-\frac{k}{k+2}}, \quad 0 < \lambda \le 1, \tag{8.2}$$

and clearly (H7) is satisfied. In view of (3.12)–(3.13b) the components of the elasticity tensor at this solution are given by: $K = k+2$, $M = (k+1)\sqrt{\nu}$, $N = k\sqrt{\nu}$, $P = 1$, and $T = 1 + (k+1)\nu$, where

$$\nu = \nu(\lambda) := \big(\mu(\lambda)\lambda^{-1}\big)^2 > 1 \ \text{ for } \ \lambda < 1 \tag{8.3}$$

and $\nu(1) = 1$. The characteristic polynomial, (3.16) with $\tau = 0$, has real roots

$$r_1 = 1, \quad r_2 = \left(\frac{1 + (k+1)\nu}{k+2}\right)^{1/2} > 1 \ \text{ for } \ \lambda < 1$$

and $r_1 = r_2 = 1$ when $\lambda = 1$. For $\lambda \in (0,1)$ Agmon's condition (see Proposition 3.2(iii)) is satisfied if and only if

$$-\frac{\det \mathbf{C}_0}{M(k+1)} = \frac{(r_2 - r_1)A(\lambda)}{k+1} = 4\nu r_2 - (1+\nu)^2 > 0. \tag{8.4}$$

This implies failure of the complementing condition (see Proposition 3.2(ii)) at $\nu = \nu_\infty$, which is the unique root greater than one of the cubic

$$\nu^3 - \left(\frac{11k+6}{k+2}\right)\nu^2 - 5\nu - 1 = 0. \tag{8.5}$$

Agmon's condition is satisfied if and only if $\lambda \in (\lambda_\infty, 1]$ and hypothesis (H8) is satisfied. In the neo-Hookean limit as $k \to \infty$ this root is approximately $\nu_\infty \approx 11.4445$, which by (8.2) and (8.3) yields $\lambda_\infty = (\nu_\infty)^{-1/4} \approx .5437$.

The linearized equations (6.2) have buckling solutions (6.3b) and barrelling solutions (6.3a). For $\nu > 1$ the side boundary conditions (6.2d) and (6.2e) become, using (6.5) (cf. Proposition 6.2), respectively,

$$B_n(\nu) := \frac{(1+\nu)^2}{4\nu r_2} - \frac{\tanh(\Lambda_n R)}{\tanh(r_2 \Lambda_n R)} = 0 \quad \text{(buckling)}.$$

$$S_n(\nu) := \frac{(1+\nu)^2}{4\nu r_2} - \frac{\tanh(r_2 \Lambda_n R)}{\tanh(\Lambda_n R)} = 0 \quad \text{(barrelling)},$$

It is straightforward to verify that $B_n(\nu)$ is negative for ν slightly greater than 1 and is positive for large ν. Then, for each $n \geq 1$, there is a smallest zero ν_n of B_n. Since for fixed ν, $B_n(\nu)$ is strictly decreasing in n, it follows that ν_n is strictly increasing and $\nu_n \to \nu_\infty$ as $n \to \infty$ by (8.4) and (8.5). Similarly there exists a strictly decreasing sequence of zeros $\tilde{\nu}_n$ of S_n that tends to ν_∞ as $n \to \infty$. Thus only buckling modes occur for $\lambda \in (\lambda_\infty, 1)$ and only barrelling modes for $\lambda \in (0, \lambda_\infty)$. By Theorem 6.3(iii) the linearized equations have a one-dimensional null space at each such ν_n. (This is also true at $\tilde{\nu}_n$). Thus we have verified (H10).

In order to verify (H11) suppose that for $\lambda = \lambda_n$ (and hence $\nu = \nu_n$) the linear equations, (6.2), have the nontrivial solution $\mathbf{v} = (v_1, v_2)^{\mathrm{T}}$. Then, for *any compressible isotropic hyperelastic material*, this solution must satisfy

$$\int_R \big((N-M)[\partial_x v_2][\partial_y v_1] - N[\partial_y v_2][\partial_x v_1]\big)d\mathbf{x} = \begin{cases} \displaystyle\int_R \big(K[\partial_x v_1]^2 + P[\partial_y v_1]^2\big)\, d\mathbf{x} \\ \displaystyle\int_R \big(P[\partial_x v_2]^2 + T[\partial_y v_2]^2\big)\, d\mathbf{x}. \end{cases} \tag{8.6}$$

Equation $(8.6)_1$ ($(8.6)_2$) is obtained by a multiplication of (6.2a) ((6.2b)) by v_1 (v_2), an integration over the region $\mathcal{R}$, and appropriate integrations by parts that involve the boundary conditions (6.2c)–(6.2e). In particular, for the special compressible neo-Hookean material we consider in this section we find that $(8.6)_2$ reduces to

$$t_n \int_R \big([\partial_x v_2][\partial_y v_1] + k[\partial_y v_2][\partial_x v_1]\big)d\mathbf{x} = -\int_R \big([\partial_x v_2]^2 + (1+(k+1)t_n^2)[\partial_y v_2]^2\big)\, d\mathbf{x}, \tag{8.7}$$

where $t_n := \sqrt{\nu_n}$. Next, we note that by (3.12)

$$\mathbf{H} : \mathsf{C}(\nabla \mathbf{f}_\lambda)\,[\mathbf{H}] = \big[1 + (k+1)t^2\big]\, H_{22}^2 + 2t(H_{12}H_{21} + kH_{11}H_{22}) + H_{12}^2 + H_{21}^2 + (k+2)H_{11}^2.$$

If we differentiate this equation with respect to t, evaluate the result at $\lambda = \lambda_n$, multiply by one-half t_n, let $\mathbf{H} = \nabla\mathbf{v}$, and integrate over the rectangle $\mathcal{R}$ we conclude, with the aid of (8.2), (8.3), and (8.7), that

$$\frac{t_n}{2}\left[\frac{d}{dt}\int_R \nabla\mathbf{v} : \mathsf{C}(\nabla\mathbf{f}_\lambda)\,[\nabla\mathbf{v}]\, d\mathbf{x}\right]\Bigg|_{\lambda=\lambda_n} = -\int_R \big([\partial_x v_2]^2 + [\partial_y v_2]^2\big)\, d\mathbf{x} < 0.$$

Thus (H11) is satisfied. Moreover, since $t = \lambda^{-\frac{2k+2}{k+2}}$ it follows that $\frac{dt}{d\lambda} < 0$, which together with the above equation implies that ρ, given by (7.6), is strictly positive. We leave the calculations[43] in Propositions 7.2 and 7.3, which determine whether the bifurcations are supercritical or subcritical as well as their energies relative to the homogeneous solution, to the interested reader.

[43] See the Appendix in [61] for the details of such a computation for other constitutive relations.

