SIAM J. APPL. MATH.
Vol. 66, No. 3, pp. 736–757

THE CONVERGENCE OF REGULARIZED MINIMIZERS FOR CAVITATION PROBLEMS IN NONLINEAR ELASTICITY*

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Abstract. Consider a nonlinearly elastic body which occupies the region $\Omega \subset \mathbb{R}^m$ $(m = 2, 3)$ in its reference state and which is held in tension under prescribed boundary displacements on $\partial\Omega$. Let $\mathbf{x}_0 \in \Omega$ be any fixed point in the body. It is known from variational arguments that, for sufficiently large boundary displacements, there may exist discontinuous weak solutions of the equilibrium equations corresponding to a hole forming at $\mathbf{x}_0$ in the deformed body (this is the phenomenon of cavitation). For each $\epsilon > 0$, define the regularized domains $\Omega_\epsilon = \Omega\backslash\overline{B_\epsilon(\mathbf{x}_0)}$ which contain a preexisting hole of radius $\epsilon > 0$ centered on $\mathbf{x}_0$. Now consider the corresponding mixed displacement/traction problem on Ω_ϵ in which the boundary $\partial\Omega$ is subject to the same boundary displacements and the deformed cavity surface (i.e., the image of ∂B_ϵ) is required to be stress-free. It follows from variational arguments that there exists a weak solution $\mathbf{u}_\epsilon$ of this problem for each $\epsilon > 0$. In this paper we prove convergence of these regularized minimizers $\mathbf{u}_\epsilon$ in the limit as $\epsilon \to 0$. In particular, we show that if $\epsilon_n \to 0$, then, passing to a subsequence, $\mathbf{u}_{\epsilon_n} \to \mathbf{u}$, where $\mathbf{u}$ is a minimizer for the original pure displacement problem on Ω.

Finally, we study the effect on cavitation of regularizing the variational problem by introducing a surface energy term which penalizes the formation and growth of cavities.

Key words. cavitation, convergence, elastic, equilibrium, regular minimizers, singular minimizers, surface energy, weak limit

AMS subject classifications. Primary, 74B20; Secondary, 49K20, 74G65

DOI. 10.1137/040618965

1. Introduction.

Let $\Omega \subset \mathbb{R}^m$ $(m = 2, 3)$ denote the region occupied by a nonlinearly elastic body in its reference configuration. A deformation of the body corresponds to a map $\mathbf{u} : \Omega \to \mathbb{R}^m$ that lies in the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^m)$, is one-to-one a.e., and satisfies

$$\det \nabla\mathbf{u}(\mathbf{x}) > 0 \quad \text{for a.e. } \mathbf{x} \in \Omega. \tag{1.1}$$

In hyperelasticity the total energy stored under such a deformation is given by

$$E(\mathbf{u}) = \int_\Omega W(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.2}$$

where $W : \overline{\Omega} \times \mathrm{M}_+^{m\times m} \to [0, \infty)$ is the stored energy function of the material and $\mathrm{M}_+^{m\times m}$ denotes the set of real $m \times m$ matrices with positive determinant. We consider

*Received by the editors November 15, 2004; accepted for publication (in revised form) September 8, 2005; published electronically February 3, 2006. This work was supported in part by the National Science Foundation under grants 0072414 and 0405646.

http://www.siam.org/journals/siap/66-3/61896.html

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the displacement problem in which we require

$$\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \text{for } \mathbf{x} \in \partial\Omega, \tag{1.3}$$

where $\mathbf{A} \in \mathrm{M}_+^{m\times m}$ is fixed.

The radial problem. Following Ball's seminal paper [1], much work has been carried out in the variational setting on the existence of discontinuous radial minimizers for (1.2) corresponding to cavitation (see, e.g., the review article [12] and the references therein). In this approach $\mathbf{A} = \lambda\mathbf{I}$, $\lambda > 0$, $\Omega = B$ is the unit ball in $\mathbb{R}^m$, and deformations are of the form

$$\mathbf{u}(\mathbf{x}) = r(R)\frac{\mathbf{x}}{|\mathbf{x}|},$$

where $r : [0,1] \to [0,\infty)$ and $R = |\mathbf{x}|$. It is known that there are large classes of physically reasonable W for which there exists a critical value of the boundary displacement λ_{crit} such that

(i) for $\lambda \le \lambda_{\text{crit}}$, the unique radial energy minimizer is the homogeneous deformation corresponding to $r(R) \equiv \lambda R$;

(ii) for $\lambda > \lambda_{\text{crit}}$, the unique radial energy minimizer satisfies $r(0) > 0$, corresponding to a deformation that produces a hole at the center of the initially perfect ball (this is the phenomenon of cavitation).

Example 1.1. A typical class of stored energy functions for which the above results hold is given by

$$W(\mathbf{x},\mathbf{F}) = c|\mathbf{F}|^p + \Gamma(\det\mathbf{F}) \quad \forall\, \mathbf{F} \in \mathrm{M}_+^{m\times m} \text{ and } \mathbf{x} \in \overline{\Omega}, \tag{1.4}$$

where $c > 0$, $1 \le p < m$, and Γ is a convex function that grows superlinearly and satisfies $\Gamma(d) \to +\infty$ as $d \to 0^+$.

The discontinuous minimizers with r as in (ii) are weak solutions of the corresponding equilibrium equations (see, e.g., [1]). Earlier approaches to cavitation had not been variational and tended to model cavitation as the growth of small preexisting voids in the material (see, e.g., [9] in the context of nonlinear elasticity and [10] in the context of elastoplasticity).

There are various ways of reconciling the two approaches. In particular, results of [18] in the variational setting (see also the example in [11]) show that radial equilibrium solutions

$$\mathbf{u}_\epsilon(\mathbf{x}) = r_\epsilon(R)\frac{\mathbf{x}}{|\mathbf{x}|}$$

for the mixed displacement/zero-traction problem in which B is replaced by $B_\epsilon = \{\mathbf{x} : \epsilon < |\mathbf{x}| < 1\}$ (i.e., a ball with a preexisting hole of radius ϵ in the reference configuration) converge to the radial minimizer for B studied in [1] as $\epsilon \to 0$. In particular, $\sup_{R\in[\epsilon,1]} |r_\epsilon(R) - r(R)| \to 0$ as $\epsilon \to 0$, where $r(R)$ is the minimizer given in (i) and (ii) above (see Figure 1 and also the discussion in [1] for the case of incompressible elasticity).

