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On irregular weak solutions of the energy–momentum equations

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Irregular mappings that are weak solutions of the energy–momentum equations are presented. One example is discontinuous at a countable number of points while the other is C^1, but not C^2. These mappings are not solutions of the usual Euler–Lagrange equations.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider the integral functional

$$E(\boldsymbol{u}) = \int_{\Omega} W(\nabla \boldsymbol{u}(\boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} \tag{1.1}$$

defined on maps $\boldsymbol{u} \in W^{1,1}(\Omega; \mathbb{R}^n)$. We suppose that W is non-negative and C^2. We also assume that W is frame indifferent and isotropic so that

$$W(\boldsymbol{F}) = W(\boldsymbol{QF}) = W(\boldsymbol{FQ}) \tag{1.2}$$

for all $\boldsymbol{Q} \in \mathrm{SO}(n)$ and all $\boldsymbol{F} \in M^{n\times n}$ with non-negative determinant, where $M^{n\times n}$ denotes the real $n \times n$ matrices and $\mathrm{SO}(n)$ denotes the $n \times n$ special orthogonal matrices. The above notation and terminology arises in the setting of nonlinear hyperelasticity, where $E(\boldsymbol{u})$ represents the total energy stored by an elastic body (occupying the region Ω in its reference configuration) when subject to a deformation $\boldsymbol{u}$.

For simplicity of exposition we consider the displacement boundary-value problem in which the maps $\boldsymbol{u}$ satisfy the boundary condition

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x} \quad \text{for all } \boldsymbol{x} \in \partial\Omega. \tag{1.3}$$

1.1. Necessary conditions for a minimizer

The usual Euler–Lagrange equations satisfied by a minimizer $\boldsymbol{u}$ of E are given by

$$\operatorname{div}\left[\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}}(\nabla\boldsymbol{u}(\boldsymbol{x}))\right]=\mathbf{0}, \tag{1.4}$$

or, in component form, by

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla\boldsymbol{u}(\boldsymbol{x}))\right]=0, \quad i=1,2,\ldots,n.$$

These equations are formally obtained by taking variations of the form

$$\boldsymbol{u}_\varepsilon=\boldsymbol{u}+\varepsilon\boldsymbol{v}, \quad \varepsilon\in\mathbb{R},\ \boldsymbol{v}\in C^1_0(\Omega;\mathbb{R}^n) \tag{1.5}$$

and setting

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}E(\boldsymbol{u}_\varepsilon)\bigg|_{\varepsilon=0}=0. \tag{1.6}$$

Other necessary conditions are obtained by taking *inner variations*, i.e. variations of the form

$$\boldsymbol{u}_\varepsilon(\boldsymbol{x})=\boldsymbol{u}(\boldsymbol{x}+\varepsilon\boldsymbol{v}(\boldsymbol{x})), \quad \boldsymbol{v}\in C^1_0(\Omega;\mathbb{R}^n),\ \varepsilon\in\mathbb{R}. \tag{1.7}$$

It was demonstrated by Ball [3] that, using variations of the form (1.7), the necessary condition (1.6) gives rise to equilibrium equations of the form

$$\operatorname{div}\boldsymbol{M}(\nabla\boldsymbol{u})=\mathbf{0}, \tag{1.8}$$

where

$$\boldsymbol{M}(\nabla\boldsymbol{u}):=W(\nabla\boldsymbol{u})\boldsymbol{I}-(\nabla\boldsymbol{u})^{\mathrm{T}}\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}}(\nabla\boldsymbol{u}), \tag{1.9}$$

or, in components, by

$$\frac{\partial}{\partial x^\alpha}\left[W(\nabla\boldsymbol{u}(\boldsymbol{x}))\delta^\beta_\alpha-\frac{\partial u^k}{\partial x^\beta}\frac{\partial W}{\partial F^k_\alpha}(\nabla\boldsymbol{u}(\boldsymbol{x}))\right]=0, \quad \beta=1,2,\ldots,n.$$