Finally, we remark that as the aspect ratio $\alpha = \pi R/L$ tends to zero the first, $n = 1$, buckling mode occurs at (cf. Ogden [44, p. 443])

$$\lambda_c = 1 - \frac{1}{3}\,\alpha^2 + O(\alpha^4). \tag{8.8}$$

The force per unit length on the top (and bottom) of the homogeneously deformed rod generally has a Taylor series expansion[44]

$$S_{22}(\lambda) = S_{22}(1) + \beta[\lambda - 1] + O([\lambda - 1]^2), \tag{8.9}$$

where β is the Young's modulus of the (two-dimensional) material. We note that $S_{22}(1) = 0$ and combine (8.8) with (8.9) to arrive at

$$S_{22}(\lambda_c) = -\frac{\beta}{3}\,\alpha^2 + O(\alpha^4), \tag{8.10}$$

as in Euler buckling (cf., e.g., Euler [45, pp. 102–103], Love [39, §264], Biot [13, p. 171], Young [65, Eqn. 6.8], Ogden [44, p. 444]). In particular, for the Euler buckling of a three-dimensional rod, Love [39, §255 and §264] computes the critical force per unit area to be

$$S_{22}(\lambda_c) = -\beta\pi^2\,\frac{r^2}{L^2}, \qquad r := \sqrt{\frac{I}{m}}, \tag{8.11}$$

where r is the radius of gyration about the load axis (the y-axis in our paper), m is the mass of the rod, and I the moment of inertia about the same axis. For our (two-dimensional) rectangle with unit density, $m = 2RL$ and $I = \int_0^L \int_{-R}^{R} y^2 dy = \frac{2}{3}R^3 L$ and hence (8.11) becomes

$$S_{22}(\lambda_c) = -\frac{\beta}{3}\,\frac{\pi^2 R^2}{L^2} = -\frac{\beta}{3}\alpha^2,$$

which is the quadratic term in our expansion (8.10).

A. Appendix

For the convenience of the reader we first gather all of our hypotheses.

(H1) *$W(\mathbf{F}) = \Phi\,(\nu_1, \nu_2)$ for all $\mathbf{F} \in \mathrm{Lin}^+$, where $\Phi \in C^2(\mathbb{R}^+ \times \mathbb{R}^+; [0, \infty))$ and ν_1 and ν_2 are the principal stretches, i.e., the eigenvalues of $\sqrt{\mathbf{F}\mathbf{F}^{\mathrm{T}}}$.*

(H2) *$W(\mathbf{F}) \to +\infty$ as $\det \mathbf{F} \to 0^+$ and also as $|\mathbf{F}| \to \infty$;*

(H3) *The reference configuration is natural:* $\mathbf{S}(\mathbf{I}) = \mathbf{0}$;

[44] In order to obtain this expansion for the stored energy given by (8.1) one first must multiply W by $\beta(k + 2)/[4(k + 1)]$.

(H4) *The strengthened tension extension inequalities:*

$$K = K(\mu,\lambda) := \Phi,_{11}(\mu,\lambda) > 0 \quad \textit{and} \quad T = T(\mu,\lambda) := \Phi,_{22}(\mu,\lambda) > 0$$

are satisfied for each $\lambda > 0$ *and each* $\mu > 0$

(H5) $\mathsf{C}_\lambda := \mathsf{C}(\nabla \mathbf{f}_\lambda)$ *satisfies the strong ellipticity condition at each* $\lambda \in (0,1]$.

(H6) $\mathsf{C}(\mathbf{I})$ *is positive definite (on symmetric tensors);*

(H7) $\dfrac{d\mu(\lambda)}{d\lambda} \leq 0$ *for* $\lambda \in (0,1]$;

(H8) *There exists* $\lambda_\infty \in (0,1)$ *such that* $A(\lambda_\infty) = 0$ *and* $A(\lambda) > 0$ *for* $\lambda \in (\lambda_\infty, 1]$;

(H9) *For some* $\lambda_0 \in (\lambda_\infty, 1)$*, some* $p \in (1,\infty)$*, and some integer* $m > 1 + 2/p$ *the stored energy* W *is of class* C^{m+2} *in a neighborhood of*

$$\mathbf{F}_{\lambda_0} := \nabla \mathbf{f}_{\lambda_0}(\mathbf{x}) \ \ \textit{for all} \ \ \mathbf{x} \in \Omega.$$

(H10) *For* λ_0 *as in (H9) the linear equations* (5.5)*, with* $\mathbf{f} = \mathbf{f}_\lambda$ *and* $\lambda = \lambda_0$*, possess a one-dimensional solution set spanned by* $\mathbf{u}_0 \in \mathcal{X}_{m,p}$.

(H11) *At* $\lambda = \lambda_0$ *as given in (H9) and (H10)*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_{\mathcal{R}} \nabla \mathbf{u}_0 : \mathsf{C}(\nabla \mathbf{f}_\lambda)\,[\nabla \mathbf{u}_0]\; d\mathbf{x} \neq 0.$$

We now prove the results in our paper that are of a purely technical nature.

Proposition 5.2. *Assume that (H1)–(H4) are satisfied. Let* $(\lambda, \mathbf{u}) \in \mathrm{Dom}_{m,p}(\mathfrak{F})$ *and define* $\mathbf{f} := \mathbf{f}_\lambda + \mathbf{u}$*. Suppose* $\mathsf{C}_\mathbf{f}(\mathbf{x})$ *satisfies the strong-ellipticity condition at every* $\mathbf{x} \in \overline{\mathcal{R}}$ *and that* $(\mathsf{C}_\mathbf{f}(\mathbf{x}), \mathbf{e}_1)$ *satisfies the complementing condition at every* $\mathbf{x} \in \overline{\mathcal{S}}$*. Then a necessary and sufficient condition for the quadratic functional* $\mathsf{Q}_\mathbf{f}$ *to be coercive is that the pair* $(\mathsf{C}_\mathbf{f}(\mathbf{x}), \mathbf{e}_1)$ *satisfies Agmon's condition at every* $\mathbf{x} \in \overline{\mathcal{S}}$.

Proof of Proposition 5.2. We note $\mathsf{C}_\mathbf{f}$ is continuous on $\overline{\mathcal{R}}$ (since $\mathbf{f} \in C^1(\overline{\Omega}; \mathbb{R}^2)$, see footnote 28). The necessity of Agmon's condition follows as in, for example, [56, Theorem 3] utilizing the reflection and periodicity properties of $\mathbf{f} = \mathbf{f}_\lambda + \mathbf{u}$ with $\mathbf{u} \in \mathcal{X}_{m,p}$.

To prove sufficiency we first note that the reflection and periodicity properties of $\mathbf{f}$ imply that $\mathsf{C}_\mathbf{f}(\mathbf{x})$ satisfies the strong-ellipticity condition at every $\mathbf{x} \in \overline{\Omega}$ and that the pair $(\mathsf{C}_\mathbf{f}(\mathbf{x}), \mathbf{e}_1)$ satisfies the complementing condition at every $\mathbf{x} \in \partial\Omega$.

Claim. For each $\mathbf{a} \in \overline{\Omega}$ there exists $k_{\mathbf{a}} > 0$ and $\delta_{\mathbf{a}} > 0$ such that, for any $\varepsilon \in (0, \delta_{\mathbf{a}})$,

$$\int_{D(\mathbf{a},\varepsilon)\cap\Omega} \nabla \mathbf{v}(\mathbf{x}) : \mathsf{C}_{\mathbf{f}}(\mathbf{x}) \left[\nabla \mathbf{v}(\mathbf{x})\right] d\mathbf{x} \geq k_{\mathbf{a}} \|\mathbf{v}\|^2_{1,2,D(\mathbf{a},\varepsilon)\cap\Omega}, \tag{A.1}$$

for any[45] $\mathbf{v} \in W^{1,2}(D(\mathbf{a}, \varepsilon) \cap \Omega; \mathbb{R}^2)$ that satisfies $\mathbf{v} = \mathbf{0}$ on $\Omega \cap \partial D(\mathbf{a}, \varepsilon)$.