The nonsymmetric case. The first results extending Ball's original variational approach to nonsymmetric situations while allowing cavitation (i.e., discontinuities) to occur at any point in the body are contained in Müller and Spector [16]. A key element in their approach is an analytical restriction on the class of admissible

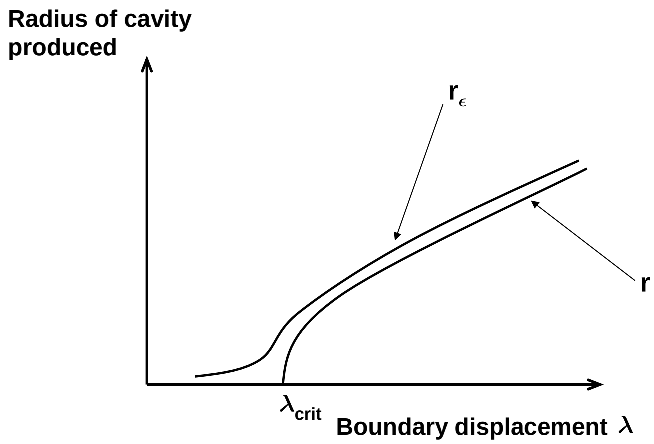

FIG. 1. *Bifurcation diagram for solid and punctured balls.*

deformations $\mathbf{u}$ called condition (INV). Condition (INV) is the requirement that $\mathbf{u}$ be monotone[1] in the sense of Lebesgue and that, roughly speaking, holes created in one part of the body cannot be filled by material from elsewhere (see section 2.3 for the precise definition). Subsequent work in [19], in the variational setting, proposed an alternative model in which cavitation could occur only at a, possibly large, number of infinitesimal flaws in the material. This was modelled mathematically by using admissible deformations whose possible point discontinuities are constrained to be at the specified flaw points: let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in \Omega$ be the flaw points and minimize the total energy (1.2) in the class of deformations that satisfy (1.1), (1.3), condition (INV), and

$$\mathrm{Det}\nabla\mathbf{u} = (\det \nabla\mathbf{u})\mathcal{L}^m + \sum_{i=0}^{n} \alpha_i \delta_{\mathbf{x}_i},$$

where $\mathrm{Det}\nabla\mathbf{u}$ denotes the distributional Jacobian of $\mathbf{u}$, $\mathcal{L}^m$ denotes m-dimensional Lebesgue measure, $\delta_{\mathbf{x}_i}$ is the Dirac measure supported at $\mathbf{x}_i$, and $\alpha_i \geq 0$ is the volume of the hole formed at $\mathbf{x}_i$ by the deformation $\mathbf{u}$ (see section 2.4 for further details). The existence of a minimizer in this class follows from [19], and it is a consequence of a result in [21] that if the matrix $\mathbf{A}$ in the boundary condition (1.3) is "sufficiently large,"[2] then any minimizer $\mathbf{u}$ must satisfy $\alpha_i > 0$ for at least one i.

In analogy with the radial problems outlined earlier, an alternative approach is to regularize by replacing the flaw points with preexisting voids in the reference configuration of maximum radius $\epsilon > 0$. The purpose of this paper is to examine the behavior of energy minimizers for these more regular problems in the limit as $\epsilon \to 0$. In particular we consider the case of one flaw point at $\mathbf{x}_0 \in \Omega$ and study the convergence of minimizers for these regularized problems to the minimizers obtained in [19]. For the convenience of the reader, we next outline the main convergence result in the case of the class of stored energy functions (1.4), deferring precise technical details to later in the paper. However, it should be noted that the results of this paper apply to much more general polyconvex energy functions (see hypotheses (H1)–(H4) in section 3).

[1]Condition (INV) requires, in particular, that $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^m)$, $p > m - 1$.

[2]This is the case, in particular, if $\mathbf{A} = t\mathbf{B}$, where $\mathbf{B} \in \mathrm{M}_+^{m\times m}$ is fixed and $t > 0$ is sufficiently large.

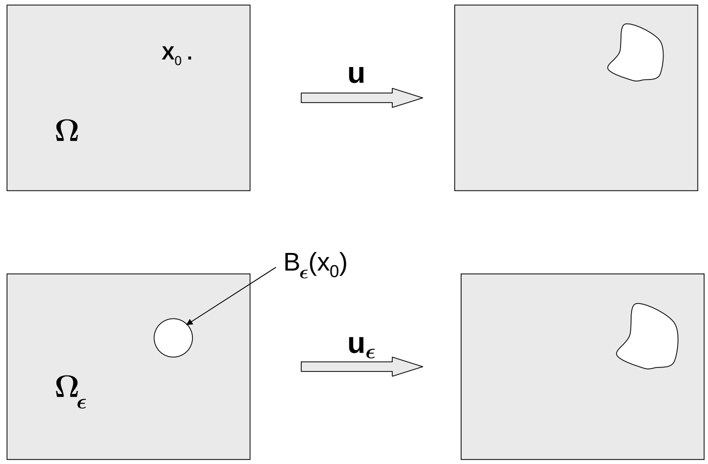

FIG. 2. *Deformations of the original and regularized domains.*

The underlying pure displacement problem. In essence, the underlying problem is to minimize the integral functional E given by (1.2) on a class of deformations[3,4] $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^m)$, $p \in (m-1, m)$, that satisfy (1.1), (1.3), condition (INV) on Ω, and

$$\mathrm{Det}\nabla\mathbf{u} = (\det\nabla\mathbf{u})\mathcal{L}^m + \alpha_0\delta_{\mathbf{x}_0},\ \ \alpha_0 \geq 0.$$

The existence of a minimizer for this problem follows from [19].

