APPLICATIONS OF ESTIMATES NEAR THE BOUNDARY TO REGULARITY OF SOLUTIONS IN LINEARIZED ELASTICITY*

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Abstract. In this paper the tame estimate of Moser [17] is used to extend the standard regularity estimate of Agmon, Douglis, and Nirenberg [4] for systems of strongly elliptic equations in linearized elasticity so that the components of the elasticity tensor need only lie in the Sobolev space $W^{m,p}(\Omega)$ for $p > n/m$, rather than $p > n$, when one obtains $W^{m+1,p}$–regularity of the solution. This improvement is necessary if one wants to prove global continuation results in such spaces for the equations of nonlinear elasticity.

Key words. elasticity, elliptic regularity, system of partial differential equations, Sobolev tame estimate

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1. Introduction. Consider a nonlinearly elastic body that occupies the region $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, in its homogeneous reference configuration. Let the boundary of the body, $\partial\Omega$, be divided into two disjoint parts $\mathcal{S}$ and $\mathcal{D}$ and suppose one is given smooth one-parameter families of boundary tractions $\mathbf{s} : \mathcal{S} \times [0, \infty) \to \mathbb{R}^n$ and boundary deformations $\mathbf{d} : \mathcal{D} \times [0, \infty) \to \mathbb{R}^n$. Assume, in addition, that one is given a smooth one-parameter family of solutions $\mathbf{f}_\lambda$, $\lambda \in [0, \lambda_0]$, for some $\lambda_0 \geq 0$, to the equations of equilibrium, with no body forces,

$$\operatorname{div} \mathbf{S}(\nabla \mathbf{f}_\lambda(\mathbf{x})) = \mathbf{0} \qquad \text{for } (\mathbf{x}, \lambda) \in \Omega \times [0, \lambda_0] \tag{1.1}$$

that satisfy the boundary conditions

$$\mathbf{S}(\nabla \mathbf{f}_\lambda(\mathbf{x}))\mathbf{n}(\mathbf{x}) = \mathbf{s}(\mathbf{x}, \lambda) \qquad \text{for } (\mathbf{x}, \lambda) \in \mathcal{S} \times [0, \lambda_0], \tag{1.2}$$

$$\mathbf{f}_\lambda(\mathbf{x}) = \mathbf{d}(\mathbf{x}, \lambda) \qquad \text{for } (\mathbf{x}, \lambda) \in \mathcal{D} \times [0, \lambda_0], \tag{1.3}$$

where $\mathbf{S}$ is the Piola-Kirchhoff stress and $\mathbf{n}$ is the outward unit normal to the region. Then it is well-known that, if $\mathcal{S}$, $\mathcal{D}$, $\mathbf{s}$, $\mathbf{d}$, and $\mathbf{S}$ are sufficiently smooth, $\mathcal{D}$ and $\mathcal{S}$ are both closed and relatively open[1], and if the linearized operator is strongly-elliptic, satisfies the complementing condition, and is bijective, then one can use the inverse or implicit function theorem, in an appropriately chosen Banach space $B(\Omega)$, to infer the existence of a solution to (1.1)–(1.3) on some interval $[\lambda_0, \lambda_0 + \epsilon)$. Moreover, the resulting one-parameter family of solutions satisfies[2] $\det \nabla \mathbf{f}_\lambda > 0$ on $\overline{\Omega} \times [0, \lambda_0 + \epsilon)$, assuming it satisfies this condition on $[0, \lambda_0]$.

The complete analysis that yields the above results can be found in, for example, the nice monograph by Valent [26]. One key ingredient in proving such results is the

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[1]Therefore, if both $\mathcal{S}$ and $\mathcal{D}$ are nonempty the region must contain a hole.

[2]Thus the solution is locally one-to-one. If, in addition, one imposes the integral constraint of Ciarlet and Nečas [5], which prevents interpenetration of matter, one will also obtain global injectivity.

fundamental regularity estimate of Agmon, Douglis, and Nirenberg [4]: For $m \in \mathbb{Z}^+$

$$\|\mathbf{u}\|_{B_t^{m+1}(\Omega)} \le N \left[\left\| \operatorname{div} \mathsf{C}[\nabla \mathbf{u}] \right\|_{B_t^{m-1}(\Omega)} + \left\| \mathsf{C}[\nabla \mathbf{u}]\mathbf{n} \right\|_{B_t^{r}(\mathcal{S})} + \|\mathbf{u}\|_{L^1(\Omega)} \right] \tag{1.4}$$

for all $\mathbf{u} \in B_t^{m+1}(\Omega)$ that satisfy $\mathbf{u} = \mathbf{0}$ on $\mathcal{D}$, where

$$\mathsf{C}_{ijkl}(\mathbf{x}) := \left[\frac{\partial S_{ij}(\mathbf{F})}{\partial F_{kl}} \right] \Bigg|_{\mathbf{F} = \nabla \mathbf{f}_\lambda(\mathbf{x})}$$

is the elasticity tensor and either (see §2)

(S) $B_t^k(\Omega)$ is the Hölder space $C^{k,t}(\Omega;\mathbb{R}^n)$, $t \in (0,1)$, and $r = m$; or

(L) $B_t^k(\Omega)$ is the Sobolev space $W^{k,t}(\Omega;\mathbb{R}^n)$, $t \in (1,\infty)$, and $r = m - \frac{1}{t}$.

In the former case, assuming sufficient differentiability of the mapping $\mathbf{F} \mapsto \mathbf{S}(\mathbf{F})$ and, for example, global strong ellipticity of the elasticity tensor, Healey and Simpson [12] have made use of degree theory and the above Schauder estimate $(1.4)_{(\mathrm{S})}$ to show one can globally continue the solution for all $\lambda \in [0,\infty)$ unless perhaps, at some finite value of λ, the solution should fail the complementing condition at an $\mathbf{x} \in \mathcal{S}$ or the local invertibility condition $\det \nabla \mathbf{f}_\lambda(\mathbf{x}) > 0$ at an $\mathbf{x} \in \overline{\Omega}$.

In the latter case, the only significant obstacle to applying the method of [12] to make use of the above L^p-estimate $(1.4)_{(\mathrm{L})}$ to obtain a similar global continuation result (see [23]) is that the best previously known[3] version of $(1.4)_{(\mathrm{L})}$ (see, e.g., [26, pp. 75–77]) requires $p := t > n$. The purpose of this paper is give a proof of $(1.4)_{(\mathrm{L})}$ under the weaker condition $p > n/m$.

2. Sobolev Inequalities. Throughout this paper, $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{Z}^+$, will be a nonempty open region. In addition, we will assume that either Ω is all of $\mathbb{R}^n$; Ω is a half-space:

$$\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n \,:\, (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}_0 < 0\};$$

or Ω is bounded with Lipschitz[4] boundary, $\partial\Omega$. We write ∇ for the gradient operator in Ω; for a vector field $\mathbf{u}$, $\nabla \mathbf{u}$ is the tensor field with components

$$(\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}.$$

We let $C^m(\Omega)$, $m \in \mathbb{N}$, denote the set of functions with m continuous derivatives in Ω. The space $C^m(\overline{\Omega})$ will denote the set of functions $\phi \in C^m(\Omega)$ for which $D^{\boldsymbol{\alpha}}\phi$ is bounded and uniformly continuous on Ω for $0 \le |\boldsymbol{\alpha}| \le m$. $C^m(\overline{\Omega})$ is a Banach space under the norm

$$\|\phi\|_{C^m(\overline{\Omega})} := \sum_{|\boldsymbol{\alpha}| \le m} \sup_{\mathbf{x} \in \Omega} |D^{\boldsymbol{\alpha}} \phi(\mathbf{x})|,$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$ and $D^{\boldsymbol{\alpha}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$.

