APPLICATIONS OF ESTIMATES NEAR THE BOUNDARY TO REGULARITY OF SOLUTIONS IN LINEARIZED ELASTICITY*

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Abstract. In this paper the tame estimate of Moser [17] is used to extend the standard regularity estimate of Agmon, Douglis, and Nirenberg [4] for systems of strongly elliptic equations in linearized elasticity so that the components of the elasticity tensor need only lie in the Sobolev space $W^{m,p}(\Omega)$ for $p > n/m$, rather than $p > n$, when one obtains $W^{m+1,p}$–regularity of the solution. This improvement is necessary if one wants to prove global continuation results in such spaces for the equations of nonlinear elasticity.

Key words. elasticity, elliptic regularity, system of partial differential equations, Sobolev tame estimate

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1. Introduction. Consider a nonlinearly elastic body that occupies the region $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, in its homogeneous reference configuration. Let the boundary of the body, $\partial\Omega$, be divided into two disjoint parts $\mathcal{S}$ and $\mathcal{D}$ and suppose one is given smooth one-parameter families of boundary tractions $\mathbf{s} : \mathcal{S} \times [0, \infty) \to \mathbb{R}^n$ and boundary deformations $\mathbf{d} : \mathcal{D} \times [0, \infty) \to \mathbb{R}^n$. Assume, in addition, that one is given a smooth one-parameter family of solutions $\mathbf{f}_\lambda$, $\lambda \in [0, \lambda_0]$, for some $\lambda_0 \geq 0$, to the equations of equilibrium, with no body forces,

$$\operatorname{div} \mathbf{S}(\nabla \mathbf{f}_\lambda(\mathbf{x})) = \mathbf{0} \qquad \text{for } (\mathbf{x}, \lambda) \in \Omega \times [0, \lambda_0] \tag{1.1}$$

that satisfy the boundary conditions

$$\mathbf{S}(\nabla \mathbf{f}_\lambda(\mathbf{x}))\mathbf{n}(\mathbf{x}) = \mathbf{s}(\mathbf{x}, \lambda) \qquad \text{for } (\mathbf{x}, \lambda) \in \mathcal{S} \times [0, \lambda_0], \tag{1.2}$$

$$\mathbf{f}_\lambda(\mathbf{x}) = \mathbf{d}(\mathbf{x}, \lambda) \qquad \text{for } (\mathbf{x}, \lambda) \in \mathcal{D} \times [0, \lambda_0], \tag{1.3}$$

where $\mathbf{S}$ is the Piola-Kirchhoff stress and $\mathbf{n}$ is the outward unit normal to the region. Then it is well-known that, if $\mathcal{S}$, $\mathcal{D}$, $\mathbf{s}$, $\mathbf{d}$, and $\mathbf{S}$ are sufficiently smooth, $\mathcal{D}$ and $\mathcal{S}$ are both closed and relatively open[1], and if the linearized operator is strongly-elliptic, satisfies the complementing condition, and is bijective, then one can use the inverse or implicit function theorem, in an appropriately chosen Banach space $B(\Omega)$, to infer the existence of a solution to (1.1)–(1.3) on some interval $[\lambda_0, \lambda_0 + \epsilon)$. Moreover, the resulting one-parameter family of solutions satisfies[2] $\det \nabla \mathbf{f}_\lambda > 0$ on $\overline{\Omega} \times [0, \lambda_0 + \epsilon)$, assuming it satisfies this condition on $[0, \lambda_0]$.

The complete analysis that yields the above results can be found in, for example, the nice monograph by Valent [26]. One key ingredient in proving such results is the

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[1] Therefore, if both $\mathcal{S}$ and $\mathcal{D}$ are nonempty the region must contain a hole.

[2] Thus the solution is locally one-to-one. If, in addition, one imposes the integral constraint of Ciarlet and Nečas [5], which prevents interpenetration of matter, one will also obtain global injectivity.