The regularized mixed displacement/traction problem. For each $\epsilon > 0$, define the domains $\Omega_\epsilon = \Omega\backslash\overline{B_\epsilon(\mathbf{x}_0)}$ which contain a preexisting hole of radius $\epsilon > 0$ centered on $\mathbf{x}_0$ (see Figure 2). Now consider the mixed displacement/traction problem on Ω_ϵ in which the outer boundary $\partial\Omega$ is subject to the same boundary displacements (1.3) and the deformed cavity surface (i.e., the image of ∂B_ϵ) is required to be stress-free. The corresponding variational problem is to minimize

$$E_\epsilon(\mathbf{u}_\epsilon) = \int_{\Omega_\epsilon} W(\mathbf{x}, \nabla\mathbf{u}_\epsilon(\mathbf{x}))\,d\mathbf{x}$$

on a class of deformations[5] $\mathbf{u}_\epsilon \in W^{1,p}(\Omega_\epsilon, \mathbb{R}^m)$, $p \in (m-1, m)$, that satisfy (1.3), condition (INV) on Ω_ϵ,

$$\det\nabla\mathbf{u}_\epsilon > 0 \text{ a.e. in } \Omega_\epsilon,$$

and

$$\mathrm{Det}\nabla\mathbf{u}_\epsilon = (\det\nabla\mathbf{u}_\epsilon)\mathcal{L}^m.$$

[3]In fact, for technical reasons, we work with the homogeneous extension $\mathbf{u}^e$ of $\mathbf{u}$ which is defined on a slightly larger domain $\Omega^e \supset \Omega$. It is obtained by extending $\mathbf{u}$ by the homogeneous deformation $\mathbf{Ax}$ (see section 2.3 and Remark 3.2).

[4]For interesting results in the borderline case, $p = m - 1$, see Conti and De Lellis [4].

[5]As in the pure displacement problem, we work with homogeneous extensions of these maps.

In Theorem 4.2 we prove convergence of minimizers for these regularized problems in the limit as $\epsilon \to 0$. In particular, we show that if $\epsilon_n \to 0$ and $(\mathbf{u}_{\epsilon_n})$ is a corresponding sequence of minimizers, then, passing to a subsequence if necessary, $\mathbf{u}_{\epsilon_n} \to \mathbf{u}$, where $\mathbf{u}$ is a minimizer for the pure displacement problem on the original domain Ω.

The above results are quite general and do not depend on the shape of the excluded regions used to obtain Ω_ϵ. In particular, the balls $B_\epsilon(\mathbf{x}_0)$ which are removed to produce Ω_ϵ could be replaced by a nested sequence of (nonspherical) regions around $\mathbf{x}_0$ whose diameters converge to zero as $\epsilon \to 0$.

In section 5, we consider the effect of regularizing the pure displacement variational problem by adding a surface energy term to the total energy which penalizes the formation and growth of cavities. In the standard model, in which surface energy is proportional to the new surface area created, cavitation is still energetically favorable for sufficiently severe boundary displacements. However, in this case we show that there is no longer bifurcation from a homogeneous deformation in the sense that there do not exist discontinuous energy minimizers producing cavities of arbitrarily small volume. Our results thus generalize the recent results of Dollhofer et al. [5], who show that the addition of surface energy in radial cavitation of an incompressible neo-Hookean material will induce the sudden formation of a single cavity of finite radius, rather than the gradual opening of a hole from zero volume.

2. Background.

2.1. Notation. Let Ω denote a nonempty, bounded, connected, open subset of $\mathbb{R}^m$ with Lipschitz boundary $\partial\Omega$ (see [6] or [13]). We denote by $L^p(\Omega)$ and $W^{1,p}(\Omega)$ the usual spaces of p-summable and Sobolev functions, respectively. We use the notation $L^p(\Omega;\mathbb{R}^m)$, etc., for vector-valued maps. A function ϕ is in $L^p_{\text{loc}}(\Omega)$ if $\phi \in L^p(U)$ for all open sets $U \subset\subset \Omega$; i.e., $U \subset K_U \subset \Omega$ for some compact set K_U. Weak convergence in these spaces will be indicated by the half arrow $\rightharpoonup$.

We denote m-dimensional Lebesgue measure by $\mathcal{L}^m$ and k-dimensional Hausdorff measure by $\mathcal{H}^k$. We write

$$B(\mathbf{z},\epsilon) := \{\mathbf{x} \in \mathbb{R}^m : |\mathbf{x}-\mathbf{z}| < \epsilon\}$$

for the open ball of radius $\epsilon > 0$ centered at $\mathbf{z} \in \mathbb{R}^m$ (we also use the notation $B_\epsilon(\mathbf{z})$ for $B(\mathbf{z},\epsilon)$).

Let $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^m)$ with $1 \le p < m$. We will be interested in pointwise properties of $\mathbf{u}$ as well as restrictions of $\mathbf{u}$ to lower-dimensional sets. We will not identify maps that are equal a.e. and choose to work with the precise representative $\mathbf{u}^* : \Omega \to \mathbb{R}^m$ defined by

$$\mathbf{u}^*(\mathbf{x}) := \begin{cases} \displaystyle\lim_{\rho\to 0^+} \frac{1}{\mathcal{L}^m(B(\mathbf{x},\rho))} \int_{B(\mathbf{x},\rho)} \mathbf{u}(\mathbf{z})\,d\mathbf{z} & \text{if the limit exists,} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

We shall make use of the fact that if $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^m)$ with $1 \le p < m$, then the above limit exists for every $\mathbf{x} \in \Omega\backslash P$, where $\mathcal{H}^{m-1}(P) = 0$. Thus, in particular, one can use the precise representative as a representative of the trace on $(m-1)$-dimensional surfaces. Moreover, if $p > m-1$, then $\mathcal{H}^1(P) = 0$, and consequently for each $\mathbf{z} \in \Omega$ the above limit is defined at *every* point on $\partial B(\mathbf{z}, r)$ for almost every $r \in (0, r_{\mathbf{z}})$, where $r_{\mathbf{z}} = \operatorname{dist}(\mathbf{z}, \partial\Omega)$. For a thorough discussion of precise representative we refer the reader to [6].