By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $1 \le p < \infty$ and $m \in \mathbb{Z}^+$, we denote the usual spaces of p-summable and Sobolev functions, respectively. We use the notation $L^p(\Omega;\mathbb{R}^n)$,

[3] The original proof in [4] uses $\mathsf{C}_{ijkl} \in C^m(\overline{\Omega})$ rather than $\mathsf{C}_{ijkl} \in W^{m,p}(\Omega)$.

[4] More precisely we assume that each $\mathbf{x} \in \partial\Omega$ has an open neighborhood whose intersection with $\partial\Omega$ is the graph of a Lipschitz function. See, e.g, [2, p. 83].

etc; for vector-valued maps. Sobolev functions on manifolds are defined by the use of local charts (see, e.g., [2, 16]). We note the norm on $W^{m,p}(\Omega)$ is

$$\|\phi\|_{m,p,\Omega}^p = \sum_{|\boldsymbol{\alpha}|\le m} \|D^{\boldsymbol{\alpha}}\phi\|_{p,\Omega}^p\,, \qquad \|\phi\|_{p,\Omega}^p := \int_\Omega |\phi(\mathbf{x})|^p \, d\mathbf{x}.$$

$W_0^{m,p}(\Omega)$ will denote those functions in $W^{m,p}(\Omega)$ that are limits of functions in $C^m(\Omega)$, each of which has support in a compact subset of Ω.

We will also make use of the space

$$C_B(\Omega) := C^0(\Omega)\cap L^\infty(\Omega),$$

which is a Banach space under the L^∞-norm. For $0<\lambda<1$ we write $C^{0,\lambda}(\overline{\Omega})$ for the Hölder spaces, i.e., the functions in $C^0(\overline{\Omega})$ that are Hölder continuous with exponent λ. $C^{0,\lambda}(\overline{\Omega})$ is a Banach space under the norm

$$\|\phi\|_{C^{0,\lambda}(\overline{\Omega})} := \sup_{\mathbf{x}\in\overline{\Omega}} |\phi(\mathbf{x})| + \sup_{\substack{\mathbf{x},\mathbf{z}\in\overline{\Omega}\\ \mathbf{x}\neq\mathbf{z}}} \frac{|\phi(\mathbf{x})-\phi(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^\lambda}.$$

We will use the following special cases of the standard Sobolev inequalities. For a proof of I–III see, for example, [2, pp. 85–86, 106–108]. Part IV can be found in Nirenberg [19] or, e.g., [9, p. 24]. See, also, Gagliardo [10].

Proposition 2.1. *Let $\Omega\subset\mathbb{R}^n$ be a nonempty, bounded open region with Lipschitz boundary. Suppose $1\le p<\infty$, $k\in\mathbb{Z}^+$, and $j\in\mathbb{N}$. Then there exists a constant $K=K(n,p,k,j,\Omega)$ that has the following properties.*

I. (Sobolev Imbedding Theorem). *If $k>n/p$ then $W^{k,p}(\Omega)\subset C_B(\Omega)$ with*

$$\sup_\Omega |\phi| \le K\,\|\phi\|_{k,p,\Omega} \text{ for all } \phi\in W^{k,p}(\Omega).$$

II. (Morrey's Inequality). *If $kp>n\ge(k-1)p$ then $W^{k,p}(\Omega)\subset C^{0,\lambda}(\overline{\Omega})$ with*

$$\|\phi\|_{C^{0,\lambda}(\overline{\Omega})} \le K\,\|\phi\|_{k,p,\Omega} \text{ for all } \phi\in W^{k,p}(\Omega)$$

and $\lambda\in(0,k-\frac{n}{p}]$ if $n>(k-1)p$ and $\lambda\in(0,1)$ if $n=(k-1)p$.

III. (Banach Algebra Property). *If $k>n/p$ then $W^{k,p}(\Omega)$ is a Banach algebra, that is,*

$$\|\phi\psi\|_{k,p,\Omega} \le K\,\|\phi\|_{k,p,\Omega}\,\|\psi\|_{k,p,\Omega} \text{ for all } \phi,\psi\in W^{k,p}(\Omega).$$

IV. (Gagliardo-Nirenberg Calculus Inequality). *Let $0<j\le k$. Then*

$$\sum_{|\boldsymbol{\alpha}|=j} \|D^{\boldsymbol{\alpha}}\phi\|_{\frac{pk}{j},\mathbb{R}^n} \le K\left(\|\phi\|_{k,p,\mathbb{R}^n}\right)^{\frac{j}{k}}\left(\|\phi\|_{\infty,\mathbb{R}^n}\right)^{1-\frac{j}{k}}$$

for all $\phi\in W^{k,p}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.

An important consequence of the above calculus inequality is the following result.

Proposition 2.2. (Moser's [17, pp. 273–274] Tame Inequality). *Suppose $1\le p<\infty$ and $k\in\mathbb{Z}^+$. Then there exists a constant $C=C(n,p,k)>0$ such that*

$$C^{-1}\,\|\phi\psi\|_{k,p,\mathbb{R}^n} \le \|\phi\|_{\infty,\mathbb{R}^n}\,\|\psi\|_{k,p,\mathbb{R}^n} + \|\psi\|_{\infty,\mathbb{R}^n}\,\|\phi\|_{k,p,\mathbb{R}^n}$$

for all $\phi, \psi \in W^{k,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Proof. One can bound $\|D^{\boldsymbol{\alpha}}(\phi\psi)\|_p$ above by the indicated terms through the use of the product rule, followed by Hölder's inequality, the Gagliardo-Nirenberg calculus inequality, and finally the arithmetic-geometric mean inequality. The desired result then follows upon summing on $|\boldsymbol{\alpha}| \le k$. See Klainerman and Majda [13, pp. 516–517] for details. ∎

We next recall the following special cases of the trace theorem, for regions with sufficiently smooth boundary, and the Rellich-Kondrachov compactness theorem.

Proposition 2.3 (Trace Theorem, see, e.g., [1, p. 216], [25, p. 330], or [14, pp. 41–43 for $p = 2$]). *Let* $\Omega \subset \mathbb{R}^n$, $n \ge 2$, *be a half-space or a nonempty, bounded open region with Lipschitz boundary* $\partial\Omega$. *Suppose* $1 < p < \infty$ *and* $k \in \mathbb{Z}^+$. *Assume* $\mathcal{S} \subset \partial\Omega$ *is a relatively open,* C^k *subset of the boundary. Then there exists a constant* $T = T(n, p, k, \mathcal{S}, \Omega)$ *such that*

$$\|\phi\|_{k-\frac{1}{p},p,\mathcal{S}} \le T\,\|\phi\|_{k,p,\Omega} \ \ \textit{for all } \phi \in W^{k,p}(\Omega), \tag{2.1}$$

where $\phi|_{\mathcal{S}}$ *is to be interpreted in the sense of trace.*

Proposition 2.4 (Rellich-Kondrachov Compactness Theorem, see, e.g., [2, p. 168]). *Let* $\Omega \subset \mathbb{R}^n$ *be a nonempty, bounded open region with Lipschitz boundary* $\partial\Omega$. *Suppose* $1 \le p < \infty$, $j \in \mathbb{N}$, *and* $k \in \mathbb{Z}^+$ *with* $kp > n$. *Then the following embedding is compact:*

$$W^{k+j,p}(\Omega) \hookrightarrow C^j(\overline{\Omega}).$$

We will use the above proposition in conjunction with the following interpolation result.