The above equations are sometimes referred to as *energy–momentum equations* and are a generalization of the Dubois–Reymond necessary condition in the one-dimensional calculus of variations. We highlight the differences between weak solutions of the two systems of equations (1.4) and (1.8). These differences arise, in part, from the following observation: if $\boldsymbol{u}$ is C^2, then expanding (1.8) using (1.9) yields

$$\operatorname{div}\boldsymbol{M}(\nabla\boldsymbol{u})=-(\nabla\boldsymbol{u})^{\mathrm{T}}\operatorname{div}\left(\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}}(\nabla\boldsymbol{u})\right)=\mathbf{0}. \tag{1.10}$$

So, smooth solutions of (1.4) give rise to solutions of (1.8). Conversely, if $\det\nabla\boldsymbol{u}\neq 0$, then $\nabla\boldsymbol{u}$ is invertible and thus solutions of (1.8) give rise to solutions of (1.4). However, if $\nabla\boldsymbol{u}$ is not invertible (and hence has eigenvalue 0) at some point, then there is the possibility that (1.10) may be satisfied at that point even if

$$\operatorname{div}\left(\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}}(\nabla\boldsymbol{u})\right)\neq\mathbf{0},$$

provided that this vector lies in the corresponding eigenspace.

We note that, for the examples given in this paper, W does not satisfy the growth condition

$$W(\boldsymbol{F}) \to \infty \quad \text{as } \det \boldsymbol{F} \to 0^+, \tag{1.11}$$

which is typically imposed on stored-energy functions in nonlinear elasticity. This condition is used to show that any deformation $\boldsymbol{u}$ with finite energy automatically satisfies the local invertibility condition

$$\det \nabla \boldsymbol{u} > 0 \quad \text{almost everywhere.}$$

However, whether or not W satisfies (1.11), one cannot in general show that a minimizer of the elastic energy E satisfies the usual Euler–Lagrange equations (1.4). The difficulty (cf. [3,4]) is that variations of the form (1.5) may have $\det \nabla \boldsymbol{u}_\varepsilon < 0$ on a set of positive measure. In our case, the energy of such a variation is undefined, while under the usual growth condition (1.11) the energy of such a variation is infinite.

We first demonstrate that if W satisfies (1.2) and $\Omega = B$ the unit ball in $\mathbb{R}^n$, then, under suitable growth conditions,

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) := \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \tag{1.12}$$

is always a weak solution of (1.8) in the sense that

$$0 = \int_B \nabla \boldsymbol{w} \colon \boldsymbol{M}(\nabla \tilde{\boldsymbol{u}}) \, \mathrm{d}\boldsymbol{x} \quad \text{for all } \boldsymbol{w} \in C_0^1(B; \mathbb{R}^n). \tag{1.13}$$

However, (1.12) is not, in general, a solution of (1.4) (see remark 2.9). (Note also that the homogeneous map $\boldsymbol{u}_{\mathrm{h}}(\boldsymbol{x}) \equiv \boldsymbol{x}$ is always a smooth solution of (1.4) and (1.8) and satisfies the same boundary condition as $\tilde{\boldsymbol{u}}$.)

We then apply the scaling construction given in [28] and use $\tilde{\boldsymbol{u}}$ to construct infinitely many weak solutions of the energy–momentum equations, each having a countably infinite number of discontinuities and satisfying the same boundary condition as $\tilde{\boldsymbol{u}}$. We also note that our approach applies to strictly convex integrands W, e.g. $W(\boldsymbol{F}) = |\boldsymbol{F}|^2$ [29].