We postpone the proof of the claim and first show how to make use of it to prove the lemma. Define $\Omega_1 := \Omega \cap \{-L < y < L\}$, $\Omega_1^e := (-2R, 2R) \times (-L, L)$, and $\mathbf{c} : \overline{\Omega}_1^e \to \mathbb{R}^3$ by

$$\mathbf{c}(x, y) := \begin{bmatrix} x \\ L\cos\left(\frac{\pi y}{L}\right) \\ L\sin\left(\frac{\pi y}{L}\right) \end{bmatrix}, \text{ so that } \mathbf{c}(\overline{\Omega}_1) = \left\{(x_1, x_2, x_3) : x_2^2 + x_3^2 = L^2,\ -R \leq x_1 \leq R\right\}$$

is the *surface of a cylinder* in $\mathbb{R}^3$. Let $\{B_i\}_{i=1}^M$ be an open cover of the compact set $\mathbf{c}(\overline{\Omega}_1)$ that consists of balls $B_i = B(\mathbf{z}_i, \varepsilon_i) \subset \mathbb{R}^3$, with $\mathbf{z}_i \in \mathbf{c}(\overline{\Omega}_1)$ and $\varepsilon_i < \min\{R, L\}$, that have the following further properties:

(i) For each i, $\mathbf{z}_i \in \mathbf{c}\left(\overline{\Omega}_1 \cap \partial\Omega\right)$ or $B(\mathbf{z}_i, \varepsilon_i) \cap \mathbf{c}\left(\overline{\Omega}_1 \cap \partial\Omega\right) = \varnothing$; and

(ii) For each i, $\mathbf{c}^{-1}\left(B(\mathbf{z}_i, \varepsilon_i) \cap \mathbf{c}(\overline{\Omega}_1^e)\right) \subset E_i$, where $D(\mathbf{a}_i, \delta_i)$ is a disk that satisfies (A.1) with $E_i := D(\mathbf{a}_i, \delta_i)$ or

$$E_i := \left(D(\mathbf{a}_i, \delta_i) \backslash H_i\right) \cup \left((0, 2L) + H_i\right), \quad H_i := \{(x, y) \in D(\mathbf{a}_i, \delta_i) : y \leq -L\}.$$

Let ψ_i be a partition of unity subordinate to $\{B_i\}$ that satisfies $\sum_{i=1}^M \psi_i^2 = 1$. Then $\mathbf{c}^{-1}\left(B(\mathbf{z}_i, \varepsilon_i) \cap \mathbf{c}(\overline{\Omega}_1^e)\right) \subset E_i$ will be an open cover of $\overline{\Omega}_1$ and $\psi_i \circ \mathbf{c}$ will be a partition of unity subordinate to this cover (and hence also subordinate to E_i) with $\sum_{i=1}^M [\psi_i(\mathbf{c}(x, y))]^2 = 1$ for every $(x, y) \in \Omega_1$.

For each $\mathbf{v} \in \mathrm{Var}$ extend $\mathbf{v}$ to be an element of $\mathcal{X}_{1,2}$ (see Lemma 4.3) and define $\varphi_i := \psi_i \circ \mathbf{c}$ and $\mathbf{v}_i := \varphi_i \mathbf{v}$. Then

$$\varphi_i^2 \nabla \mathbf{v} : \mathsf{C}_{\mathbf{f}}\left[\nabla \mathbf{v}\right] = \nabla \mathbf{v}_i : \mathsf{C}_{\mathbf{f}}\left[\nabla \mathbf{v}_i\right] - \left[2\varphi_i \nabla \mathbf{v} + \mathbf{v} \otimes \nabla\varphi_i\right] : \mathsf{C}_{\mathbf{f}}\left[\mathbf{v} \otimes \nabla\varphi_i\right]$$

and hence

$$\begin{aligned} \int_{\Omega_1} \nabla \mathbf{v} : \mathsf{C}_{\mathbf{f}}\left[\nabla \mathbf{v}\right] d\mathbf{x} &= \int_{\Omega_1} \sum_{i=1}^M \varphi_i^2 \nabla \mathbf{v} : \mathsf{C}_{\mathbf{f}}\left[\nabla \mathbf{v}\right] d\mathbf{x} \\ &= \sum_{i=1}^M \int_{\Omega_1 \cap E_i} \nabla \mathbf{v}_i : \mathsf{C}_{\mathbf{f}}\left[\nabla \mathbf{v}_i\right] d\mathbf{x} \\ &\quad - \int_{\Omega_1} \sum_{i=1}^M \left[2\varphi_i \nabla \mathbf{v} + \mathbf{v} \otimes \nabla\varphi_i\right] : \mathsf{C}_{\mathbf{f}}\left[\mathbf{v} \otimes \nabla\varphi_i\right] d\mathbf{x}. \end{aligned} \tag{A.2}$$

[45] As the proof will make clear, if $\mathbf{a} \in \Omega$ then $D(\mathbf{a}, \varepsilon) \cap \Omega = D(\mathbf{a}, \varepsilon)$, while if $\mathbf{a} \in \partial\Omega$ then $D(\mathbf{a}, \varepsilon) \cap \Omega$ will be a half-disk whose flat edge lies on the boundary of Ω.

We are now ready to apply the claim to the integral in the second line of (A.2). By hypotheses (i) and (ii) above, for each i the set $\Omega_1 \cap E_i$ is either a disk contained in the interior of Ω, a half-disk contained in the interior of Ω whose flat side lies on $\partial\Omega$, or two disjoint regions whose y-translation (by $2L$ to, say, the lower region) will result in one of the above cases. However, since $\mathbf{v}$ is $2L$–periodic in y and since the partition of unity is defined on the cylinder, for such i we may treat E_i as if it were a disk.

Thus, by the claim, the continuity of $\mathsf{C}_{\mathbf{f}}$, and the uniform boundedness of the derivatives of a partition of unity, there are constants $k_i > 0$ and $c_1 > 0$ such that, for any $\eta > 0$,

$$\int_{\Omega_1} \nabla\mathbf{v} : \mathsf{C}_{\mathbf{f}}\,[\nabla\mathbf{v}]\,d\mathbf{x} \geq \sum_{i=1}^{M} k_i \,\|\mathbf{v}_i\|^2_{1,2,\Omega_1\cap E_i} - c_1 \int_{\Omega_1} \left[|\nabla\mathbf{v}||\mathbf{v}| + |\mathbf{v}|^2\right] d\mathbf{x} \tag{A.3a}$$

$$\geq k\,\|\mathbf{v}\|^2_{1,2,\Omega_1} - c_1\,\|\mathbf{v}\|_{1,2,\Omega_1}\,\|\mathbf{v}\|_{0,2,\Omega_1} - c_1\,\|\mathbf{v}\|^2_{0,2,\Omega_1} \tag{A.3b}$$

$$\geq (k-\eta)\,\|\mathbf{v}\|^2_{1,2,\Omega_1} - (C_\eta + c_1)\,\|\mathbf{v}\|^2_{0,2,\Omega_1} \tag{A.3c}$$

for some $C_\eta > 0$, where $k := c_2^{-1}\min\{k_i\}$ and we have used the inequalities

$$\|\mathbf{v}\|_{1,2,\Omega_1} = \left\|\sum_{i=1}^{M}\varphi_i^2\mathbf{v}\right\|_{1,2,\Omega_1} \leq c_2 \sum_{i=1}^{M}\|\mathbf{v}_i\|_{1,2,\Omega_1\cap E_i}\,,$$

$$\|\mathbf{v}\|_{1,2,\Omega_1}\,\|\mathbf{v}\|_{0,2,\Omega_1} \leq \eta\,\|\mathbf{v}\|^2_{1,2,\Omega_1} + C_\eta\,\|\mathbf{v}\|^2_{0,2,\Omega_1}$$

to obtain (A.3b) from (A.3a) and (A.3c) from (A.3b), respectively.