Proposition 2.5 (Ehrling's Lemma [8], see, e.g., [14, p. 102] or [16, p. 85]). *Let* X, Y, *and* Z *be Banach spaces with*[5] $X \hookrightarrow Y$, $Y \subset Z$, *and* $\|y\|_Z \le C\|y\|_Y$ *for all* $y \in Y$ *and some* $C > 0$. *Then for every* $\varepsilon > 0$ *there exists* $\Lambda_\varepsilon > 0$ *such that*

$$\|x\|_Y \le \varepsilon\|x\|_X + \Lambda_\varepsilon\|x\|_Z \ \ \textit{for every } x \in X.$$

3. Half-Balls and Further Properties of Sobolev Spaces. Of fundamental importance to estimates at the boundary for systems of linear elliptic partial differential equations are Sobolev spaces on balls and half-balls. With this in mind, for $\mathbf{x}_0 \in \mathbb{R}^n$ and $R > 0$ we let

$$\mathrm{B}_R(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| < R\} \tag{3.1}$$

denote the open ball of radius R centered at $\mathbf{x}_0$. Given $\mathbf{n}_0 \in \mathbb{R}^n$ with $|\mathbf{n}_0| = 1$ we write

$$\mathcal{H} = \mathcal{H}(\mathbf{x}_0, \mathbf{n}_0) := \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{x}_0)\cdot\mathbf{n}_0 < 0\} \tag{3.2}$$

for the open half-space with outward unit normal $\mathbf{n}_0$ and $\mathbf{x}_0 \in \partial\mathcal{H}$. The open half-ball $\mathrm{B}_R(\mathbf{x}_0) \cap \mathcal{H}$ will be denoted by

$$\mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| < R,\ (\mathbf{x} - \mathbf{x}_0)\cdot\mathbf{n}_0 < 0\}. \tag{3.3}$$

[5] We use the notation $X \hookrightarrow Y$ to denote that X is compactly imbedded in Y.

Note that the relative interior of the flat portion of the boundary of $\mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0)$ is given by

$$\mathrm{B}_R(\mathbf{x}_0) \cap \partial\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| < R,\ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}_0 = 0\}.$$

We define[6]

$$C^m_{0,\mathcal{C}}\left(\mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0)\right) := \{\phi \in C^m(\mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0))\ :\ \overline{\mathrm{spt}\,\phi} \subset\subset \mathrm{B}_R(\mathbf{x}_0)\}.$$

Each such function will be thus be **zero** in an open neighborhood of the **curved** portion of the boundary of a half-ball. We then define the Sobolev space

$$W^{m,p}_{0,\mathcal{C}}\left(\mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0)\right) := \text{ closure of } C^m_{0,\mathcal{C}}\left(\mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0)\right) \text{ in } W^{m,p}\left(\mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0)\right).$$

This space satisfies $W^{m,p}_0 \subset W^{m,p}_{0,\mathcal{C}} \subset W^{m,p}$, with each containment a closed subspace, from which one can deduce many of its properties. Further, if we let $\mathbb{E} : W^{m,p}\left(\mathcal{H}\left(\mathbf{x}_0, \mathbf{n}_0\right)\right) \to W^{m,p}(\mathbb{R}^n)$ be the standard extension[7] operator it is clear that if we restrict the domain of $\mathbb{E}$ to $W^{m,p}_{0,\mathcal{C}}(\mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0))$ its range will be contained in $W^{m,p}_0(\mathrm{B}_R(\mathbf{x}_0))$. Thus we can also view $W^{m,p}_{0,\mathcal{C}}(\mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0))$ as a closed subspace of $W^{m,p}_0(\mathrm{B}_R(\mathbf{x}_0))$.

For the remainder of this section we assume that $\mathbf{x}_0 \in \mathbb{R}^n$ *and a unit vector* $\mathbf{n}_0 \in \mathbb{R}^n$ *are given and we define* $\mathrm{B}_R := \mathrm{B}_R(\mathbf{x}_0)$, $\mathrm{HB}_R := \mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0)$, *and* $\mathcal{H} := \mathcal{H}(\mathbf{x}_0, \mathbf{n}_0)$. We note that each $\phi \in W^{m,p}_{0,\mathcal{C}}(\mathrm{HB}_R)$ has a natural extension $\phi^{\mathrm{e}} \in W^{m,p}(\mathcal{H})$, i.e.,

$$\phi^{\mathrm{e}}(\mathbf{x}) := \begin{cases} \phi(\mathbf{x}), & \text{if } \mathbf{x} \in \mathrm{HB}_R, \\ 0, & \text{if } \mathbf{x} \in \mathcal{H} \setminus \mathrm{HB}_R \end{cases} \tag{3.4}$$

with[8]

$$\|\phi^{\mathrm{e}}\|_{m,p,\mathcal{H}} = \|\phi\|_{m,p,\mathrm{HB}_R}\,, \qquad \|\phi^{\mathrm{e}}\|_{m-\frac{1}{p},p,\partial\mathcal{H}} \geq \|\phi\|_{m-\frac{1}{p},p,\mathrm{B}_R\cap\partial\mathcal{H}}\,. \tag{3.5}$$

Of particular interest is the following special case of Moser's tame inequality on half-balls.

PROPOSITION 3.1. *Let* $p \in [1, \infty)$ *and* $k \in \mathbb{Z}^+$. *Then there is a constant* $C = C(n, p, k) > 0$ *such that, for any* $\mathbf{x_0} \in \mathbb{R}^n$ *and* $R_0 > 0$,

$$C^{-1}\,\|\phi\psi\|_{k,p,\mathrm{HB}_{R_0}} \leq \|\phi\|_{\infty,\mathrm{HB}_{R_0}}\,\|\psi\|_{k,p,\mathrm{HB}_{R_0}} + \|\psi\|_{\infty,\mathrm{HB}_{R_0}}\,\|\phi\|_{k,p,\mathrm{HB}_{R_0}}$$

for all $\phi, \psi \in W^{k,p}_{0,\mathcal{C}}(\mathrm{HB}_{R_0}) \cap L^\infty(\mathrm{HB}_{R_0})$.

Proof. If $\phi, \psi \in W^{k,p}_{0,\mathcal{C}}(\mathrm{HB}_{R_0}) \cap L^\infty(\mathrm{HB}_{R_0})$ then $\phi^{\mathrm{e}}, \psi^{\mathrm{e}} \in W^{k,p}(\mathcal{H})$. We can next use the aforementioned standard extension, $\mathbb{E}$, to obtain functions defined on all of $\mathbb{R}^n$ with support in B_{R_0}. The desired result then follows from Proposition 2.2. ☐

Also of interest is the following simple corollary to the trace theorem.

COROLLARY 3.2. *Let* $n \geq 2$, $1 < p < \infty$, *and* $m \in \mathbb{Z}^+$. *Then for any* $R > 0$

$$\|\phi^{\mathrm{e}}\|_{m-\frac{1}{p},p,\partial\mathcal{H}} \leq \widehat{T}\,\|\phi\|_{m,p,\mathrm{HB}_R} \ \textit{ for all } \phi \in W^{m,p}_{0,\mathcal{C}}\left(\mathrm{HB}_R\right),$$

[6] As usual we write $U \subset\subset \Omega$ for the requirement that U be contained in a compact subset of Ω.

[7] $\mathbb{E}$ is defined as a suitable linear combination of scaled reflections of the function across the hyperplane perpendicular to $\mathbf{n}_0$, see, e.g., [2, p. 148].

[8] The asserted inequality in the fractional-order norms is clear from their intrinsic definition (see, e.g., [1, pp. 208–214]). Neither this inequality nor a reverse inequality is needed here since, following ADN [3, 4], we instead use the fractional-order norm of the extended function ϕ^{e} on $\partial\mathcal{H}$.

where $\widehat{T}$ *is the constant from the trace theorem on* $\mathcal{H}$ *with* $\mathcal{S} = \partial\mathcal{H}$*. Thus* $\widehat{T} = \widehat{T}(n, p, m)$ *is independent of* R*.*

Proof. This result follows immediately from $(3.5)_1$ and Proposition 2.3 with $\Omega = \mathcal{H}$ and $\mathcal{S} = \partial\mathcal{H}$. □

We will also make use of the following corollary to the Rellich-Kondrachov theorem and Ehrling's lemma. Once again the fact the constant is independent of R will be important in our estimates.