It is known [1, 17] that if W is strongly elliptic and $\boldsymbol{u}$ is a piecewise C^1 weak solution of (1.4) for which $\nabla \boldsymbol{u}$ jumps across a smooth surface $\Gamma \subset \Omega$, then the jump in $\nabla \boldsymbol{u}$ across Γ must be zero. In § 3 we show that the corresponding result is false for weak solutions of (1.8); we give an example of a piecewise C^1 weak solution of the energy–momentum equations which is not C^2 across a smooth surface. We do this using an example of degenerate cavitation studied in [25] and show that the discontinuous minimizer obtained in [25] is a weak solution of the energy–momentum equations which, apart from being discontinuous at the origin of B, is piecewise C^1 but not C^2 on $B \backslash \{\mathbf{0}\}$. This should be contrasted with the results of [5] which show that, for a class of polyconvex stored-energy functions, any $C^{1,\beta}$ weak solution of the energy–momentum equations is automatically a smooth solution of the Euler–Lagrange equations.

The results of this paper should also be taken in conjunction with known regularity results[1] (see, for example, [7, 9, 12–14, 18]) for the system (1.4), which show

[1]These results are obtained for a class of integrands which are incompatible with (1.11).

that if W is uniformly quasiconvex, then minimizers of (1.1) are $C^{1,\alpha}$ off a closed set of n-dimensional measure 0. See also the interesting results of [21] and also [30] which give a $W^{1,\infty}$ weak solution of (1.4) that is nowhere C^1. In the context of nonlinear elasticity, it is known that energy minimizers may develop discontinuities corresponding to cavitation (see, for example, [2, 15, 20, 24, 27]). These cavitation solutions are weak solutions of system (1.8) (see, for example, [2, 28]). In [28], the present authors give an explicit construction of infinitely many singular weak solutions for systems (1.4) and (1.8) using radial cavitation solutions as building blocks. In [16], Knops and Stuart prove the uniqueness result (under the assumption of strict quasiconvexity) that the only C^2 solutions of (1.4) satisfying homogeneous boundary conditions are the homogeneous maps (see also [25] for an alternative proof). Taheri [31] shows that the same conclusion holds for C^1 weak solutions of (1.8). See also the examples [10,11] of singular minimizers (exhibiting the Lavrentiev phenomenon) for two-dimensional problems of elasticity.

2. $\boldsymbol{x}/|\boldsymbol{x}|$ satisfies the energy–momentum equations

In this section we show that the mapping

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) := \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$$

is a weak solution of the energy–momentum equations whenever $\tilde{\boldsymbol{u}}$ has finite energy. We start by gathering some basic results. The first is well known to workers in nonlinear elasticity (see, for example, [8]), while the second can be found, for example, in [2].

Lemma 2.1. *Let W be frame indifferent and isotropic. Then there exists a symmetric function Φ such that*

$$W(\boldsymbol{F}) = \Phi(\nu_1, \nu_2, \ldots, \nu_n), \tag{2.1}$$

where $\nu_1, \nu_2, \ldots, \nu_n$ are the eigenvalues of $\sqrt{\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}}$, which are known in elasticity terminology as the principal stretches.

Lemma 2.2. *Let $1 \leqslant p < n$. Then $\tilde{\boldsymbol{u}} \in W^{1,p}(B;\mathbb{R}^n)$ with weak derivative*

$$\nabla\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}|}\left(\boldsymbol{I} - \frac{\boldsymbol{x}\otimes\boldsymbol{x}}{|\boldsymbol{x}|^2}\right). \tag{2.2}$$

Moreover, for $\boldsymbol{x} \neq \boldsymbol{0}$ the principal stretches of $\tilde{\boldsymbol{u}}$ are given by

$$\nu_1 = 0,\ \nu_2 = \frac{1}{|\boldsymbol{x}|},\ \nu_3 = \frac{1}{|\boldsymbol{x}|},\ \ldots,\ \nu_n = \frac{1}{|\boldsymbol{x}|}. \tag{2.3}$$

Lemma 2.3. *For $\boldsymbol{x} \neq \boldsymbol{0}$,*

$$\operatorname{div}\left(\frac{\mathrm{dW}}{\mathrm{d}\boldsymbol{F}}(\nabla\tilde{\boldsymbol{u}})\right) = \left(\frac{\mathrm{d}}{\mathrm{d}R}\Phi_{,1} + \frac{n-1}{R}[\Phi_{,1} - \Phi_{,2}]\right)\frac{\boldsymbol{x}}{|\boldsymbol{x}|},$$

where $\Phi_{,i}$ denotes the partial derivative of Φ with respect to ν_i and evaluated at (2.3).