Finally, by the reflection symmetry[46] we have

$$\nabla\mathbf{v}(x,-y) : \mathsf{C}(\nabla\mathbf{f}(x,-y))\,[\nabla\mathbf{v}(x,-y)] = \nabla\mathbf{v}(x,y) : \mathsf{C}(\nabla\mathbf{f}(x,y))\,[\nabla\mathbf{v}(x,y)]$$

for all $(x,y) \in \Omega$ and hence

$$\int_{\Omega_1} \nabla\mathbf{v} : \mathsf{C}_{\mathbf{f}}\,[\nabla\mathbf{v}]\,d\mathbf{x} = 2\int_{\mathcal{R}} \nabla\mathbf{v} : \mathsf{C}_{\mathbf{f}}\,[\nabla\mathbf{v}]\,d\mathbf{x}$$

and, similarly,

$$\|\mathbf{v}\|^2_{1,2,\Omega_1} = 2\,\|\mathbf{v}\|^2_{1,2,\mathcal{R}}\,, \qquad \|\mathbf{v}\|^2_{0,2,\Omega_1} = 2\,\|\mathbf{v}\|^2_{0,2,\mathcal{R}}\,.$$

Therefore (A.3) implies (5.3) for any $\mathbf{v} \in \mathrm{Var}$.

Proof of the Claim. We will prove that there exist $k_{\mathbf{a}} > 0$ and $\varepsilon^*_{\mathbf{a}} > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*_{\mathbf{a}})$,

$$\int_{D(\mathbf{a},\varepsilon)\cap\Omega} \nabla\mathbf{v}(\mathbf{x}) : \mathsf{C}_{\mathbf{a}}\,[\nabla\mathbf{v}(\mathbf{x})]\,d\mathbf{x} \geq 2k_{\mathbf{a}}\,\|\mathbf{v}\|^2_{1,2,D(\mathbf{a},\varepsilon)\cap\Omega}\,, \tag{A.4}$$

[46] For $\mathbf{Q} = \mathrm{diag}\{1,-1\}$ hypothesis (H1) and (3.2) yield $W(\mathbf{QFQ}) = W(\mathbf{F})$ and $\nabla\mathbf{v}(x,-y) = \mathbf{Q}\nabla\mathbf{v}(x,y)\mathbf{Q}$.

for all $\mathbf{v}$ as in the claim where $\mathsf{C}_{\mathbf{a}} := \mathsf{C}(\nabla\mathbf{f}(\mathbf{a}))$ is a *constant* tensor for each $\mathbf{a} \in \overline{\Omega}$. The desired result, (A.1), then follows from the continuity of $\mathsf{C}_{\mathbf{f}}$. If $\mathbf{a} \in \Omega$ then (A.4) follows from the strong-ellipticity of $\mathsf{C}_{\mathbf{a}}$ (see van Hove [64]). If instead $\mathbf{a} \in \partial\Omega$ then (A.4) with $k_{\mathbf{a}} = 0$ follows from [57, Theorem 1] (see also [56, Theorem 3]). We next indicate how to extend Theorem 1 in [57] to get (A.4) with $k_{\mathbf{a}} > 0$ for some $\varepsilon^*_{\mathbf{a}} > 0$.

We use the terminology of [57]. Let $\mathcal{H} := \{(x,y) \in \mathbb{R}^2 : x < R\}$ be a half-space, $\mathsf{C}_0 := \mathsf{C}(\nabla\mathbf{f}(\mathbf{a}))$, and $\mathbf{n} = \mathbf{e}_1$. Fix $\xi \neq 0$ so that $\mathbf{t} = (0,\xi) \perp \mathbf{n}$ and define the quadratic form Q^ξ_α by (5.11) in [57]. Then the strong ellipticity of C_0 implies that $\mathbf{M}$ is *strictly* positive definite. Then Lemma 6.3 in [57] implies that Q^ξ_α is nonnegative for all α sufficiently large and is regular for all $\alpha \geq 0$ (by strong-ellipticity). Since Agmon's condition is satisfied, Proposition 6.2 then yields Q^ξ_α nonnegative for all $\alpha > 0$ and the continuity of $\alpha \mapsto Q^\xi_\alpha$ then implies Q^ξ_α nonnegative for all $\alpha \geq 0$. Since both the complementing condition and Agmon's condition are satisfied we can now apply Proposition 6.1 to conclude Q^ξ_α is coercive for all $\alpha \geq 0$; in fact there exists $k > 0$ such that $Q^\xi_0[\mathbf{z}] \geq k\,\|\mathbf{z}\|^2_{1,2,\mathbb{R}^+}$ for all $\mathbf{z} \in W^{1,2}(\mathbb{R}^+;\mathbb{R}^2)$ and $|\xi| = 1$. A scaling argument then yields

$$Q^\xi_0[\mathbf{z}] \geq k \int_0^\infty \left[\left| \frac{d\mathbf{z}}{ds} \right|^2 + |\xi|^2 |\mathbf{z}|^2 \right] ds$$

for all such $\mathbf{z}$ and $\xi \neq 0$. A Fourier transform of Q^ξ_0 in the y–variable (as in Lemma 5.3) yields

$$\int_{\mathcal{H}} \nabla\mathbf{v}(\mathbf{x}) : \mathsf{C}_0\,[\nabla\mathbf{v}(\mathbf{x})]\,d\mathbf{x} \geq k \int_{\mathcal{H}} |\nabla\mathbf{v}(\mathbf{x})|^2 d\mathbf{x} \quad \text{for all} \quad \mathbf{v} \in W^{1,2}(\mathcal{H};\mathbb{R}^2).$$

Then, choosing any $\varepsilon^*_{\mathbf{a}} \in (0,R)$, Poincare's inequality yields (A.4). □

Proposition 5.3. *Fix $\lambda \in (0,1]$ and suppose C_λ satisfies the strong-ellipticity condition and $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies the complementing condition. Then:*

(i) There exists a constant $\omega > 0$, which depends only on m, p, $\mathcal{R}$, and C_λ, such that

$$\|\mathbf{v}\|_{\mathcal{X}_{m,p}} \leq \omega \left(\|\mathfrak{L}_\lambda[\mathbf{v}]\|_{\mathcal{Y}_{m,p}} + \|\mathbf{v}\|_{0,p,\mathcal{R}_e} \right) \text{ for all } \mathbf{v} \in \mathcal{X}_{m,p}. \tag{A.5}$$

Thus $\mathfrak{L}_\lambda \in \mathrm{BL}(\mathcal{X}_{m,p};\mathcal{Y}_{m,p})$ has finite dimensional null space and closed range in $\mathcal{Y}_{m,p}$. Moreover, $\mathfrak{D}_\lambda$ (with domain $\mathrm{Dom}_{m,p}(\mathfrak{D}_\lambda)$) is a closed operator in $\mathcal{X}_{m-2,p}$ with finite dimensional null space and closed range in $\mathcal{X}_{m-2,p}$.

In addition, if $m \geq 3$ (or $m = 2$ and $p \geq 2$) and the spectrum of $\mathfrak{D}_\lambda$ is real then:

(ii) $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies Agmon's condition if and only if the spectrum is bounded below.