COROLLARY 3.3. *Let* $1 \le p < \infty$*,* $j \in \mathbb{N}$*, and* $k \in \mathbb{Z}^+$ *with* $kp > n$*. Then for every* $\varepsilon > 0$ *there exists* $\Lambda_\varepsilon = \Lambda_\varepsilon(n, p, k, j) > 0$ *such that, for every* $R \in (0, 1]$*,*

$$\|\phi\|_{C^j(\overline{\mathrm{HB}}_R)} \le \varepsilon \|\phi\|_{k+j,p,\mathrm{HB}_R} + \Lambda_\varepsilon \|\phi\|_{p,\mathrm{HB}_R} \quad \textit{for every } \phi \in W^{k+j,p}_{0,\mathcal{C}}(\mathrm{HB}_R).$$

Proof. Let $1 \le p < \infty$, $k \in \mathbb{Z}^+$, $j \in \mathbb{N}$, and $R \in (0, 1)$. As in the previous proofs we note that each $\phi \in W^{k+j,p}_{0,\mathcal{C}}(\mathrm{HB}_R)$ can be extended (by zero) to a function $\phi^{\mathrm{e}} \in W^{k+j,p}_{0,\mathcal{C}}(\mathrm{HB}_1)$ in such a manner that the extension preserves the norm. Moreover, this extension also preserves the L^p and C^j-norms, provided each is finite on the original half-ball HB_R. Thus the desired results will follow once we prove them on the unit half-ball.

On the unit half-ball the result now follows immediately from Ehrling's lemma with $X = W^{k+j,p}_{0,\mathcal{C}}(\mathrm{HB}_1)$, $Y = C^j(\overline{\mathrm{HB}}_1)$, and $Z = L^p(\mathrm{HB}_1)$ since $W^{k+j,p}(\mathrm{HB}_1) \hookrightarrow C^j(\overline{\mathrm{HB}}_1)$ by the Rellich-Kondrachov theorem and $C^j(\overline{\mathrm{HB}}_1) \subset L^p(\mathrm{HB}_1)$. □

Finally, we note a useful consequence of Morrey's inequality (II of Proposition 2.1) on half-balls.

COROLLARY 3.4. *Let* $1 \le p < \infty$*,* $n \ge 2$*, and* $m \in \mathbb{Z}^+$ *satisfy* $mp > n \ge (m-1)p$*. Define* $\lambda > 0$ *by* $\lambda = \lambda(n, p, m) := m - n/p$*, if* $n > (m-1)p$*, and* $\lambda := \frac{1}{2}$*, if* $n = (m-1)p$*. Fix* $R_l > 0$*. Then there is a constant* $M = M(n, p, m, R_l)$ *such that for every* $R \in (0, R_l]$*,*

$$\sup_{\mathbf{x} \in \overline{\mathrm{HB}}_R} |\phi(\mathbf{x}) - \phi(\mathbf{x}_0)| \le M R^\lambda \|\phi\|_{m,p,\mathrm{HB}_{R_l}} \quad \textit{for all } \phi \in W^{m,p}(\mathrm{HB}_{R_l}).$$

Proof. Let $1 \le p < \infty$, $n \ge 2$, and $m \in \mathbb{Z}^+$ satisfy $mp > n \ge (m-1)p$. Then by Morrey's inequality (II of Proposition 2.1) applied to HB_{R_l} there is a constant $M = M(n, p, m, R_l)$ such that

$$\|\phi\|_{C^{0,\lambda}(\overline{\mathrm{HB}}_{R_l})} \le M \|\phi\|_{m,p,\mathrm{HB}_{R_l}} \quad \text{for all } \ \phi \in W^{m,p}(\mathrm{HB}_{R_l}).$$

Thus, in particular, by the definition of the Hölder norm,

$$|\phi(\mathbf{x}) - \phi(\mathbf{x}_0)| \le M |\mathbf{x} - \mathbf{x}_0|^\lambda \|\phi\|_{m,p,\mathrm{HB}_{R_l}} \le M R^\lambda \|\phi\|_{m,p,\mathrm{HB}_{R_l}}$$

for every $\mathbf{x} \in \overline{\mathrm{HB}}_R$, from which the desired result follows. □

4. The Elasticity Tensor; Strong Ellipticity; The Complementing Condition. We let $\mathrm{Lin}^n = \mathrm{Lin}(\mathbb{R}^n; \mathbb{R}^n)$ denote the space of all linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$ with inner product and norm, respectively, given by:

$$\mathbf{G} : \mathbf{H} := \mathrm{trace}(\mathbf{G}\mathbf{H}^{\mathrm{T}}), \qquad |\mathbf{G}|^2 := \mathbf{G} : \mathbf{G},$$

where $\mathbf{H}^{\mathrm{T}}$ denotes the transpose of $\mathbf{H}$. We write (see Del Piero [6]) $\mathrm{LinLin}^n = \mathrm{Lin}(\mathrm{Lin}^n;\mathrm{Lin}^n)$ for the space of all linear transformations from Lin^n into Lin^n; thus, in components, if $\mathsf{C} \in \mathrm{LinLin}^n$ and $\mathbf{A} \in \mathrm{Lin}^n$

$$(\mathsf{C}[\mathbf{A}])_{ij} = \sum_{k,l=1}^{n} \mathsf{C}_{ijkl} A_{kl}.$$

Although LinLin^n is also an inner product space we will not make use of the inner product structure here. Instead we will use the equivalent operator norm

$$|\mathsf{C}| := \max_{\substack{\mathbf{A}\in\mathrm{Lin}^n\\ |\mathbf{A}|=1}} \left|\mathsf{C}[\mathbf{A}]\right| \quad \text{for } \mathsf{C} \in \mathrm{LinLin}^n.$$

We denote by $\mathbf{a}\otimes\mathbf{b}$ the tensor product of any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$; in components $(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j$. We write div for the divergence operators in $\mathbb{R}^n$; for a tensor field $\mathbf{S}$, $\operatorname{div}\mathbf{S}$ is the vector field with components

$$(\operatorname{div}\mathbf{S})_i = \sum_{j=1}^{n} \frac{\partial S_{ij}}{\partial x_j}.$$

Let $\mathsf{C}_0 \in \mathrm{LinLin}^n$. We say C_0 satisfies the *strong-ellipticity condition* provided there is a constant $k_0 > 0$ such that

$$\mathbf{a}\otimes\mathbf{b} : \mathsf{C}_0[\mathbf{a}\otimes\mathbf{b}] \geq k_0 |\mathbf{a}|^2 |\mathbf{b}|^2 \quad \text{for all } \ \mathbf{a}, \mathbf{b} \in \mathbb{R}^n.$$

Let $\mathbf{x}_0 \in \mathbb{R}^n$ and suppose $\mathbf{n}_0 \in \mathbb{R}^n$ with $|\mathbf{n}_0| = 1$ is the outward unit normal to the half-space $\mathcal{H} = \mathcal{H}(\mathbf{x}_0, \mathbf{n}_0)$ given by (3.2). Consider the problem: Find $\mathbf{w} : \overline{\mathcal{H}} \to \mathbb{R}^n$ that satisfies

$$\begin{aligned} \operatorname{div}\mathsf{C}_0[\nabla\mathbf{w}] &= \mathbf{0} \quad \text{in } \mathcal{H}, \\ \mathsf{C}_0[\nabla\mathbf{w}]\mathbf{n}_0 &= \mathbf{0} \quad \text{on } \partial\mathcal{H}. \end{aligned} \tag{4.1}$$

We seek solutions of (4.1) that are *bounded exponentials*, i.e.,

$$\mathbf{w}(\mathbf{x}) = \mathbf{z}\left(-(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}_0\right)\exp\left(\mathrm{i}\,(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{t}\right) \tag{4.2}$$

for some nontrivial $\mathbf{t} \in \mathbb{R}^n$ that is tangent to $\partial\mathcal{H}$ (i.e., $\mathbf{t}\cdot\mathbf{n}_0 = 0$ and $\mathbf{t} \neq \mathbf{0}$) and some $\mathbf{z} \in C^2([0,\infty);\mathbb{C}^n)$ that satisfies $\sup\{|\mathbf{z}(s)| : s \in [0,\infty)\} < \infty$. We say the pair $(\mathsf{C}_0, \mathbf{n}_0)$ satisfies the *complementing condition* if (4.1) has no nontrivial bounded exponential solution.[9] We note the existence of exponential solutions of (4.1) is determined solely by the components of C_0 and $\mathbf{n}_0$ and as such the complementing condition is an *algebraic condition*.