Proof. First note that

$$\frac{\partial W}{\partial F^i_\alpha}(\nabla\tilde{\boldsymbol{u}}) = \Phi_{,1}\,\frac{x^i x^\alpha}{R^2} + \Phi_{,2}\left[\delta^i_\alpha - \frac{x^i x^\alpha}{R^2}\right]. \tag{2.4}$$

From this, an easy calculation yields

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial W}{\partial F^i_\alpha}(\nabla\tilde{\boldsymbol{u}})\right) = \left(\frac{\mathrm{d}}{\mathrm{d}R}\Phi_{,1} + \frac{n-1}{R}[\Phi_{,1} - \Phi_{,2}]\right)\frac{x^i}{R}, \tag{2.5}$$

where the arguments of Φ and its derivatives are given by (2.3). □

LEMMA 2.4. *Suppose that $\boldsymbol{M}(\nabla\tilde{\boldsymbol{u}}) \in L^1(B; M^{n\times n})$ and that*

$$\lim_{\varepsilon\to 0}\varepsilon^n\Phi\left(0, \frac{1}{\varepsilon}, \frac{1}{\varepsilon}, \dots, \frac{1}{\varepsilon}\right) = 0. \tag{2.6}$$

Then $\tilde{\boldsymbol{u}}$ is a weak solution of the energy–momentum equations (i.e. it satisfies (1.13)).

Proof. First note that, by (1.10), (2.2) and lemma 2.3,

$$\operatorname{div}\boldsymbol{M}(\nabla\tilde{\boldsymbol{u}}) = \boldsymbol{0} \quad \text{in } B\setminus B_\varepsilon.$$

Thus, for any $\boldsymbol{w} \in C^1_0(B;\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{B\setminus B_\varepsilon}\nabla\boldsymbol{w} : \boldsymbol{M}(\nabla\tilde{\boldsymbol{u}})\,\mathrm{d}\boldsymbol{x} &= \int_{B\setminus B_\varepsilon}\operatorname{div}(\boldsymbol{M}(\nabla\tilde{\boldsymbol{u}})\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\partial B_\varepsilon}\boldsymbol{w}\cdot\boldsymbol{M}(\nabla\tilde{\boldsymbol{u}})\boldsymbol{n}\,\mathrm{d}S\\ &= \Phi\left(0, \frac{1}{\varepsilon}, \dots, \frac{1}{\varepsilon}\right)\int_{\partial B_\varepsilon}\boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}S\\ &= \Phi\left(0, \frac{1}{\varepsilon}, \dots, \frac{1}{\varepsilon}\right)\int_{B_\varepsilon}\operatorname{div}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{2.7}$$

where we have made use of (1.9) and (2.1)–(2.4) in obtaining the penultimate equality. The desired result then follows from (2.6), (2.7) and the dominated convergence theorem applied to a sequence $\varepsilon_j \to 0$. □

THEOREM 2.5. *Let the stored-energy function W be isotropic and frame indifferent. Suppose that the symmetric function Φ given in lemma 2.1 satisfies (2.6). Assume either that W satisfies*

$$\left|\boldsymbol{A}^{\mathrm{T}}\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}}(\boldsymbol{A})\right| \leqslant K[1 + W(\boldsymbol{A})], \quad W(\boldsymbol{A}) \leqslant K[1 + |\boldsymbol{A}|^p] + h(\det\boldsymbol{A}) \tag{2.8}$$

for all $\boldsymbol{A} \in M^{n\times n}$ with $\det\boldsymbol{A} \geqslant 0$ or, equivalently, that Φ satisfies