2.2. The topological image. In this section we briefly recall some facts about the Brouwer degree (see, e.g., [7] or [22] for more details). Let $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^m$ be a C^1 map. If $\mathbf{y}_0 \in \mathbb{R}^m \backslash \mathbf{u}(\partial\Omega)$ is such that $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathbf{u}^{-1}(\mathbf{y}_0)$, then the degree is defined by

$$\deg(\mathbf{u}, \Omega, \mathbf{y}_0) := \sum_{\mathbf{x} \in \mathbf{u}^{-1}(\mathbf{y}_0)} \operatorname{sgn}[\det \nabla \mathbf{u}(\mathbf{x})], \tag{2.1}$$

where $\operatorname{sgn}(t) = 1$ for $t > 0$ and $\operatorname{sgn}(t) = -1$ for $t < 0$. In particular, if $\mathbf{g} : \overline{\Omega} \to \mathbb{R}^m$ is a diffeomorphism with $\det \nabla \mathbf{g} > 0$ on Ω, then from (2.1) we conclude that

$$\deg(\mathbf{g}, \Omega, \mathbf{y}_0) = \begin{cases} 1 & \text{if } \mathbf{y}_0 \in \mathbf{g}(\Omega), \\ 0 & \text{if } \mathbf{y}_0 \in \mathbb{R}^m \backslash \mathbf{g}(\overline{\Omega}). \end{cases}$$

If ϕ is a C^∞ function supported in the connected component of $\mathbb{R}^m \backslash \mathbf{u}(\partial\Omega)$ that contains $\mathbf{y}_0$, then one can show that

$$\int_\Omega \phi(\mathbf{u}(\mathbf{x})) \det \nabla \mathbf{u}(\mathbf{x})\, d\mathbf{x} = \deg(\mathbf{u}, \Omega, \mathbf{y}_0) \int_{\mathbb{R}^m} \phi(\mathbf{y})\, d\mathbf{y}.$$

One can now define $\deg(\mathbf{u}, \Omega, \mathbf{y})$ for any continuous function $\mathbf{u} : \Omega \to \mathbb{R}^m$ and any $\mathbf{y} \in \mathbb{R}^m \backslash \mathbf{u}(\partial\Omega)$ by using this formula and approximating by C^∞ functions. Moreover, the degree depends only on $\mathbf{u}|_{\partial\Omega}$. Accordingly we can write $\deg(\mathbf{u}, \partial\Omega, \mathbf{y})$ instead of $\deg(\mathbf{u}, \Omega, \mathbf{y})$.

Definition 2.1. *Let $B(\mathbf{z}, r) \subset \Omega$ and suppose that $\overline{\mathbf{u}} : \partial B(\mathbf{z}, r) \to \mathbb{R}^m$ is continuous. We define the topological image of $B(\mathbf{z}, r)$ under $\overline{\mathbf{u}}$ by*

$$\operatorname{im}_{\mathrm{T}}(\overline{\mathbf{u}}, B(\mathbf{z}, r)) := \{\mathbf{y} \in \mathbb{R}^m \backslash \overline{\mathbf{u}}(\partial B(\mathbf{z}, r)) : \deg(\overline{\mathbf{u}}, \partial B(\mathbf{z}, r), \mathbf{y}) \neq 0\}. \tag{2.2}$$

Remark 2.2. Let $\mathbf{g} : \overline{B(\mathbf{z}, r)} \to \mathbb{R}^m$ be a homeomorphism. If $\overline{\mathbf{u}} : \partial B(\mathbf{z}, r) \to \mathbb{R}^m$ is such that $\overline{\mathbf{u}}(\partial B(\mathbf{z}, r)) = \mathbf{g}(\partial B(\mathbf{z}, r))$, then

$$\operatorname{im}_{\mathrm{T}}(\overline{\mathbf{u}}, B(\mathbf{z}, r)) = \mathbf{g}(B(\mathbf{z}, r)).$$

2.3. Invertibility condition (INV). In nonlinear elasticity one is interested in globally invertible maps since, in general, matter cannot interpenetrate itself. We say that $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^m)$, $p \geq 1$, is *one-to-one* a.e. if there is a Lebesgue null set $N \subset \Omega$ such that $\mathbf{u}|_{\Omega \backslash N}$ is injective. Unfortunately, if $p < m$, the weak limit of a sequence of maps which are one-to-one a.e. need not be one-to-one a.e. (see, e.g., [16, section 11]). A property that is slightly stronger than one-to-one a.e. is therefore needed.

Definition 2.3. *Let $r_{\mathbf{z}} = \operatorname{dist}(\mathbf{z}, \partial\Omega)$. We say that $\mathbf{u} : \Omega \to \mathbb{R}^m$ satisfies invertibility condition* (INV) *on Ω, provided that for every $\mathbf{z} \in \Omega$ there exists an $\mathcal{L}^1$ null set $N_{\mathbf{z}}$ such that, for all $r \in (0, r_{\mathbf{z}}) \backslash N_{\mathbf{z}}$,*

(o) $\mathbf{u}|_{\partial B(\mathbf{z}, r)}$ *is continuous;*

(i) $\mathbf{u}(\mathbf{x}) \in \operatorname{im}_{\mathrm{T}}(\mathbf{u}, B(\mathbf{z}, r)) \cup \mathbf{u}(\partial B(\mathbf{z}, r))$ *for $\mathcal{L}^m$ a.e.* $\mathbf{x} \in \overline{B(\mathbf{z}, r)}$;

(ii) $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^m \backslash \operatorname{im}_{\mathrm{T}}(\mathbf{u}, B(\mathbf{z}, r))$ *for $\mathcal{L}^m$ a.e.* $\mathbf{x} \in \Omega \backslash \overline{B(\mathbf{z}, r)}$.

The next results show that condition (INV) is preserved under weak convergence and that mappings that satisfy condition (INV) and have nonzero Jacobian a.e. are one-to-one a.e.

Proposition 2.4 (see [16, Lemma 3.3]). *Let $p > m-1$ and suppose that $(\mathbf{u}_n^*)$ is a sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$ that satisfies condition* (INV). *Suppose also that*

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \ \text{ in } \ W^{1,p}(\Omega;\mathbb{R}^m).$$

Then $\mathbf{u}^*$ satisfies condition (INV).