In this paper we will need to make the algebraic nature of this condition more precise by recalling the *minor constant*, $\Delta_0 := \Delta(\mathsf{C}_0, \mathbf{n}_0)$, of Agmon, Douglis, and Nirenberg [4, pp. 42–43] that measures how well the boundary condition actually *complements* the differential equation in the half-space $\mathcal{H}$.

Given $\mathsf{C}_0 : \mathrm{Lin}^n \to \mathrm{Lin}^n$, $\mathbf{x}_0 \in \mathbb{R}^n$, a unit vector $\mathbf{n}_0 \in \mathbb{R}^n$, and the half-space $\mathcal{H}$ given by (3.2) let $\mathbf{t} \in \mathbb{R}^n$ with $|\mathbf{t}| = 1$ satisfy $\mathbf{t}\cdot\mathbf{n}_0 = 0$, so $\mathbf{t}$ is a unit vector lying in $\partial\mathcal{H}$, and define (cf. [22]) $\mathbf{M}, \mathbf{N}_{\mathbf{t}}, \mathbf{P}_{\mathbf{t}} \in \mathrm{Lin}^n$ by

$$\mathbf{M}\mathbf{a} := \mathsf{C}_0\left[\mathbf{a}\otimes\mathbf{n}_0\right]\mathbf{n}_0, \quad \mathbf{N}_{\mathbf{t}}\mathbf{a} := \mathsf{C}_0\left[\mathbf{a}\otimes\mathbf{t}\right]\mathbf{n}_0, \quad \mathbf{P}_{\mathbf{t}}\mathbf{a} := \mathsf{C}_0\left[\mathbf{a}\otimes\mathbf{t}\right]\mathbf{t}$$

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[9] For a physical interpretation of these conditions in terms of Rayleigh waves in a half-space and dynamic stability see [21, 24].

for $\mathbf{a} \in \mathbb{R}^n$. Then (4.1) and (4.2) reduce to the system of ordinary differential equations and boundary condition:

$$-\mathbf{M}\ddot{\mathbf{z}} + \mathrm{i}(\mathbf{N_t} + \mathbf{N_t}^{\mathrm{T}})\dot{\mathbf{z}} + \mathbf{P_t}\mathbf{z} = \mathbf{0} \quad \text{on} \quad (0,\infty),$$
$$\mathbf{M}\dot{\mathbf{z}}(0) - \mathrm{i}\mathbf{N_t}\mathbf{z}(0) = \mathbf{0}, \tag{4.3}$$

where $\mathbf{z} : [0,\infty) \to \mathbb{C}^n$.

If C_0 satisfies the strong ellipticity condition then $\mathbf{M}$ is strictly positive definite and hence, by the standard theory (see, e.g., [7]) for such systems of ordinary differential equations, $(4.3)_1$ has exactly n bounded, linearly independent solutions $\mathbf{z}^k \in C^\infty([0,\infty);\mathbb{C}^n)$, $k = 1, 2, \ldots, n$, each of which is contained in $L^2((0,\infty);\mathbb{C}^n)$. Assume the solutions are normalized so that (see [4, pp. 43–44]), e.g., $\mathbf{z}^k(0) = \mathbf{e}_k$ for $k = 1, 2, \ldots, n$, where $\{\mathbf{e}_k\}$ is the standard basis for $\mathbb{R}^n$.

The complementing condition is then the requirement that no (nontrivial) linear combination of these solutions satisfy the boundary condition $(4.3)_2$. To measure how well this condition is satisfied, for each unit tangent vector $\mathbf{t}$ define $\mathbf{L_t} \in C^\infty([0,\infty);\mathrm{Lin}(\mathbb{R}^n;\mathbb{C}^n))$ by[10]

$$\mathbf{L_t}(s)\mathbf{e}_k = \mathbf{M}\dot{\mathbf{z}}^k(s) - \mathrm{i}\mathbf{N_t}\mathbf{z}^k(s) \quad \text{for} \quad k = 1, 2, \ldots, n.$$

The complementing condition is then equivalent to the requirement that $\mathbf{L_t}(0)$ be nonsingular for each unit[11] vector $\mathbf{t} \perp \mathbf{n}_0$ or, equivalently,

$$\Delta_0 = \Delta(\mathsf{C}_0, \mathbf{n}_0) := \min_{\substack{\mathbf{t}\perp\mathbf{n}_0 \\ |\mathbf{t}|=1}} |\det \mathbf{L_t}(0)| > 0. \tag{4.4}$$

The following result is due to Agmon, Douglis, and Nirenberg [3, 4].

Proposition 4.1 (ADN Estimate for Constant Coefficients [4, Theorem 10.2]). *Let $1 < p < \infty$, $m \in \mathbb{Z}^+$ and $\mathbf{n}_0 \in \mathbb{R}^n$ with $|\mathbf{n}_0| = 1$. Then there exists a constant $A = A(n, p, m, k_0, \Delta_0, |\mathsf{C}_0|)$ such that*

$$\|\mathbf{u}\|_{m+1,p,\mathrm{HB}_1} \le A\left[\left\|\mathrm{div}(\mathsf{C}_0[\nabla\mathbf{u}])\right\|_{m-1,p,\mathrm{HB}_1} + \left\|\mathsf{C}_0[\nabla\mathbf{u}^{\mathrm{e}}]\mathbf{n}_0\right\|_{m-\frac{1}{p},p,\partial\mathcal{H}}\right]$$

for all $\mathbf{u} \in W^{m+1,p}_{0,\mathcal{C}}(\mathrm{HB}_1;\mathbb{R}^n)$. Here $\mathrm{B}_1 = \mathrm{B}_1(\mathbf{x}_0)$, $\mathrm{HB}_1 = \mathrm{HB}_1(\mathbf{x}_0, \mathbf{n}_0)$, $\mathcal{H} = \mathcal{H}(\mathbf{x}_0, \mathbf{n}_0)$, and $\mathbf{u}^{\mathrm{e}}$ is given by (4.6) with $R = 1$.

We note the proof of Corollary 3.3 immediately yields the following corollary to the above result.