Proof of Proposition 5.3. (i). Let $\{B_\alpha\}$ be a locally finite cover of Ω with either balls contained in Ω or half-balls in $\overline{\Omega}$ with centers on $\partial\Omega$. The use of a partition of unity subordinate to $\{B_\alpha\}$ shows that it suffices to establish (A.5) on B_α for all $\mathbf{v} \in W^{m,p}(B_\alpha;\mathbb{R}^2)$ that satisfy $\mathbf{v} = \mathbf{0}$ on $\partial B_\alpha \cap \overline{\Omega}$. However, since the coefficients of $\mathfrak{D}_\lambda$ are constants (in $\mathbf{x}$) and since the complementing condition holds on $\partial\Omega$, the estimates of Agmon, Douglis, and Nirenberg [5, Theorems 10.3 and 10.4] yield a constant $\omega_\alpha > 0$ (depending only on m, p, B_α and C_λ) such that

$$\|\mathbf{v}\|_{m,p,B_\alpha} \le \omega_\alpha \left(\|\mathfrak{D}_\lambda[\mathbf{v}]\|_{m-2,p,B_\alpha} + \|\mathfrak{B}_\lambda[\mathbf{v}]\|_{m-1-\frac{1}{p},p,\overline{B}_\alpha\cap\partial\Omega} + \|\mathbf{v}\|_{0,p,B_\alpha} \right)$$

for all $\mathbf{v}$ as above. Thus, with the partition of unity,

$$\|\mathbf{v}\|_{m,p,\mathcal{R}} \le \omega' \left(\|\mathfrak{D}_\lambda[\mathbf{v}]\|_{m-2,p,\mathcal{R}_e} + \|\mathfrak{B}_\lambda[\mathbf{v}]\|_{m-1-\frac{1}{p},p,\mathcal{S}_e} + \|\mathbf{v}\|_{m-1,p,\mathcal{R}_e} \right) \tag{A.6}$$

for all $\mathbf{v} \in W^{m,p}(\mathcal{R}_e;\mathbb{R}^2)$ ($\mathcal{R}_e$ and $\mathcal{S}_e$ defined in (4.6) and (4.7), respectively). Now consider $\mathbf{v} \in \mathcal{X}_{m,p}$. Then the reflection symmetry and periodicity of $\mathbf{v}$ imply $\|\mathbf{v}\|_{m,p,\mathcal{R}_e} = 3^{-\frac{1}{p}} \|\mathbf{v}\|_{m,p,\mathcal{R}}$ and hence (A.6) is also valid for all $\mathbf{v} \in \mathcal{X}_{m,p}$. Finally, Ehrling's Lemma (see, e.g., [42, p. 85]) implies that, for all $\mathbf{v} \in W^{m,p}(\mathcal{R}_e;\mathbb{R}^2)$,

$$\|\mathbf{v}\|_{m-1,p,\mathcal{R}_e} \le \varepsilon \|\mathbf{v}\|_{m,p,\mathcal{R}_e} + c(\varepsilon) \|\mathbf{v}\|_{0,p,\mathcal{R}_e}$$

with ε arbitrarily small, which together with (A.6) yields (A.5).

Next, we note that, by the compactness of the imbedding $W^{m,p}(\mathcal{R}_e;\mathbb{R}^2) \hookrightarrow L^p(\mathcal{R}_e;\mathbb{R}^2)$ and Lemma 3 in Peetre [46], inequality (A.5) is equivalent to $\mathfrak{L}_\lambda$ having a finite dimensional null space and a closed range in $\mathcal{Y}_{m,p}$. Thus $\mathfrak{L}_\lambda$ is a semi-Fredholm operator. Similarly, from (A.5), we have

$$\|\mathbf{v}\|_{m,p,\mathcal{R}_e} \le \omega \left(\|\mathfrak{D}_\lambda[\mathbf{v}]\|_{m-2,p,\mathcal{R}_e} + \|\mathbf{v}\|_{0,p,\mathcal{R}_e} \right) \text{ for all } \mathbf{v} \in \mathrm{Dom}_{m,p}(\mathfrak{D}_\lambda). \tag{A.7}$$

Thus $\mathfrak{D}_\lambda$ is a closed operator in $\mathcal{X}_{m-2,p}$ and (again by [46, Lemma 3]) is semi-Fredholm. Since (A.5) and (A.7) are also valid if $\mathfrak{D}_\lambda$ is replaced by $\mathfrak{D}_\lambda - \mu\mathfrak{I}$, for any $\mu \in \mathbb{C}$ (and a different constant ω), it follows that $(\mathfrak{D}_\lambda - \mu\mathfrak{I}, \mathfrak{B}_\lambda) \in \mathrm{BL}(\mathcal{X}^c_{m,p};\mathcal{Y}^c_{m,p})$ and $\mathfrak{D}_\lambda - \mu\mathfrak{I}$ is semi-Fredholm ($\mathfrak{I} : \mathcal{X}^c_{m-2,p} \to \mathcal{X}^c_{m-2,p}$ is the identity map). We recall the index of a semi-Fredholm operator equals the dimension of its null space minus the codimension of its range; this is an integer or $-\infty$. In particular, if $\mathfrak{D}_\lambda - \mu\mathfrak{I}$ is a bijection then its index is zero.

(ii). Suppose either $m \ge 3$ or $p \ge 2$ and $m = 2$. Also suppose the spectrum $\Sigma(\lambda)$ of $\mathfrak{D}_\lambda$ is real. By part (i), for each $\mu \in \mathbb{C}$, $\mathfrak{D}_\lambda - \mu\mathfrak{I}$ is a semi-Fredholm operator. However its index is zero for $\mathrm{Im}\{\mu\} \ne 0$ and therefore for all $\mu \in \mathbb{C}$ by stability of the Fredholm index, (the index is locally constant in μ by Kato [36, Chapter IV, Theorems 2.14 and 5.17]). Let $\widehat{\mathcal{D}}_\lambda$ denote the operator $\mathfrak{D}_\lambda$ mapping domain $\mathrm{D} := \mathrm{Dom}_{2,2}(\mathfrak{D}_\lambda)$ into $\mathcal{X}_{0,2}$; $\widehat{\mathcal{D}}_\lambda$ is closed by part (i). Moreover, by (3.7) and (5.8), $\widehat{\mathcal{D}}_\lambda$ is symmetric: for all $\mathbf{v}, \mathbf{w} \in \mathrm{D}$

$$\left\langle \mathbf{w}, \widehat{\mathcal{D}}_\lambda[\mathbf{v}] \right\rangle_{0,2,\mathcal{R}} = \left\langle \widehat{\mathcal{D}}_\lambda[\mathbf{w}], \mathbf{v} \right\rangle_{0,2,\mathcal{R}},$$

where $\big\langle \cdot, \cdot \big\rangle_{0,2,\mathcal{R}}$ denotes the $L^2(\mathcal{R})$-inner product on $\mathcal{X}_{0,2}$.

To show $\widehat{\mathcal{D}}_\lambda$ is self-adjoint let $\mu \in \mathbb{C}$ with $\mathrm{Im}\{\mu\} \neq 0$. By Kato [36, Chapter V, Equation 3.13], $\widehat{\mathcal{D}}_\lambda - \mu\widehat{\mathfrak{T}}$ is injective and its range is closed in $\mathcal{X}^c_{0,2}$ ($\widehat{\mathfrak{T}} : \mathcal{X}^c_{0,2} \to \mathcal{X}^c_{0,2}$ is the identity map). Next, since $\mu \notin \Sigma(\lambda)$ the range of $\mathfrak{D}_\lambda - \mu\mathfrak{T}$ is equal to $\mathcal{X}^c_{m-2,p}$. Our assumptions on m and p imply $\mathcal{X}^c_{m,p} \subset \mathcal{X}^c_{2,2}$. Thus $\mathrm{Dom}^c_{m,p}(\mathfrak{D}_\lambda) \subset \mathrm{D}$ and hence $\mathcal{X}^c_{m-2,p}$ is contained in the range of $\widehat{\mathcal{D}}_\lambda - \mu\widehat{\mathfrak{T}}$. However, $\mathcal{X}^c_{m-2,p}$ is dense in $\mathcal{X}^c_{0,2}$ and so $\widehat{\mathcal{D}}_\lambda - \mu\widehat{\mathfrak{T}}$ is surjective. Therefore, $\widehat{\mathcal{D}}_\lambda$ is self-adjoint by Kato [36, Chapter V, Theorem 3.16]. (Again, by part (i), we see $\widehat{\mathcal{D}}_\lambda - \mu\widehat{\mathfrak{T}}$ is Fredholm with index zero for all $\mu \in \mathbb{C}$.)