$$|\boldsymbol{z}\cdot\nabla_{\boldsymbol{z}}\Phi(\boldsymbol{z})| \leqslant K[1 + \Phi(\boldsymbol{z})], \qquad \Phi(\boldsymbol{z}) \leqslant K[1 + |\boldsymbol{z}|^p] + h\left(\prod_{i=1}^{n} z^i\right)$$

for all $\boldsymbol{z} \in \mathbb{R}^n$ *with* $z^i \geqslant 0$, $i = 1, 2, \ldots, n$. *Here* $h \in C([0,\infty);[0,\infty))$, $K > 0$ *and* $1 \leqslant p < n$. *Then* $\tilde{\boldsymbol{u}}$ *is a weak solution of the energy–momentum equations.*

Remark 2.6. Condition $(2.8)_1$ has been used in [3, 4] to show that, for stored-energy functions that satisfy (1.11), every energy minimizer must satisfy the energy–momentum equations. In [4] Ball also shows that $(2.8)_1$ implies that

$$W(\boldsymbol{A}) \leqslant K(|\boldsymbol{A}|^s + |\boldsymbol{A}^{-1}|^s)$$

for some $s > 0$ and $K > 0$.

Proof of theorem 2.5. We first note that $(2.8)_2$ and the hypothesis that $h(0)$ is finite imply that $\tilde{\boldsymbol{u}}$ has finite energy. Hypothesis $(2.8)_1$ then yields $\boldsymbol{M}(\nabla\tilde{\boldsymbol{u}}) \in L^1(B; M^{n\times n})$. The desired result now follows from the previous lemma. □

Example 2.7. In particular, if $\mu > 0$, $1 \leqslant p < n$ and $h \in C([0,\infty);[0,\infty)) \cap C^2((0,\infty))$ satisfies $th'(t) \leqslant C[1 + h(t)]$ for all $t \in [0,\infty)$, then the Ogden [22, 23] materials

$$\Phi(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{\mu}{p}\sum_{i=1}^{n} \lambda_i^p + h\biggl(\prod_{i=1}^{n} \lambda_i\biggr)$$

satisfy the hypotheses of the previous theorem. Moreover, if $h'(1) + \mu = 0$, then the reference configuration, $\boldsymbol{F} = \boldsymbol{I}$, is stress free and if, in addition, $t \mapsto h(t^n)$ is convex, then $\boldsymbol{u}_{\mathrm{h}}(\boldsymbol{x}) \equiv \boldsymbol{x}$ is the unique global minimizer of the energy that satisfies the boundary condition (1.3). Moreover, if $h'(t) \to -\infty$ as $t \to 0$ then, although it does not require infinite energy to compress the material to zero volume, such compressions will generate infinite stresses.

Corollary 2.8. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. Suppose that the stored-energy function* W *satisfies the hypotheses of theorem 2.5. Then, as a consequence of theorem 2.5, one can construct infinitely many weak solutions of the energy–momentum equations that satisfy (1.3). Each solution in the construction has a countable number of singularities and is contained in* $W^{1,p}(\Omega)$ *for all* $1 \leqslant p < n$, *but is not contained in* $W^{1,n}(\Omega)$ *or* $C(\Omega)$.

Proof. Let $\Omega \approx \bigcup_{i=1}^{\infty} \bar{B}(\boldsymbol{x}_i, \delta_i)$ up to a set of n-dimensional Lebesgue measure zero, where $\bar{B}(\boldsymbol{x}_i, \delta_i) \subset \Omega$ are pairwise-disjoint, closed balls of radius $\delta_i > 0$ centred at $\boldsymbol{x}_i$. (Many such decompositions can be obtained by the Vitali covering theorem.) Then, as shown in [28],

$$\hat{\boldsymbol{u}}(\boldsymbol{x}) := \begin{cases} \delta_i \dfrac{\boldsymbol{x} - \boldsymbol{x}_i}{|\boldsymbol{x} - \boldsymbol{x}_i|} + \boldsymbol{x}_i & \text{if } \boldsymbol{x} \in \bar{B}(\boldsymbol{x}_i, \delta_i), \\ \boldsymbol{x} & \text{otherwise} \end{cases}$$