Corollary 4.2. *Let p, m, $\mathbf{n}_0$, and $A = A(n, p, m, k_0, \Delta_0, |\mathsf{C}_0|)$ be as in Proposition 4.1. Then for all $R \in (0, 1]$*

$$\|\mathbf{u}\|_{m+1,p,\mathrm{HB}_R} \le A\left[\left\|\mathrm{div}(\mathsf{C}_0[\nabla\mathbf{u}])\right\|_{m-1,p,\mathrm{HB}_R} + \left\|\mathsf{C}_0[\nabla\mathbf{u}^{\mathrm{e}}]\mathbf{n}_0\right\|_{m-\frac{1}{p},p,\partial\mathcal{H}}\right] \tag{4.5}$$

for all $\mathbf{u} \in W^{m+1,p}_{0,\mathcal{C}}(\mathrm{HB}_R;\mathbb{R}^n)$. Here $\mathrm{B}_R = \mathrm{B}_R(\mathbf{x}_0)$, $\mathrm{HB}_R = \mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0)$, $\mathcal{H} = \mathcal{H}(\mathbf{x}_0, \mathbf{n}_0)$, and

$$\mathbf{u}^{\mathrm{e}}(\mathbf{x}) := \begin{cases} \mathbf{u}(\mathbf{x}), & \text{if } \mathbf{x} \in \mathrm{HB}_R, \\ \mathbf{0}, & \text{if } \mathbf{x} \in \mathcal{H} \setminus \mathrm{HB}_R. \end{cases} \tag{4.6}$$

[10] For each tangent vector $\mathbf{t}$ and $s \ge 0$ the operator $\mathbf{L_t}(s)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{C}^n$. In particular, the k-th column of the matrix $\mathbf{L_t}(0)$ will consist of the boundary condition evaluated at $\mathbf{z}^k$.

[11] Condition (4.4) for unit vectors $\mathbf{t} \perp \mathbf{n}_0$ implies (4.4) for all vectors $\alpha\mathbf{t} \perp \mathbf{n}_0$ since $\mathbf{L}_{\alpha\mathbf{t}}(0) = \alpha\mathbf{L_t}(0)$.

5. The ADN Estimate with Sobolev Coefficients on Balls and Half-Balls. Recall $\mathrm{LinLin}^n = \mathrm{Lin}(\mathrm{Lin}^n; \mathrm{Lin}^n)$, $\mathrm{Lin}^n = \mathrm{Lin}(\mathbb{R}^n; \mathbb{R}^n)$, and for any $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{n}_0 \in \mathbb{R}^n$ with $|\mathbf{n}_0| = 1$ the half-space $\mathcal{H} := \mathcal{H}(\mathbf{x}_0, \mathbf{n}_0)$ is given by (3.2) and the ball $\mathrm{B}_R := \mathrm{B}_R(\mathbf{x}_0)$ and half-ball $\mathrm{HB}_R := \mathrm{HB}_R(\mathbf{x}_0, \mathbf{n}_0)$ are given by (3.1) and (3.3), respectively, for any $R > 0$.

LEMMA 5.1. *Let $p \in (1, \infty)$ and suppose $m \in \mathbb{Z}^+$ satisfies $mp > n$. Let $k_0 > 0$, $\delta_0 > 0$, $\mu > 0$, and $R_l \in (0, 1]$ be given. Then there exist constants $R_\sigma = R_\sigma(n, p, m, k_0, \delta_0, \mu, R_l)$, $0 < R_\sigma < R_l$, and $D = D(n, p, m, k_0, \delta_0, \mu, R_l) > 0$ such that any elasticity tensor $\mathsf{C} \in W^{m,p}(\mathrm{HB}_{R_l}; \mathrm{LinLin}^n)$ that satisfies*

$$\|\mathsf{C}\|_{m,p,\mathrm{HB}_{R_l}} \leq \mu, \tag{5.1}$$

$$\mathbf{a} \otimes \mathbf{b} : \mathsf{C}(\mathbf{x}_0)[\mathbf{a} \otimes \mathbf{b}] \geq k_0 |\mathbf{a}|^2 |\mathbf{b}|^2 \quad \textit{for all} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \ \textit{and} \tag{5.2}$$

$$\Delta(\mathsf{C}(\mathbf{x}_0), \mathbf{n}_0) \geq \delta_0, \tag{5.3}$$

for some $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{n}_0 \in \mathbb{R}^n$ with $|\mathbf{n}_0| = 1$, will also satisfy

$$\|\mathbf{u}\|_{m+1,p,\mathrm{HB}_R} \leq D \left[\left\| \operatorname{div} \mathsf{C}[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{HB}_R} + \left\| \mathsf{C}[\nabla \mathbf{u}^{\mathrm{e}}] \mathbf{n}_0 \right\|_{m-\frac{1}{p},p,\partial\mathcal{H}} + \|\mathbf{u}\|_{p,\mathrm{HB}_R} \right] \tag{5.4}$$

for all $R \in (0, R_\sigma]$ and $\mathbf{u} \in W^{m+1,p}_{0,\mathcal{C}}(\mathrm{HB}_R; \mathbb{R}^n)$, where $\mathbf{u}^{\mathrm{e}}$ is given by (4.6).

Proof. First, fix $1 < p < \infty$ and $m \in \mathbb{Z}^+$ that satisfy $mp > n$. Let $R_l \in (0, 1]$, $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{n}_0 \in \mathbb{R}^n$ with $|\mathbf{n}_0| = 1$. Then, by Corollary 3.2 (corollary to the trace theorem) there is a $\widehat{T} = \widehat{T}(n, p, m)$ such that for all $R \in (0, R_l]$

$$\|\mathbf{v}^{\mathrm{e}}\|_{m-\frac{1}{p},p,\partial\mathcal{H}} \leq \widehat{T}\, \|\mathbf{v}\|_{m,p,\mathrm{HB}_R} \quad \text{for all } \mathbf{v} \in W^{m,p}_{0,\mathcal{C}}(\mathrm{HB}_R; \mathbb{R}^n). \tag{5.5}$$

Next, fix $k_0 > 0$, $\delta_0 > 0$, and $\mu > 0$. We will construct D and R_σ, *which only depend on n, p, m, k_0, δ_0, μ, and R_l*, such that (5.4) is satisfied. Suppose $\mathsf{C} \in W^{m,p}(\mathrm{HB}_{R_l}; \mathrm{LinLin}^n)$ satisfies (5.1)–(5.3). Then by the corollary to the ADN estimate with constant coefficients, (4.5),

$$\|\mathbf{u}\|_{m+1,p,\mathrm{HB}_R} \leq A \left[\left\| \operatorname{div} \mathsf{C}_0[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{HB}_R} + \left\| \mathsf{C}_0[\nabla \mathbf{u}^{\mathrm{e}}] \mathbf{n}_0 \right\|_{m-\frac{1}{p},p,\partial\mathcal{H}} \right] \tag{5.6}$$

for all $R \in (0, R_l]$ and $\mathbf{u} \in W_0(\mathrm{HB}_R) := W^{m+1,p}_{0,\mathcal{C}}(\mathrm{HB}_R; \mathbb{R}^n)$, where $\mathsf{C}_0 := \mathsf{C}(\mathbf{x}_0)$ and $A = A(n, p, m, k_0, \delta_0, |\mathsf{C}_0|)$.

Define $\mathsf{J} \in W^{m,p}(\mathrm{HB}_{R_l}; \mathrm{LinLin}^n)$ by $\mathsf{J}(\mathbf{x}) := \mathsf{C}_0 - \mathsf{C}(\mathbf{x})$ so that $\mathsf{C}_0 = \mathsf{C} + \mathsf{J}$. Then by the triangle inequality and (5.5) with $\mathbf{v} = \mathsf{J}[\nabla \mathbf{u}^{\mathrm{e}}]\mathbf{n}_0$

$$\begin{aligned} \left\| \mathsf{C}_0[\nabla \mathbf{u}^{\mathrm{e}}] \mathbf{n}_0 \right\|_{m-\frac{1}{p},p,\partial\mathcal{H}} &\leq \left\| \mathsf{C}[\nabla \mathbf{u}^{\mathrm{e}}] \mathbf{n}_0 \right\|_{m-\frac{1}{p},p,\partial\mathcal{H}} + \left\| \mathsf{J}[\nabla \mathbf{u}^{\mathrm{e}}] \mathbf{n}_0 \right\|_{m-\frac{1}{p},p,\partial\mathcal{H}} \\ &\leq \left\| \mathsf{C}[\nabla \mathbf{u}^{\mathrm{e}}] \mathbf{n}_0 \right\|_{m-\frac{1}{p},p,\partial\mathcal{H}} + \widehat{T} \left\| \mathsf{J}[\nabla \mathbf{u}] \right\|_{m,p,\mathrm{HB}_R} \end{aligned} \tag{5.7}$$