Proposition 2.5 (see [16, Lemma 3.4]). *Let $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^m)$ with $p > m-1$. Suppose that $\det \nabla \mathbf{u} \neq 0$ a.e. and that $\mathbf{u}^*$ satisfies condition* (INV). *Then $\mathbf{u}$ is one-to-one a.e.*

Definition 2.6. *We say that the function $\mathbf{u}$ satisfies condition* (INV) *on $\Omega\backslash\{\mathbf{x}_0\}$ if $\mathbf{u}$ satisfies condition* (INV) *on $\Omega\backslash\overline{B(\mathbf{x}_0,\delta)}$ for every sufficiently small $\delta > 0$.*

Homogeneous extensions of maps. Let $\Omega \subset\subset \Omega^{\mathrm{e}}$, where Ω^{e} is a bounded, open, connected set with smooth boundary and suppose that $\mathbf{u}^{\mathrm{h}} : \Omega^{\mathrm{e}} \to \mathbb{R}^m$ is the orientation preserving homogeneous map

$$\mathbf{u}^{\mathrm{h}}(\mathbf{x}) \equiv \mathbf{A}\mathbf{x},$$

where $\mathbf{A} \in \mathrm{M}_+^{m\times m}$. If $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^m)$ satisfies $\mathbf{u} = \mathbf{u}^{\mathrm{h}}$ on $\partial\Omega$, then we define its homogeneous extension $\mathbf{u}^e : \Omega^{\mathrm{e}} \to \mathbb{R}^m$ by

$$\mathbf{u}^e(\mathbf{x}) := \begin{cases} \mathbf{u}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega, \\ \mathbf{A}\mathbf{x} & \text{if } \mathbf{x} \in \Omega^{\mathrm{e}}\backslash\Omega \end{cases} \tag{2.3}$$

and note that $\mathbf{u}^e \in W^{1,p}(\Omega^{\mathrm{e}};\mathbb{R}^m)$. More generally, let $\mathbf{x}_0 \in \Omega$ and $\epsilon \geq 0$ satisfy $B(\mathbf{x}_0,\epsilon) \subset\subset \Omega \subset\subset \Omega^{\mathrm{e}}$ and define

$$\Omega_\epsilon := \Omega\backslash\overline{B(\mathbf{x}_0,\epsilon)}, \qquad \Omega_\epsilon^{\mathrm{e}} := \Omega^{\mathrm{e}}\backslash\overline{B(\mathbf{x}_0,\epsilon)} \tag{2.4}$$

(this corresponds to a preexisting void of radius ϵ in the reference configuration). If $\mathbf{u}_\epsilon \in W^{1,p}(\Omega_\epsilon;\mathbb{R}^m)$ satisfies $\mathbf{u}_\epsilon = \mathbf{u}^{\mathrm{h}}$ on $\partial\Omega$, then we define its homogeneous extension $\mathbf{u}_\epsilon^e : \Omega_\epsilon^{\mathrm{e}} \to \mathbb{R}^m$ by

$$\mathbf{u}_\epsilon^e(\mathbf{x}) := \begin{cases} \mathbf{u}_\epsilon(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_\epsilon, \\ \mathbf{A}\mathbf{x} & \text{if } \mathbf{x} \in \Omega^{\mathrm{e}}\backslash\Omega \end{cases} \tag{2.5}$$

and note that[6] $\mathbf{u}_\epsilon^e \in W^{1,p}(\Omega_\epsilon^{\mathrm{e}};\mathbb{R}^m)$. The use of such an extension allows us to obtain the following result, whose proof we omit since it is similar to that of Theorem TL in [17].

Lemma 2.7. *Let $\mathbf{u}_\epsilon \in W^{1,p}(\Omega_\epsilon;\mathbb{R}^m)$, $p > m-1$, $\epsilon \geq 0$, satisfy $\mathbf{u}_\epsilon = \mathbf{u}^{\mathrm{h}}$ on $\partial\Omega$. Suppose that its homogeneous extension $\mathbf{u}_\epsilon^e$, given by* (2.5), *is one-to-one a.e. on $\Omega_\epsilon^{\mathrm{e}}$. Then*

$$\mathbf{u}_\epsilon(\mathbf{x}) \in \mathbf{u}^{\mathrm{h}}(\overline{\Omega}) \quad \text{for a.e. } \ \mathbf{x} \in \Omega_\epsilon.$$

[6] If $\epsilon = 0$, then there is no preexisting hole. In this case, $\Omega_0 = \Omega$, $\Omega_0^{\mathrm{e}} = \Omega^{\mathrm{e}}$, and thus $\mathbf{u}_0 = \mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $\mathbf{u}_0^e = \mathbf{u}^e \in W^{1,p}(\Omega^{\mathrm{e}};\mathbb{R}^m)$ in accordance with the earlier definition (2.3).

2.4. The distributional Jacobian. Given a mapping $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^m)$, with $p > \frac{m^2}{m+1}$, it follows (using the Sobolev embedding theorem) that the distributional Jacobian defined by

$$(\mathrm{Det}\nabla\mathbf{u})(\phi) := -\int_\Omega \frac{1}{m}\left([\mathrm{adj}\,\nabla\mathbf{u}]\mathbf{u}\right)\cdot\nabla\phi\,d\mathbf{x} \quad \forall\phi \in C_0^\infty(\Omega) \tag{2.6}$$

is a well-defined distribution (where $\mathrm{adj}\,\nabla\mathbf{u}$ denotes the adjugate matrix of $\nabla\mathbf{u}$, that is, the transposed matrix of cofactors of $\nabla\mathbf{u}$). The definition follows from the well-known formula for expressing $\det\nabla\mathbf{u}$ as a divergence (see, e.g., [14] for further details and references).

Next, suppose that $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^m)$, $p > m-1$, satisfies (INV) on Ω. Then $\mathbf{u} \in L^\infty_{\mathrm{loc}}(\Omega)$, and hence $\mathrm{Det}\,\nabla\mathbf{u}$ is again a well-defined distribution. Moreover, it follows from [16, Lemma 8.1] that if $\mathbf{u}$ further satisfies $\det\nabla\mathbf{u} > 0$ a.e., then $\mathrm{Det}\,\nabla\mathbf{u}$ is a Radon measure and

$$\mathrm{Det}\,\nabla\mathbf{u} = (\det\nabla\mathbf{u})\mathcal{L}^m + \mu^s,$$

where μ^s is singular with respect to $\mathcal{L}^m$. In this paper we will be interested in the case when μ^s is a Dirac measure[7] of the form $\alpha\delta_{\mathbf{x}_0}$ (where $\alpha > 0$ and $\mathbf{x}_0 \in \Omega$) which corresponds to $\mathbf{u}$ creating a cavity of volume α at the point $\mathbf{x}_0$.[8]