has the required properties. □

Remark 2.9. Straightforward examples show that, in general, $\tilde{\boldsymbol{u}}$ is not a solution of the Euler–Lagrange equations (1.4). For example, take $W(\boldsymbol{F}) = |\boldsymbol{F}|^2$. Then

$$\Phi = \sum_{i=1}^{n} (\nu_i)^2$$

and an easy calculation shows that (2.5) takes the form

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial W}{\partial F^i_\alpha}(\nabla\tilde{\boldsymbol{u}})\right) = \left(\frac{n-1}{R}[-\Phi_{,2}]\right)\frac{x^i}{R} = -2(n-1)\frac{x^i}{R^3} \neq 0.$$

REMARK 2.10. It is interesting to note the connection between the Lagrange multiplier equations and the energy–momentum equations (1.8). Consider the constrained variational problem in which we minimize the energy (1.1) subject to the pointwise constraint $G(\boldsymbol{u}) = 0$ almost everywhere. (For example, in some studies of liquid crystals (see, for example, [6, 19]),

$$E(\boldsymbol{u}) = \int_B |\nabla\boldsymbol{u}|^2$$

and the vector fields $\boldsymbol{u}$ are constrained to be unit vectors so that $G(\boldsymbol{u}) = |\boldsymbol{u}|^2 - 1 = 0$ almost everywhere.) The Lagrange multiplier equations for this problem are formally given by

$$\operatorname{div}\left(\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}}(\nabla\boldsymbol{u})\right) + \lambda(\boldsymbol{x})\nabla_{\boldsymbol{u}}G(\boldsymbol{u}) = \boldsymbol{0}, \quad G(\boldsymbol{u}) = 0, \tag{2.9}$$

where $\lambda(\boldsymbol{x})$ is a Lagrange multiplier corresponding to the constraint. Note that, in this case, on differentiating the constraint, we obtain

$$(\nabla\boldsymbol{u})^{\mathrm{T}}\nabla_{\boldsymbol{u}}G(\boldsymbol{u}) = \boldsymbol{0}, \tag{2.10}$$

and hence, on multiplying $(2.9)_1$ by $(\nabla\boldsymbol{u})^{\mathrm{T}}$, we obtain

$$(\nabla\boldsymbol{u})^{\mathrm{T}}\operatorname{div}\left(\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}}(\nabla\boldsymbol{u})\right) = \boldsymbol{0}.$$

Thus, by (1.10), it follows that any (smooth) solution of the Lagrange multiplier equations is also a solution of the energy–momentum equations. This connection between solutions is also apparent from the fact that the inner variations (1.7) which are used to derive the energy–momentum equations also preserve the constraint $G(\boldsymbol{u}) = 0$ almost everywhere.

Conversely, if $\boldsymbol{u}$ is a smooth solution of the energy–momentum equations which also satisfies the constraint $G(\boldsymbol{u}) = 0$ and if $\nabla_{\boldsymbol{u}}G(\boldsymbol{u}) \neq \boldsymbol{0}$ and $\nabla\boldsymbol{u}$ has rank $n-1$, then a similar argument shows that there exists a $\lambda(\boldsymbol{x})$ such that the Lagrange multiplier equations hold. Of course, if any of these assumptions do not hold, then the reverse implication fails. (Note in particular that, for the liquid crystal problem mentioned above, these assumptions do hold.)

3. C^1 solutions of the energy–momentum equations need not be C^2

In this section we show that a continuously differentiable weak solution of the energy–momentum equations need not be C^2.