Next, let $\widehat{\Sigma}(\lambda)$ denote the (real) spectrum of $\widehat{\mathcal{D}}_\lambda$ and suppose that $\mu \in \mathbb{C}\backslash\Sigma(\lambda)$. Then, as just above, $\widehat{\mathcal{D}}_\lambda - \mu\widehat{\mathfrak{T}}$ is surjective and therefore bijective. Thus $\mu \in \mathbb{C}\backslash\widehat{\Sigma}(\lambda)$ and so $\widehat{\Sigma}(\lambda) \subset \Sigma(\lambda)$.

To prove (ii), suppose $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies Agmon's condition. By Proposition 5.2, Q_λ is coercive. Next, we note that since $\mathfrak{D}_\lambda$ is a Fredholm operator with index zero, its spectrum consists solely of eigenvalues. Let $\mu \in \Sigma(\lambda)$. Then $\mathfrak{D}_\lambda - \mu\mathfrak{T}$ is not injective and hence there exists $\mathbf{v} \in \mathrm{Dom}_{m,p}(\mathfrak{D}_\lambda)$ with $\mathbf{v} \neq \mathbf{0}$ such that $(\mathfrak{D}_\lambda - \mu\mathfrak{T})[\mathbf{v}] = \mathbf{0}$. It then follows from (5.3) and (5.8) that

$$-k_1 \|\mathbf{v}\|^2_{0,2,\mathcal{R}} \leq \mathrm{Q}_\lambda[\mathbf{v}] = \int_{\mathcal{R}} \mathbf{v} \cdot \mathfrak{D}_\lambda[\mathbf{v}] \, \mathrm{d}\mathbf{x} = \mu \|\mathbf{v}\|^2_{0,2,\mathcal{R}}$$

and hence $\mu \geq -k_1$. Thus $\Sigma(\lambda)$ is bounded below.

Conversely, suppose $\Sigma(\lambda)$ is bounded below. Then $\widehat{\Sigma}(\lambda)$ is also bounded below. By Kato [36, Chapter V, §3.10],

$$\inf_{\substack{\mathbf{v}\in \mathrm{D} \\ \|\mathbf{v}\|_{0,2,\mathcal{R}}=1}} \Big\langle \mathbf{v}, \widehat{\mathcal{D}}_\lambda[\mathbf{v}] \Big\rangle_{0,2,\mathcal{R}} = \inf \widehat{\Sigma}(\lambda) := \widehat{\sigma}.$$

Therefore for all $\mathbf{v} \in \mathrm{D}$ and $\mu \in \mathbb{R}$

$$\Big\langle \mathbf{v}, (\widehat{\mathcal{D}}_\lambda - \mu\widehat{\mathfrak{T}})[\mathbf{v}] \Big\rangle_{0,2,\mathcal{R}} \geq (\widehat{\sigma} - \mu)\big\|\mathbf{v}\big\|^2_{0,2,\mathcal{R}}$$

and hence

$$\Big\|(\widehat{\mathcal{D}} - \mu\widehat{\mathfrak{T}})[\mathbf{v}]\Big\|_{0,2,\mathcal{R}} \geq \frac{1}{2}|\mu|\big\|\mathbf{v}\big\|_{0,2,\mathcal{R}}$$

for all $\mathbf{v} \in \mathrm{D}$ and negative μ with $|\mu|$ sufficiently large. Finally, by Agmon [4, Theorem 2.1] we see that $(\mathsf{C}_\lambda, \mathbf{e}_1)$ satisfies Agmon's condition (in fact, the argument there is applicable in a neighborhood of any point of $\partial\Omega$). (See also [56, Theorems 3–5].) $\square$

Proposition 5.4. *Suppose that* $\mathbf{S}(\mathbf{I}) = \mathbf{0}$ *and* $\mathsf{C}(\mathbf{I})$ *is positive definite (on symmetric tensors). Then the operator* $\mathfrak{D}_1$ *is a bijection from* $\mathrm{Dom}_{m,p}(\mathfrak{D}_1)$ *onto* $\mathcal{X}_{m-2,p}$ *and consequently a Fredholm operator with index zero.*

Proof of Proposition 5.4. Assume $\mathbf{S}(\mathbf{I}) = \mathbf{0}$ and that $\mathsf{C}(\mathbf{I})$ is positive definite. First note that $\mathfrak{D}_1 : \mathrm{Dom}_{m,p}(\mathfrak{D}_1) \to \mathcal{X}_{m-2,p}$ is injective since if $\mathfrak{D}_1[\mathbf{v}] = 0$ and $\mathbf{v} \in \mathrm{Dom}_{m,p}(\mathfrak{D}_1)$ then (5.8), the positive definiteness of $\mathsf{C}(\mathbf{I})$, and Korn's (2.2) inequality imply

$$0 = \int_{\mathcal{R}} \nabla\mathbf{v} : \mathsf{C}(\mathbf{I})\,[\nabla\mathbf{v}]\ d\mathbf{x} \geq k \int_{\mathcal{R}} |(\nabla\mathbf{v})_s|^2\ d\mathbf{x} \geq k\hat{k}\,\|\mathbf{v}\|^2_{1,2,\mathcal{R}} \tag{A.8}$$

and hence $\mathbf{v} = \mathbf{0}$. Thus $\mathfrak{D}_1$ and consequently $\mathfrak{L}_1$ are injective. However, Proposition 5.3(i) implies that $\mathfrak{L}_1$ and $\mathfrak{D}_1$ are semi-Fredholm operators. Thus if can we show that $\mathfrak{L}_1$ has index zero it will follow that $\mathfrak{L}_1$ and therefore $\mathfrak{D}_1$ are bijective.

To show that $\mathfrak{L}_1$ has index zero we construct a homotopy that connects $\mathfrak{L}_1$ to an operator that is known to have index zero (cf., e.g., [35, §3]). Define, for $t \in [0,1]$,

$$\begin{aligned}
&\mathfrak{L}^{(t)} := t\mathfrak{L}_1 + (1-t)\big(-\Delta, \frac{\partial}{\partial n}\big) \in \mathrm{BL}(\mathcal{X}_{m,p}; \mathcal{Y}_{m,p}),\\
&\mathsf{C}^{(t)} := t\mathsf{C}(\mathbf{I}) + (1-t)\mathbb{I} \in \mathrm{LinLin},\\
&\frac{\partial\mathbf{v}}{\partial n} := (\nabla\mathbf{v})\,\mathbf{n} \text{ on } \partial\Omega,
\end{aligned}$$

so that, for $\mathbf{v} \in \mathcal{X}_{m,p}$,

$$\mathfrak{L}^{(t)}[\mathbf{v}] = \Big(-\mathrm{div}\,\mathsf{C}^{(t)}[\nabla\mathbf{v}], \mathsf{C}^{(t)}[\nabla\mathbf{v}]\mathbf{n}\Big).$$

Here $\mathbb{I}$ is the 4-tensor $\mathbb{I}[\mathbf{H}] = \mathbf{H}$ for $\mathbf{H} \in \mathrm{Lin}$. It is easily verified that, for each t, $\mathsf{C}^{(t)}$ is both strongly-elliptic and positive definite (on symmetric tensors when $t = 1$). Thus, as above, Korn's (2.2) inequality, implies that the quadratic form

$$\mathrm{Q}^{(t)}[\mathbf{v}] = \int_{\mathcal{R}} \nabla\mathbf{v}(\mathbf{x}) : \mathsf{C}^{(t)}\,[\nabla\mathbf{v}(\mathbf{x})]\,d\mathbf{x}$$

is uniformly positive for all $\mathbf{v} \in \mathrm{Var}$ and $t \in [0,1]$ (see also (3.23)). Moreover, Theorem 1 (or Theorem 3) in [56] shows that $(\mathsf{C}^{(t)}, \mathbf{e}_1)$ satisfies the complementing condition at each $t \in [0,1]$ (see also Proposition 3.2(iv)). Again, from Proposition 5.3(i), $\mathfrak{L}^{(t)}$ therefore satisfies an estimate of the form (5.9) and, consequently, $\mathfrak{L}^{(t)}$ is semi-Fredholm for each $t \in [0,1]$. Furthermore, the index of $\mathfrak{L}^{(t)}$ is independent of t (Kato [36, Chapter IV, Theorems 2.14 and 5.17]). We will now prove that $\mathfrak{L}^{(0)}$ is bijective and therefore that $\mathfrak{L}^{(0)}$ and hence $\mathfrak{L}_1$ have index zero.