Example 2.8. Let $\Omega = B$ be the unit ball in $\mathbb{R}^3$ (i.e., $m = 3$) and let

$$\mathbf{u}(\mathbf{x}) = (|\mathbf{x}| + c)\frac{\mathbf{x}}{|\mathbf{x}|}, \quad c > 0. \tag{2.7}$$

Then $\mathbf{u}$ produces a hole of radius c at the center of the deformed ball. In this case it can be shown that

$$\mathrm{Det}\,\nabla\mathbf{u} = (\det\nabla\mathbf{u})\mathcal{L}^3 + \frac{4}{3}\pi c^3\delta_{\mathbf{0}}. \tag{2.8}$$

The last expression (2.8) is to be interpreted in the sense of distributions so that

$$(\mathrm{Det}\,\nabla\mathbf{u})(\phi) = \int_\Omega (\det\nabla\mathbf{u}(\mathbf{x}))\phi(\mathbf{x})\,d\mathbf{x} + \frac{4}{3}\pi c^3\phi(0) \quad \forall\phi \in C_0^\infty(\Omega)$$

(notice that the coefficient of $\delta_{\mathbf{0}}$ in (2.8) is the volume of the hole that is formed at the origin under the deformation (2.7)).

3. The energy: Existence of minimizers. We consider an m-dimensional elastic body which, in its reference state, occupies the region $\Omega \subset \mathbb{R}^m$. We let $W \in C(\overline{\Omega} \times \mathrm{M}^{m\times m}_+; [0,\infty))$ denote the stored energy function for the body. For ease of exposition we state the following conditions on W in the case of three dimensions (i.e., $m = 3$). Let $p > 2 = m - 1$, $D = \mathrm{M}^{3\times3} \times \mathrm{M}^{3\times3} \times (0,\infty)$; then we will refer to the following hypotheses on W:

(H1) (polyconvexity) there exists $\Phi : \Omega \times D \to \overline{\mathbb{R}}$ such that for a.e. $\mathbf{x} \in \Omega$

$$W(\mathbf{x},\mathbf{F}) = \Phi(\mathbf{x}, (\mathbf{F}, \mathrm{adj}\,\mathbf{F}, \det\mathbf{F})) \quad \forall\mathbf{F} \text{ such that } \det\mathbf{F} > 0,$$

where $\Phi(\mathbf{x},\cdot) : D \to \overline{\mathbb{R}}$ is convex for a.e. $\mathbf{x} \in \Omega$;

[7] Other assumptions on the support of the singular measure μ^s may be relevant for modelling different forms of fracture. See also [15] for further results on the singular support of the distributional Jacobian.

[8] Note that such a cavity need not be spherical.

(H2) (continuity) $\Phi(\mathbf{x},\cdot) : D \to \overline{\mathbb{R}}$ is continuous for a.e. $\mathbf{x} \in \Omega$ and $\Phi(\cdot, N) : \Omega \to \overline{\mathbb{R}}$ is measurable for every $N \in D$;

(H3) (coercivity) $W(\mathbf{x},\mathbf{F}) \geq C|\mathbf{F}|^p + \Gamma(\det \mathbf{F}) + K$ for a.e. $\mathbf{x} \in \Omega$, where $C > 0$, K are constants, and $\Gamma : (0,\infty) \to \mathbb{R}$ is a convex function satisfying $\Gamma(t)/t \to +\infty$ as $t \to +\infty$;

(H4) $\Gamma(t) \to +\infty$ as $t \to 0^+$.

Now fix $\mathbf{x}_0 \in \Omega$ and for each $0 \leq \epsilon < \operatorname{dist}(\mathbf{x}_0, \partial\Omega)$ recall the definition of the homogeneous extension of a map given in section 2.3 (see (2.3)–(2.5) in particular). For each such ϵ we seek a minimizer for the total elastic energy

$$E_\epsilon(\mathbf{u}_\epsilon) = \int_{\Omega_\epsilon} W(\mathbf{x}, \nabla \mathbf{u}_\epsilon(\mathbf{x}))\, d\mathbf{x}$$

in the class of admissible functions

$$\mathcal{A}_\epsilon(\mathbf{x}_0) = \{\mathbf{u}_\epsilon \in W^{1,p}(\Omega_\epsilon; \mathbb{R}^m) : \mathbf{u}_\epsilon|_{\partial\Omega} = \mathbf{u}^{\mathrm{h}},\ (\mathbf{u}^e_\epsilon)^* \text{ satisfies (INV) on } \Omega^{\mathrm{e}}_\epsilon, \\ \det \nabla \mathbf{u}_\epsilon > 0 \text{ a.e.},\quad \operatorname{Det} \nabla \mathbf{u}^e_\epsilon = (\det \nabla \mathbf{u}^e_\epsilon)\mathcal{L}^m\}$$

if $\epsilon > 0$, and in the class

$$\mathcal{A}_0(\mathbf{x}_0) = \{\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^m) : \mathbf{u}|_{\partial\Omega} = \mathbf{u}^{\mathrm{h}},\ (\mathbf{u}^e)^* \text{ satisfies (INV) on } \Omega^{\mathrm{e}}, \\ \det \nabla \mathbf{u} > 0 \text{ a.e.},\quad \operatorname{Det} \nabla \mathbf{u}^e = (\det \nabla \mathbf{u}^e)\mathcal{L}^m + \alpha_{\mathbf{u}}\delta_{\mathbf{x}_0}\} \tag{3.1}$$

if $\epsilon = 0$, where $\alpha_{\mathbf{u}} \geq 0$ is a scalar depending on the map $\mathbf{u}$ and $\delta_{\mathbf{x}_0}$ denotes the Dirac measure with support at $\mathbf{x}_0$. Thus, $\mathcal{A}_0(\mathbf{x}_0)$ contains maps $\mathbf{u}$ that produce a cavity of volume $\alpha_{\mathbf{u}}$ located at $\mathbf{x}_0 \in \Omega$.