Let $B \subset \mathbb{R}^3$ denote the unit ball in $\mathbb{R}^3$. A study of degenerate cavitation [25] considers the model problem of minimizing a functional of the form[2]

$$E(\boldsymbol{u}) = \int_B [\tfrac{1}{2}|\nabla \boldsymbol{u}|^2 + \tfrac{4}{3}\kappa \det \nabla \boldsymbol{u}]\,\mathrm{d}\boldsymbol{x} \tag{3.1}$$

on the admissible set of deformations

$$\mathcal{A} = \{\boldsymbol{u} \in W^{1,2}(B;\mathbb{R}^n)\colon\ \det \nabla \boldsymbol{u} \geqslant 0 \text{ a.e.},\ \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x} \text{ for all } \boldsymbol{x} \in \partial B\},$$

where $\kappa \geqslant 0$ is a scalar. In the class of radial deformations of the form

$$\boldsymbol{u}(\boldsymbol{x}) = r(R)\frac{\boldsymbol{x}}{R}, \quad R = |\boldsymbol{x}|, \tag{3.2}$$

it follows from theorem 1.3 of [25] that

(i) if $\kappa \in [0,1]$, then $r(R) \equiv R$ is the radial minimizer,

(ii) if $\kappa \in (1, \frac{3}{2})$, then the radial minimizer is given by

$$r(R) = \begin{cases} \kappa^{-1}R + \dfrac{\kappa - 1}{\kappa R^2} & \text{if } R \in [R_0, 1], \\ \frac{3}{2}\kappa^{-1}R_0 & \text{if } R \in [0, R_0], \end{cases}$$

where

$$R_0 = [2(\kappa - 1)]^{1/3},$$

(iii) if $\kappa \geqslant \frac{3}{2}$, then $r(R) \equiv 1$ is the radial minimizer.

Therefore, for $\kappa > 1$, the above maps produce a hole at the centre of the deformed ball. (Note that these deformations are degenerate in the sense that $\det \nabla \boldsymbol{u} = 0$ on a set of non-zero measure.)

Formally, the Euler–Lagrange equations for (3.1) are given by $\Delta \boldsymbol{u} = \boldsymbol{0}$ since $\det \nabla \boldsymbol{u}$ is a null Lagrangian. However, these equations are not satisfied throughout B by the above radial minimizers; in particular, they do not hold on B_{R_0} (in case (ii)) and on B (in case (iii)), where B_{R_0} denotes the open ball of radius R_0 centred at the origin. Despite this, we demonstrate next that these radial minimizers do satisfy the weak form of the energy–momentum equations on B. We consider case (ii) only (as case (iii) has already been discussed).

An easy calculation shows that, for the stored-energy function in (3.1), the corresponding energy–momentum tensor is given by

$$\boldsymbol{M}(\nabla \boldsymbol{u}) = |\nabla \boldsymbol{u}|^2 \boldsymbol{I} - 2(\nabla \boldsymbol{u})^{\mathrm{T}} \nabla \boldsymbol{u}. \tag{3.3}$$

We next observe that the radial maps given in (ii) are C^1 on $\bar{B} \setminus \{\boldsymbol{0}\}$ and are C^2 in each of $B \setminus \overline{B}_{R_0}$ and $B_{R_0} \setminus \{\boldsymbol{0}\}$. To prove our claim we will first demonstrate that the energy–momentum equations (1.8) hold in each of $B \setminus \overline{B}_{R_0}$ and $B_{R_0} \setminus \{\boldsymbol{0}\}$. To see this, observe that, by (1.10), for C^2 maps, these equations are given by

$$\operatorname{div} \boldsymbol{M}(\nabla \boldsymbol{u}) = -2(\nabla \boldsymbol{u})^{\mathrm{T}} \Delta \boldsymbol{u} = \boldsymbol{0}. \tag{3.4}$$

[2]Results on degenerate cavitation in the radial case for more general energy functions are contained in [26].