The injectivity of $\mathfrak{L}^{(0)}$ follows as in (A.8) from the uniform positivity of $\mathrm{Q}^{(0)}$. It is also surjective since the Poisson problem,

$$\Delta\mathbf{v} = \mathbf{h} \ \text{ on } \ \Omega, \qquad \frac{\partial\mathbf{v}}{\partial n} = \mathbf{g} \ \text{ on } \partial\Omega,$$

has a solution $\mathbf{v} \in \mathcal{X}_{m,p}$ for any $(\mathbf{h}, \mathbf{g}) \in \mathcal{Y}_{m,p}$. To see this consider the set S of finite linear combinations of order pairs $(\mathbf{h}, \mathbf{g})$ that satisfy $Y_{(i)}$ (in §4) and are of the form

$$\mathbf{h}^{(n)}(x,y) = \begin{bmatrix} h_1^{(n)}(x)\cos(\Lambda_n y) \\ h_2^{(n)}(x)\sin(\Lambda_n y) \end{bmatrix}, \qquad \mathbf{g}^{(n)}(x,y) = \begin{bmatrix} g_1^{(n)}(x)\cos(\Lambda_n y) \\ g_2^{(n)}(x)\sin(\Lambda_n y) \end{bmatrix},$$

where $\Lambda_n = \frac{n\pi}{L}$, $n \in \mathbb{N}$, and, for $i = 1, 2$, $h_i^{(n)} \in C^\infty([-R, R])$ and $g_i^{(n)}(\pm R) \in \mathbb{R}$. Then by the theory of Fourier series the set S is dense in $\mathcal{Y}_{m,p} \cap (C^\infty(\overline{\Omega}; \mathbb{R}^2) \times C^\infty(\partial\Omega; \mathbb{R}^2))$, which in turn is dense in $\mathcal{Y}_{m,p}$.

Now take $(\mathbf{h}, \mathbf{g}) \in$ S and, as in (6.3) and (6.4), solve

$$\frac{d^2 v_i^{(n)}}{dx^2} - \Lambda_n^2 v_i^{(n)} = -h_i^{(n)} \text{ on } [-R, R], \qquad \frac{dv_i^{(n)}}{dx}(\pm R) = g_i^{(n)}(\pm R), \quad i = 1, 2,$$

to conclude that $\mathbf{v}$, which is contained in $\mathcal{X}_{m,p}$, is a finite linear combination of

$$\mathbf{v}^{(n)}(x,y) = \begin{bmatrix} v_1^{(n)}(x)\cos(\Lambda_n y) \\ v_2^{(n)}(x)\sin(\Lambda_n y) \end{bmatrix}.$$

Moreover, $\mathfrak{L}^{(0)}[\mathbf{v}] = (\mathbf{h}, \mathbf{g})$ (when $n = 0$, a constant can be added to v_1 so that $X_{(i)}$ of §4 is satisfied). Therefore S is contained in the range of $\mathfrak{L}^{(0)}$. However, the latter is closed so it must be equal to $\mathcal{Y}_{m,p}$. This proves $\mathfrak{L}^{(0)}$ is bijective. □

Lemma 6.4. *Assume (H1)–(H5), (H7), and (H8). Then there exists* $\mathbf{v} \in \mathrm{Var}$ *and* $\lambda \in (\lambda_\infty, 1)$ *such that* $\mathrm{Q}_\lambda[\mathbf{v}] < 0$.

Proof of Lemma 6.4. Fix $n \in \mathbb{Z}^+$. For $\lambda \in (\lambda_\infty, 1)$ let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ be two linearly independent solutions of (6.2)(a–c) of the form

$$\mathbf{u}^{(i)} = \mathrm{e}^{r_i \Lambda_n (x-R)} \begin{bmatrix} a_1^{(i)} \cos(\Lambda_n y) \\ a_2^{(i)} \sin(\Lambda_n y) \end{bmatrix}, \quad i = 1, 2,$$

if $r_1 \neq r_2$ and, if $r_1 = r_2$, replace $\mathbf{u}^{(2)}$ with $\partial\mathbf{u}^{(1)}/\partial r_1$. Here $a_j^{(i)} \in \mathbb{C}$ and r_i are the roots of (3.16), at $\tau = 0$, with positive real part. If we substitute $\mathbf{u} := c_1\mathbf{u}^{(1)} + c_2\mathbf{u}^{(2)}$ ($\mathbf{c} := (c_1, c_2)^{\mathrm{T}} \in \mathbb{C}^2$) into the complexified form of (5.1), i.e.,

$$\mathbb{Q}_\lambda[\mathbf{v}] := \int_{\mathcal{R}} \overline{\nabla\mathbf{v}(\mathbf{x})} : \mathsf{C}_\lambda\left[\nabla\mathbf{v}(\mathbf{x})\right] d\mathbf{x} \quad \text{ for } \ \mathbf{v} \in W^{1,2}(\mathcal{R}; \mathbb{C}^2),$$

then we find with the aid of (5.8) and (6.2) that

$$\mathbb{Q}_\lambda^n(\mathbf{c}) := \mathbb{Q}_\lambda[\mathbf{u}] = \overline{\mathbf{c}} \cdot \left[\mathbf{B}(\lambda) - \overline{\mathbf{E}(\lambda)}^{\mathrm{T}}\mathbf{B}(\lambda)\mathbf{E}(\lambda)\right]\mathbf{c}, \tag{A.9}$$

where

$$\mathbf{B}(\lambda)_{ij} := \int_0^L \overline{\mathbf{u}^{(i)}} \cdot \mathfrak{B}_\lambda\big[\mathbf{u}^{(j)}\big]\, dy$$

(at $x = R$) are the components of a 2×2 matrix $\mathbf{B}(\lambda)$, and

$$\mathbf{E}(\lambda) := \begin{cases} \operatorname{diag}[\mathrm{e}^{-2r_1\Lambda_n R}, \mathrm{e}^{-2r_2\Lambda_n R}] & \text{if} \quad r_1 \neq r_2, \\ \mathrm{e}^{-2r_1\Lambda_n R} \begin{bmatrix} 1 & -2\Lambda_n R \\ 0 & 1 \end{bmatrix} & \text{if} \quad r_1 = r_2. \end{cases}$$

Note that $\mathbf{B}(\lambda)$ is Hermitian by (A.9) and since C_λ is symmetric.