PROPOSITION 3.1 (see [19, Theorem 4.1]). *Let $p > m-1$ and $\epsilon \geq 0$. Suppose that $W \in C(\overline{\Omega} \times \mathrm{M}^{m\times m}_+; [0,\infty))$ satisfies hypotheses* (H1)–(H4). *Then E_ϵ attains its infimum on $\mathcal{A}_\epsilon(\mathbf{x}_0)$.*[9]

Thus both the *mixed displacement/traction problem* ($\epsilon > 0$) and the *pure displacement problem* ($\epsilon = 0$) have energy minimizers. In the next section we will show that a subsequence of the minimizers of the mixed problems converges to a minimizer of the displacement problem as the preexisting hole size shrinks to zero.

Remark 3.2. The reasons for requiring that the homogeneous extensions $\mathbf{u}^e, \mathbf{u}^e_\epsilon$, rather than the original maps $\mathbf{u}, \mathbf{u}_\epsilon$, satisfy condition (INV) are to, first, prevent the phenomenon of cavitation at the boundary of Ω (see [16, p. 55]) and, second, to prevent leakage at the boundary (see [16, p. 56] and [17, p. 975]). For interesting related results see Swanson and Ziemer [25, 26].

Remark 3.3. Suppose that in the above theorem $\mathbf{A} = \lambda\mathbf{I}$ and $\lambda > \lambda_{\mathrm{crit}}$ (where λ_{crit} is the critical boundary displacement after which radial cavitation occurs). Suppose further that the energy grows sufficiently slowly with respect to $|\mathbf{F}|$; e.g., W is given by (1.4) for some $c > 0$ and $p \in (m-1, m)$ with Γ as in (H3) and (H4). It then follows from results on radial cavitation (see [20]) that any minimizer $\mathbf{u}$ given by the above result with $\epsilon = 0$ must satisfy $\alpha_{\mathbf{u}} > 0$; i.e., it must form a new cavity.

4. Convergence of minimizers.

In this section we show that, as $\epsilon \to 0^+$, minimizers $\mathbf{u}_\epsilon$ of the mixed displacement/traction problem given by Proposition 3.1 converge to a minimizer of the pure displacement problem $\mathbf{u} \in \mathcal{A}_0(\mathbf{x}_0)$. More precisely, we prove the following main theorem of the paper.

[9] We refer the reader to [4] for interesting analytical difficulties that arise in the borderline case $p = m - 1$.

THEOREM 4.1. *Let $\mathbf{x}_0 \in \Omega$ be fixed and let $W \in C(\overline{\Omega} \times \mathrm{M}^{m\times m}_+; [0,\infty))$ satisfy* (H1)–(H4). *Suppose that (ϵ_n) is a monotone decreasing sequence converging to zero and let $(\mathbf{u}_{\epsilon_n})$ be a corresponding sequence of minimizers whose existence is given by Proposition* 3.1. *Then there exists $\mathbf{u} \in \mathcal{A}_0(\mathbf{x}_0)$ and a subsequence $(\mathbf{u}_{\epsilon_{n_j}})$ such that*

$$\mathbf{u}_{\epsilon_{n_j}} \rightharpoonup \mathbf{u} \quad \text{as} \quad j \to \infty \quad \text{in} \quad W^{1,p}(\Omega^{\mathrm{e}}_\delta; \mathbb{R}^m)$$

for any $\delta > 0$. Moreover, $\mathbf{u}$ is a minimizer of E_0 on $\mathcal{A}_0(\mathbf{x}_0)$.

The proof of this theorem is contained in the remainder of this section. Throughout this section (ϵ_n) will denote a fixed monotone decreasing sequence converging to zero, while $(\mathbf{u}_{\epsilon_n})$ will denote a corresponding sequence of minimizers whose existence is given by Proposition 3.1.

The convergence proof is split into three parts: We first identify a map $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^m)$ such that for a subsequence $(\mathbf{u}_{\epsilon_{n_j}})$ of $(\mathbf{u}_{\epsilon_n})$ we have $\mathbf{u}_{\epsilon_{n_j}} \rightharpoonup \mathbf{u}$ in $W^{1,p}(\Omega_\delta; \mathbb{R}^m)$ as $j \to \infty$ for any $\delta > 0$ sufficiently small. It then follows from Proposition 2.4 and Definition 2.6 that $\mathbf{u}$ satisfies condition (INV) on $\Omega\backslash\{\mathbf{x}_0\}$. In section 4.2 we show in Theorem 4.3 that it then follows that $\mathbf{u}$ satisfies (INV) on Ω and in Lemma 4.5 that $\operatorname{Det} \nabla\mathbf{u} = (\det \nabla\mathbf{u})\mathcal{L}^m + \alpha_{\mathbf{u}}\delta_{\mathbf{x}_0}$. Thus we can conclude that $\mathbf{u} \in \mathcal{A}_0(\mathbf{x}_0)$. Finally, in section 4.3 in Theorem 4.6 we prove that $\mathbf{u}$ is a minimizer of the energy E_0 on $\mathcal{A}_0(\mathbf{x}_0)$ using lower semicontinuity arguments.

4.1. Identifying a weak limit for the sequence of minimizers.

THEOREM 4.2. *Let $(\mathbf{u}_{\epsilon_n})$ be a sequence of minimizers given by Proposition* 3.1. *Then there is a subsequence $(\mathbf{u}_{\epsilon_{n_j}})$ and a mapping $\mathbf{u} : \Omega \to \mathbb{R}^m$ such that, for any $\delta > 0$, $\mathbf{u} \in W^{1,p}(\Omega\backslash\overline{B_\delta}; \mathbb{R}^m)$ and*

$$\mathbf{u}_{\epsilon_{n_j}} \rightharpoonup \mathbf{u} \quad \text{as} \quad j \to \infty \quad \text{in} \quad W^{1,p}(\Omega\backslash\overline{B(\mathbf{x}_0,\delta)}; \mathbb{R}^m).$$

Proof. Let $(\mathbf{u}_{\epsilon_n})$ be a sequence of minimizers given by Proposition 3.1. We first note that, since W is continuous and nonnegative on the compact set $\overline{\Omega} \times \{\mathbf{A}\}$, the homogeneous deformation $\mathbf{u}^{\mathrm{h}}$ has finite energy. Hence for any $n \in \mathbb{N}$,

$$E_{\epsilon_n}(\mathbf{u}^{\mathrm{h}}) = \int_{\Omega_{\epsilon_n}} W(\mathbf{x}, \mathbf{A})\, d\mathbf{x} \leq \int_{\Omega} W(\mathbf{x}, \mathbf{A})\, d\mathbf{x} < \infty. \tag{4.1}$$