for all $R \in (0, R_l]$ and $\mathbf{u} \in W_0(\mathrm{HB}_R)$, where we have also used the fact that $\mathbf{n}_0$ is a constant unit vector. Similarly, the triangle inequality yields for each $R \in (0, R_l]$

$$\begin{aligned} \left\| \operatorname{div} \mathsf{C}_0[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{HB}_R} &\leq \left\| \operatorname{div} \mathsf{C}[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{HB}_R} + \left\| \operatorname{div} \mathsf{J}[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{HB}_R} \\ &\leq \left\| \operatorname{div} \mathsf{C}[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{HB}_R} + \left\| \mathsf{J}[\nabla \mathbf{u}] \right\|_{m,p,\mathrm{HB}_R} \end{aligned} \tag{5.8}$$

for all $\mathbf{u} \in W_0(\mathrm{HB}_R)$. If we combine (5.6)–(5.8) we find, for all such R and $\mathbf{u}$,

$$\begin{aligned} \|\mathbf{u}\|_{m+1,p,\mathrm{HB}_R} \leq\ & A \left[\left\| \operatorname{div} \mathsf{C}[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{HB}_R} + \left\| \mathsf{C}[\nabla \mathbf{u}^{\mathrm{e}}] \mathbf{n}_0 \right\|_{m-\frac{1}{p},p,\partial\mathcal{H}} \right] \\ & + A(\widehat{T} + 1) \left\| \mathsf{J}[\nabla \mathbf{u}] \right\|_{m,p,\mathrm{HB}_R}. \end{aligned} \tag{5.9}$$

We will show that, for all R sufficiently small, the last term in (5.9) can be bounded above by an arbitrarily small constant times the $W^{m+1,p}$-norm of $\mathbf{u}$ plus a (large) constant times its L^p-norm, which will establish the desired result, (5.4). With this in mind let R_σ, to be determined later, satisfy $R_\sigma \in (0, R_l/2)$ and suppose that $\phi_\sigma \in C^\infty(\mathbb{R}^n; [0,1])$ satisfies

$$\phi_\sigma(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \mathrm{HB}_{R_\sigma}, \\ 0, & \text{if } \mathbf{x} \in \mathbb{R}^n \setminus \mathrm{HB}_{2R_\sigma}. \end{cases}$$

Then $\phi_\sigma \mathsf{J} \in W^{m,p}_{0,\mathcal{C}}(\mathrm{HB}_{R_l}; \mathrm{LinLin}^n)$ and, for any $R \in (0, R_\sigma]$,

$$\|\mathsf{J}[\nabla \mathbf{u}]\|_{m,p,\mathrm{HB}_R} = \|(\phi_\sigma \mathsf{J})[\nabla \mathbf{u}]\|_{m,p,\mathrm{HB}_R} = \|(\phi_\sigma \mathsf{J})[\nabla \mathbf{u}^e]\|_{m,p,\mathrm{HB}_{R_l}} \tag{5.10}$$

for all $\mathbf{u} \in W_0(\mathrm{HB}_R)$.

We note that, since $m > n/p$, the Sobolev imbedding theorem yields $J, \nabla\mathbf{u} \in L^\infty$. Thus we may apply Proposition 3.1 (with $R_0 = R_l$) to deduce the existence of a constant $C = C(n,p,m)$ such that, for every $R \in (0, R_\sigma]$ and $\mathbf{u} \in W_0(\mathrm{HB}_R)$,

$$\begin{aligned} C^{-1}&\|(\phi_\sigma \mathsf{J})[\nabla \mathbf{u}^e]\|_{m,p,\mathrm{HB}_{R_l}} \\ &\le \left(\sup_{\mathrm{HB}_{R_l}} |\phi_\sigma \mathsf{J}| \right) \|\nabla \mathbf{u}^e\|_{m,p,\mathrm{HB}_{R_l}} + \left(\sup_{\mathrm{HB}_{R_l}} |\nabla \mathbf{u}^e| \right) \|\phi_\sigma \mathsf{J}\|_{m,p,\mathrm{HB}_{R_l}} \\ &= \left(\sup_{\mathrm{HB}_{2R_\sigma}} |\phi_\sigma \mathsf{J}| \right) \|\nabla \mathbf{u}\|_{m,p,\mathrm{HB}_R} + \left(\sup_{\mathrm{HB}_R} |\nabla \mathbf{u}| \right) \|\phi_\sigma \mathsf{J}\|_{m,p,\mathrm{HB}_{R_l}} \\ &\le \left(\sup_{\mathrm{HB}_{2R_\sigma}} |\mathsf{J}| \right) \|\nabla \mathbf{u}\|_{m,p,\mathrm{HB}_R} + Q_\sigma \|\mathsf{J}\|_{m,p,\mathrm{HB}_{R_l}} \left(\sup_{\mathrm{HB}_R} |\nabla \mathbf{u}| \right), \end{aligned} \tag{5.11}$$

where, by the Banach algebra property of $W^{m,p}$, $Q_\sigma = Q_\sigma(\phi_\sigma) > 0$ is proportional to the $W^{m,p}$-norm of ϕ_σ on the half-ball HB_{R_l}. We further note that by the Sobolev imbedding theorem and (5.1)

$$\|\mathsf{C}_0\|^p_{m,p,\mathrm{HB}_{R_l}} = \frac{\omega_n}{2} R_l^n |\mathsf{C}(\mathbf{x}_0)|^p \le \frac{\omega_n}{2} \left(\sup_{\mathrm{HB}_{R_l}} |\mathsf{C}| \right)^p \le \frac{\omega_n}{2} \widehat{K}^p \mu^p \tag{5.12}$$

(since $R_l \le 1$), where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ and $\widehat{K} = \widehat{K}(n,p,m,R_l)$. Thus, $\mathsf{J} = \mathsf{C}_0 - \mathsf{C}$ satisfies

$$\|\mathsf{J}\|_{m,p,\mathrm{HB}_{R_l}} \le \|\mathsf{C}_0\|_{m,p,\mathrm{HB}_{R_l}} + \|\mathsf{C}\|_{m,p,\mathrm{HB}_{R_l}} \le \mu N, \tag{5.13}$$

where $N = N(n,p,m,R_l) := 1 + \widehat{K}[\omega_n/2]^{\frac{1}{p}}$.

Now $m > n/p$ and hence the integer $\widehat{m} := [\![\frac{n}{p}]\!] + 1$ satisfies $m \ge \widehat{m} > \frac{n}{p} \ge \widehat{m} - 1$. Thus, if we let $\alpha > 0$ and $\varepsilon > 0$ be (small) parameters to be determined later, (5.13), Corollary 3.3 (with $j = 1$), and Corollary 3.4, imply that there exist $\Lambda_\varepsilon = \Lambda_\varepsilon(n,p,m) > 0$ and $R_\sigma = R_\sigma(\alpha, n, p, m, R_l, \mu) \in (0, R_l/2)$ such that, for any $R \in (0, R_\sigma]$,

$$\sup_{\mathrm{HB}_{2R_\sigma}} |\mathsf{J}| < \alpha, \qquad \sup_{\mathrm{HB}_R} |\nabla \mathbf{u}| \le \varepsilon \|\mathbf{u}\|_{m+1,p,\mathrm{HB}_R} + \Lambda_\varepsilon \|\mathbf{u}\|_{p,\mathrm{HB}_R} \tag{5.14}$$

for every $\mathbf{u} \in W_0(\mathrm{HB}_R)$. Therefore, (5.10)–(5.14) together with the fact $\|\nabla \mathbf{u}\|_{m,p} \leq \|\mathbf{u}\|_{m+1,p}$ yield, for all $R \in (0, R_\sigma]$ and $\mathbf{u} \in W_0(\mathrm{HB}_R)$,