To establish the lemma we will show that $\mathbb{Q}^n_{\lambda_\infty}(\mathbf{c}) < 0$ for some $\mathbf{c} \in \mathbb{C}^2$. We first note that since the complementing condition fails at λ_∞, there exists $\mathbf{c} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ such that $\mathbf{v} = c_1\mathbf{u}^{(1)} + c_2\mathbf{u}^{(2)}$ satisfies the boundary conditions (6.2d) and (6.2e) at $x = R$ and hence $\mathbf{B}(\lambda_\infty)\mathbf{c} = \mathbf{0}$. We next show that $\mathbf{c} \neq \mathbf{e}_1$ and (if $r_1 \neq r_2$) $\mathbf{c} \neq \mathbf{e}_2$ so that $\mathbf{E}(\lambda_\infty)\mathbf{c}$ is not parallel to $\mathbf{c}$. We also show that $\mathbf{B}(\lambda_\infty)$ is not the zero matrix. Suppose $\mathbf{c} = \mathbf{e}_1$. Then $\mathfrak{B}_{\lambda_\infty}[\mathbf{u}^{(1)}] = \mathbf{0}$ at $x = R$ and hence $p_1(r_1) = p_2(r_1) = 0$ ($\lambda = \lambda_\infty$ in the following). Therefore,

$$r_1^2 = \frac{NP}{K(N-M)} = \frac{P^2 - M(M-N)}{PK},$$

which is impossible since $M > 0$ and $P \neq |N - M|$ (this follows from (3.13b), (4.5), and (H7), $\mu > \lambda$). A similar argument shows that $\mathbf{c} \neq \mathbf{e}_2$, when $r_1 \neq r_2$, and $\mathbf{B}(\lambda_\infty) \neq \mathbf{0}$.

Next, let 0 and σ denote the eigenvalues of $\mathbf{B}(\lambda_\infty)$. If $\sigma > 0$ then $\mathbf{d} := \mathbf{E}(\lambda_\infty)\mathbf{c}$ is not parallel to $\mathbf{c}$ and hence $\mathbb{Q}^n_{\lambda_\infty}(\mathbf{c}) = -\overline{\mathbf{d}} \cdot \mathbf{B}(\lambda_\infty)\mathbf{d} < 0$. If instead $\sigma < 0$ then $\mathbf{e} := \mathbf{E}(\lambda_\infty)^{-1}\mathbf{c}$ is not parallel to $\mathbf{c}$ and so $\mathbb{Q}^n_{\lambda_\infty}(\mathbf{c}) = \overline{\mathbf{e}} \cdot \mathbf{B}(\lambda_\infty)\mathbf{e} < 0$.

It therefore follows that $\mathbb{Q}_\lambda[\mathbf{v}] < 0$ for all λ slightly greater than λ_∞. Finally we note that we may assume $\mathbf{v}$ is real-valued since, by the symmetry of C_λ,

$$\mathbb{Q}_\lambda[\mathbf{v}] = \mathrm{Q}_\lambda\big[\mathrm{Re}\{\mathbf{v}\}\big] + \mathrm{Q}_\lambda\big[\mathrm{Im}\{\mathbf{v}\}\big];$$

so $\mathbf{v} \in \mathrm{Var}$. □

Lemma 6.5. *Assume (H1)–(H6) and (H8). Then for each* $k \in \mathbb{Z}^+$ *there exists* $\mathbf{z} \in \mathrm{Dom}_{m,p}(\mathfrak{D}_{\lambda_k}) \cap \mathrm{Var}_k$*, such that* $\mathrm{Q}_{\lambda_k}[\mathbf{z}] = 0$ *and* $\mathbf{z}$ *is a nontrivial solution of* (6.2) *at* $\lambda = \lambda_k$.

Proof of Lemma 6.5. We first note that by (H8) and Proposition 5.2 the quadratic form Q_{λ_k} is coercive on Var, i.e., satisfies (5.3). Next, clearly $\mathrm{Q}_{\lambda_k} \geq 0$ on Var_k. Now suppose that Q_{λ_k} is strictly positive on $\mathrm{Var}_k \setminus \{\mathbf{0}\}$. If this is so we will show that Q_{λ_k} is uniformly positive, i.e, satisfies $(5.2)_2$. It will then follow from $(5.2)_2$ that Q_λ is uniformly positive for λ slightly below λ_k, which contradicts (6.7), the definition of λ_k. This will show that Q_{λ_k} cannot be strictly positive.

In order to prove $(5.2)_2$ given the strict positivity of Q_{λ_k} we assume, for the sake of contradiction, $(5.2)_2$ is not true and, in particular, that there is a sequence $\mathbf{z}_n \in \mathrm{Var}_k$ with $\|\mathbf{z}_n\|_{1,2,\mathcal{R}} = 1$ and $\mathrm{Q}_{\lambda_k}[\mathbf{z}_n] \to 0$ as $n \to \infty$. Then by the Rellich compactness theorem we may assume that, for a subsequence, $\mathbf{z}_n \rightharpoonup \mathbf{z}$ weakly in $W^{1,2}(\mathcal{R};\mathbb{R}^2)$ and strongly in $L^2(\mathcal{R};\mathbb{R}^2)$ with $\mathbf{z} \in \mathrm{Var}_k$. Therefore,

$$\mathrm{Q}_{\lambda_k}[\mathbf{z}_n] = \mathrm{Q}_{\lambda_k}[\mathbf{z}_n - \mathbf{z}] + \int_{\mathcal{R}} \nabla(\mathbf{z}_n - \mathbf{z}) : \mathsf{C}_{\lambda_k}[\nabla \mathbf{z}]\, d\mathbf{x} + \int_{\mathcal{R}} \nabla \mathbf{z} : \mathsf{C}_{\lambda_k}[\nabla \mathbf{z}_n]\, d\mathbf{x}$$

and hence, in view of the positivity of Q_{λ_k} and the weak convergence of $\mathbf{z}_n$ to $\mathbf{z}$ in $W^{1,2}(\mathcal{R};\mathbb{R}^2)$,

$$\lim_{n\to\infty} \mathrm{Q}_{\lambda_k}[\mathbf{z}_n] \geq \mathrm{Q}_{\lambda_k}[\mathbf{z}].$$

Consequently, $\mathrm{Q}_{\lambda_k}[\mathbf{z}] = 0$ and so $\mathbf{z} = \mathbf{0}$. Thus, $\|\mathbf{z}_n\|_{0,2,\mathcal{R}} \to 0$ and hence (5.3) (with $\mathbf{v} = \mathbf{z_n}$) yields $\|\mathbf{z}_n\|_{1,2,\mathcal{R}} \to 0$, which is a contradiction. Therefore, Q_{λ_k} is not strictly positive on $\mathrm{Var}_k \setminus \{\mathbf{0}\}$. (See [24, Theorem 4.1] or [56, Theorem 1] for alternate proofs.)

Let $\mathbf{z} \in \mathrm{Var}_k$ with $\mathbf{z} \neq \mathbf{0}$ satisfy $\mathrm{Q}_{\lambda_k}[\mathbf{z}] = 0$. If we write

$$\mathbf{z}(\mathbf{x}) = \begin{bmatrix} z_1(x)\cos(\Lambda_k y) \\ z_2(x)\sin(\Lambda_k y) \end{bmatrix} \quad \text{then} \quad \tilde{\mathbf{z}}(x) := \begin{bmatrix} z_1(x) \\ z_2(x) \end{bmatrix}$$

is a weak solution of the ordinary differential and boundary operators associated with (6.2)(a,b,d,e). Regularity theory for ordinary differential equations implies $\tilde{\mathbf{z}} \in C^\infty[-R,R]$. Therefore $\mathbf{z} \in \mathrm{Dom}_{m,p}(\mathfrak{D}_{\lambda_k})$ and $\mathbf{z}$ is a nontrivial solution of (6.2) at $\lambda = \lambda_k$. $\square$

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