Now note that, in case (ii), the radial map satisfies $\Delta \boldsymbol{u} = \boldsymbol{0}$ on $B \setminus \bar{B}_{R_0}$, and hence the above equations hold in this region. Next observe that in B_{R_0}, the radial minimizer is of the form

$$\boldsymbol{u} = c\frac{\boldsymbol{x}}{|\boldsymbol{x}|},$$

where c is a constant. Thus, by lemma 2.2,

$$\nabla \boldsymbol{u} = \frac{c}{|\boldsymbol{x}|}\left[\boldsymbol{I} - \frac{\boldsymbol{x} \otimes \boldsymbol{x}}{|\boldsymbol{x}|^2}\right], \tag{3.5}$$

and an easy calculation yields

$$\Delta \boldsymbol{u} = -2c\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}$$

and so (3.4) also holds in $B_{R_0} \setminus \{\boldsymbol{0}\}$. Thus, for $\boldsymbol{v} \in C_0^1(B; \mathbb{R}^3)$,

$$\begin{aligned}\int_B \nabla \boldsymbol{v} : \boldsymbol{M} \, \mathrm{d}\boldsymbol{x} &= \int_{B\setminus B_{R_0}} \operatorname{div}(\boldsymbol{M}\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} + \lim_{\varepsilon \to 0} \int_{B_{R_0}\setminus B_\varepsilon} \operatorname{div}(\boldsymbol{M}\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \\ &= \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} \boldsymbol{v} \cdot \boldsymbol{M}\boldsymbol{n} \, \mathrm{d}S,\end{aligned}$$

where we have used (1.8), the divergence theorem and the fact that $\boldsymbol{u}$ is C^1 across ∂B_{R_0}. Next, using (3.3) and (3.5), it follows that the above expression is equal to

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} \boldsymbol{v} \cdot \left(-\frac{c^2}{|\boldsymbol{x}|^2}\right)\boldsymbol{n} \, \mathrm{d}S &= -c^2 \lim_{\varepsilon \to 0}\left(\frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon} \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S\right) \\ &= -c^2 \lim_{\varepsilon \to 0}\left(\frac{1}{\varepsilon^2}\int_{B_\varepsilon} \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}\right) \\ &= 0,\end{aligned}$$

and hence the radial minimizers given by (3.2) and (ii) satisfy the weak form of the energy–momentum equations. Note, however, that this radial solution is not C^2 across ∂B_{R_0} since $r''(R)$ is discontinuous at $R = R_0$.

Remark 3.1. In the general radial-cavitation problem studied in [2], the energy functional (1.1) is minimized in a class of deformations of the unit ball B of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{r(R)}{R}\boldsymbol{x}, \quad R = |\boldsymbol{x}|,$$

and it is shown that there exist minimizers satisfying $r(0) > 0$ corresponding to a cavity forming at the centre of the deformed ball (see [15, 20, 27] for generalizations to non-symmetric problems). It is interesting to note that in these problems,

$$\nabla \boldsymbol{u}(\boldsymbol{x}) = r'(R)\frac{\boldsymbol{x} \otimes \boldsymbol{x}}{R^2} + \frac{r(R)}{R}\left(\boldsymbol{I} - \frac{\boldsymbol{x} \otimes \boldsymbol{x}}{R^2}\right)$$

and (by calculations analogous to those in lemma 2.3)

$$\operatorname{div}\left(\frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{F}}(\nabla \boldsymbol{u})\right) = f(R)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}.$$

There may be situations in which $f(R) \neq 0$ at some points but with $r'(R) = 0$ at these points. This would then generate a solution of the energy–momentum equations (1.8), (1.9) which is not a solution of the equilibrium equations (1.4). (Note also that the radial cavitation equilibria studied in [2] satisfy $r'(R) \to 0$ as $R \to 0$ and that $f(0)$ is typically undefined.)

REMARK 3.2. The example of degenerate cavitation has been extended in [32] to the case of integral functionals of the form:

$$E(\boldsymbol{u}) = \int_B [|\nabla \boldsymbol{u}|^p + \alpha \det \nabla \boldsymbol{u}] \, \mathrm{d}\boldsymbol{x},$$

where $p \in [2, 3)$, $\alpha > 0$ and B is the unit ball centred at the origin in $\mathbb{R}^3$. In this case it is shown in [32] that an exactly analogous phenomenon of degenerate cavitation occurs.

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