$$\|\mathsf{J}[\nabla \mathbf{u}]\|_{m,p,\mathrm{HB}_R} \leq C \left[\left(\alpha + \mu N Q_\sigma \varepsilon \right) \|\mathbf{u}\|_{m+1,p,\mathrm{HB}_R} + \mu N Q_\sigma \Lambda_\varepsilon \|\mathbf{u}\|_{p,\mathrm{HB}_R} \right] \tag{5.15}$$

and hence, in view of (5.9),

$$F \|\mathbf{u}\|_{m+1,p,\mathrm{HB}_R} \leq A \left[\left\| \operatorname{div} \mathsf{C}[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{HB}_R} + \left\| \mathsf{C}[\nabla \mathbf{u}^{\mathrm{e}}] \mathbf{n}_0 \right\|_{m-\frac{1}{p},p,\partial\mathcal{H}} \right] + \mu C N Q_\sigma \Lambda_\varepsilon A(\widehat{T}+1) \|\mathbf{u}\|_{p,\mathrm{HB}_R},$$

where

$$F := 1 - A(\widehat{T}+1)C\big(\alpha + \mu N Q_\sigma \varepsilon\big).$$

Finally, define $\alpha := \frac{1}{3}[A(\widehat{T}+1)C]^{-1}$. This determines $R_\sigma \in (0, R_l/2)$ that satisfies $(5.14)_1$. It also fixes ϕ_σ and hence Q_σ. Then, define $\varepsilon := \frac{1}{3}[\mu A C N Q_\sigma(\widehat{T}+1)]^{-1}$, which determines Λ_ε. It follows that $F = \frac{1}{3} > 0$, which completes the proof. ∎

To finish the section we note the following result, whose proof is essentially identical to the previous result.

Lemma 5.2. *Let $1 < p < \infty$ and suppose $m \in \mathbb{Z}^+$ satisfies $mp > n$. Let $k_0 > 0$, $\mu > 0$, and $R_l \in (0, 1]$ be given. Then there exist constants $D = D(n, p, m, k_0, \mu, R_l) > 0$ and $R_\sigma = R_\sigma(n, p, m, k_0, \mu, R_l) \in (0, R_l)$ such that each of the following holds.*

A. *Any elasticity tensor $\mathsf{C} \in W^{m,p}(\mathrm{B}_{R_l}; \mathrm{LinLin}^n)$ that satisfies* (5.1) *(with HB_{R_l} replaced by B_{R_l}) and* (5.2) *for some $\mathbf{x}_0 \in \mathbb{R}^n$ will also satisfy*

$$\|\mathbf{u}\|_{m+1,p,\mathrm{B}_R} \leq D \left[\left\| \operatorname{div} \mathsf{C}[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{B}_R} + \|\mathbf{u}\|_{p,\mathrm{B}_R} \right]$$

for all $R \in (0, R_\sigma]$ and $\mathbf{u} \in W_0^{m+1,p}(\mathrm{B}_R; \mathbb{R}^n)$.

B. *Any elasticity tensor $\mathsf{C} \in W^{m,p}(\mathrm{HB}_{R_l}; \mathrm{LinLin}^n)$ that satisfies* (5.1) *and* (5.2)*, for some $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{n}_0 \in \mathbb{R}^n$ with $|\mathbf{n}_0| = 1$, will also satisfy*

$$\|\mathbf{u}\|_{m+1,p,\mathrm{HB}_R} \leq D \left[\left\| \operatorname{div} \mathsf{C}[\nabla \mathbf{u}] \right\|_{m-1,p,\mathrm{HB}_R} + \|\mathbf{u}\|_{p,\mathrm{HB}_R} \right]$$

for all $R \in (0, R_\sigma]$ and $\mathbf{u} \in W_0^{m+1,p}(\mathrm{HB}_R; \mathbb{R}^n)$.

6. The ADN Estimate with $\mathbf{W^{m,p}}$-Coefficients for the Displacement, Traction, and Mixed Problems. We now assume $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a nonempty, bounded open set with boundary

$$\partial\Omega = \mathcal{D} \cup \mathcal{S}, \qquad \mathcal{D} \cap \mathcal{S} = \varnothing,$$

where

$\mathcal{D}$ and $\mathcal{S}$ are both closed and relatively open.

Consequently, if both are nonempty the region must contain a hole. We note that a standard covering argument together with a partition of unity and a local flattening of the boundary allows one to make use of Lemmas 5.1 and 5.2 to arrive at the following improvement to the well-known regularity results for the equations of linearized elasticity. For a detailed proof see Agmon, Douglis, and Nirenberg [3, Theorem 15.2].

See also, e.g., [9, Theorem 17.2], [11, §8.4], or [16, Theorem 6.3.9]. If $p > n$ rather than $p > n/m$ this result has been previously proven[12] by Valent [26].

THEOREM 6.1. *Let* $\partial\Omega$ *be* C^{m+1}, $1 < p < \infty$, *and suppose* $m \in \mathbb{Z}^+$ *satisfies* $mp > n$. *Suppose further* $k > 0$, $\delta > 0$, *and* $\mu > 0$ *are given. Then there exists a constant* $N = N(n, p, m, k, \delta, \mu, \Omega, \mathcal{S}) > 0$ *such that any elasticity tensor* $\mathsf{C} \in W^{m,p}(\Omega; \mathrm{LinLin}^n)$ *that satisfies*

$$\|\mathsf{C}\|_{m,p,\Omega} \leq \mu, \tag{6.1}$$

$$\mathbf{a}\otimes\mathbf{b} : \mathsf{C}(\mathbf{x})[\mathbf{a}\otimes\mathbf{b}] \geq k|\mathbf{a}|^2|\mathbf{b}|^2 \quad \textit{for all} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \ \mathbf{x} \in \overline{\Omega}, \ \textit{and} \tag{6.2}$$

$$\Delta(\mathsf{C}(\mathbf{x}), \mathbf{n}(\mathbf{x})) \geq \delta \ \textit{for every}\ \mathbf{x} \in \mathcal{S}, \tag{6.3}$$

will also satisfy

$$\|\mathbf{u}\|_{m+1,p,\Omega} \leq N\left[\left\|\operatorname{div}\mathsf{C}[\nabla\mathbf{u}]\right\|_{m-1,p,\Omega} + \left\|\mathsf{C}[\nabla\mathbf{u}]\mathbf{n}\right\|_{m-\frac{1}{p},p,\mathcal{S}} + \|\mathbf{u}\|_{p,\Omega}\right]$$

for all $\mathbf{u} \in W^{m+1,p}(\Omega; \mathbb{R}^n)$ *such that* $\mathbf{u} = \mathbf{0}$ *on* $\mathcal{D}$. *Here* $\mathbf{n}$ *is the outward unit normal to* Ω *and* $\Delta(\mathsf{C}(\mathbf{x}), \mathbf{n}(\mathbf{x}))$ *is given by* (4.4).

Thus, given n and p, in order insure the solution has $(m+1)$–weak derivatives in the region, $m > n/p$, we require the elasticity tensor field be contained in the Sobolev space $W^{m,p}$, (6.1), the elasticity tensor field be *uniformly strongly elliptic*, (6.2), *the complementing condition be satisfied uniformly on* $\mathcal{S}$, (6.3), and the boundary have local parameterizations that are of class C^{m+1}. Regularity results with less boundary smoothness have been given by Nečas [18] and Maz′ya and Shaposhnikova [15, Chapter 7]. Note $m > n/p$ together with the assumed boundary smoothness implies the elasticity tensor field is assumed to be continuous on $\overline{\Omega}$. For *interior* regularity under weaker smoothness hypotheses on the elasticity tensor field see, e.g., Ragusa [20] and the references therein.

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