Next, by the convexity of Γ and its growth at zero and infinity ((H3) and (H4)) it follows that Γ is bounded below. Thus, by the coercivity condition (H3) and the Poincaré inequality, we find that for any $N \in \mathbb{N}$ and $n > N$

$$E_{\epsilon_N}(\mathbf{u}_{\epsilon_n}) \geq C_1 \|\mathbf{u}_{\epsilon_n}\|^p_{W^{1,p}(\Omega_{\epsilon_N})} - C_2\mathcal{L}^m(\Omega_{\epsilon_N}), \tag{4.2}$$

where C_1 and C_2 are positive constants. In addition, W is nonnegative and $\Omega_{\epsilon_N} \subset \Omega_{\epsilon_n}$ so that

$$\begin{aligned} E_{\epsilon_N}(\mathbf{u}_{\epsilon_n}) &= \int_{\Omega_{\epsilon_N}} W(\mathbf{x}, \nabla\mathbf{u}_{\epsilon_n}(\mathbf{x}))\, d\mathbf{x} \\ &\leq \int_{\Omega_{\epsilon_n}} W(\mathbf{x}, \nabla\mathbf{u}_{\epsilon_n}(\mathbf{x}))\, d\mathbf{x} \quad = \quad E_{\epsilon_n}(\mathbf{u}_{\epsilon_n}). \end{aligned} \tag{4.3}$$

Also, $\mathbf{u}^{\mathrm{h}} \in \mathcal{A}_{\epsilon_n}(\mathbf{x}_0)$ and $\mathbf{u}_{\epsilon_n}$ is a minimizer of E_{ϵ_n} on $\mathcal{A}_{\epsilon_n}(\mathbf{x}_0)$, and so

$$E_{\epsilon_n}(\mathbf{u}_{\epsilon_n}) \leq E_{\epsilon_n}(\mathbf{u}^{\mathrm{h}}). \tag{4.4}$$

Therefore by (4.1)–(4.4) the sequence $(\mathbf{u}_{\epsilon_n})$ is bounded in the reflexive Banach space $W^{1,p}(\Omega_{\epsilon_N};\mathbb{R}^m)$, and consequently there exists a subsequence, still labeled $(\mathbf{u}_{\epsilon_n})$, converging weakly to some function $\mathbf{u}^N$ in $W^{1,p}(\Omega_{\epsilon_N};\mathbb{R}^m)$; that is,

$$\mathbf{u}_{\epsilon_n} \rightharpoonup \mathbf{u}^N \quad \text{in} \quad W^{1,p}(\Omega_{\epsilon_N};\mathbb{R}^m) \quad \text{as} \quad n \to \infty.$$

Now inductively take successive subsequences with $N = 1, 2, 3, \dots$ and then choose a diagonal sequence to obtain a subsequence, labeled $(\mathbf{u}_{\epsilon_{n_j}})$, of $(\mathbf{u}_{\epsilon_n})$, that satisfies

$$\mathbf{u}_{\epsilon_{n_j}} \rightharpoonup \mathbf{u} \quad \text{in} \quad W^{1,p}(\Omega_{\epsilon_N};\mathbb{R}^m) \quad \text{as} \quad j \to \infty,$$

where $\mathbf{u} : \Omega\backslash\{\mathbf{x}_0\} \to \mathbb{R}^m$ is defined by

$$\mathbf{u}(\mathbf{x}) := \begin{cases} \mathbf{u}^1(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_{\epsilon_1}, \\ \mathbf{u}^N(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_{\epsilon_N}\backslash\Omega_{\epsilon_{N-1}}, \end{cases}$$

for $N = 2, 3, 4, \dots$. By construction $\mathbf{u} \in W^{1,p}(\Omega_{\epsilon_N};\mathbb{R}^m)$ for any N. ∎

Note that $\mathbf{u}$ is well defined by the uniqueness of weak limits. We will henceforth use $(\mathbf{u}_{\epsilon_n})$ to denote the subsequence of minimizers $(\mathbf{u}_{\epsilon_{n_j}})$ obtained in Theorem 4.2. It now follows from standard arguments that the limit function $\mathbf{u}$ identified in the above theorem lies in $W^{1,p}(\Omega;\mathbb{R}^m)$ for $p > m-1$. For example, let $\psi \in C^\infty(\mathbb{R})$ be a fixed monotone increasing function that satisfies

$$\psi(t) = \begin{cases} 0 & \text{if } \ t \le 1, \\ 1 & \text{if } \ t \ge 4/3. \end{cases}$$

For each $n \in \mathbb{N}$, extend $\mathbf{u}_{\epsilon_n} : \Omega_{\epsilon_n} \to \mathbb{R}^m$ to a mapping $\tilde{\mathbf{u}}_{\epsilon_n} : \Omega \to \mathbb{R}^m$ by defining

$$\tilde{\mathbf{u}}_{\epsilon_n}(\mathbf{x}) := \begin{cases} \psi\left(\frac{2|\mathbf{x}-\mathbf{x}_0|}{|\mathbf{x}-\mathbf{x}_0|+\epsilon_n}\right)\mathbf{u}_{\epsilon_n}(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega_{\epsilon_n}, \\ 0 & \text{otherwise.} \end{cases}$$

Then verify that $\tilde{\mathbf{u}}_{\epsilon_n} \rightharpoonup \mathbf{u}$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ as $n \to \infty$.

4.2. The weak limit u lies in $\mathcal{A}_0(\mathbf{x}_0)$. In this section we prove that the weak limit $\mathbf{u}$ of the sequence $(\mathbf{u}_{\epsilon_n})$ is in the class of admissible functions $\mathcal{A}_0(\mathbf{x}_0)$. Since $\mathbf{u}_{\epsilon_n} \rightharpoonup \mathbf{u}$ in $W^{1,p}(\Omega_{\epsilon_N};\mathbb{R}^m)$ for any N, standard results imply that $\mathbf{u} = \mathbf{u}^{\mathrm{h}}$ on $\partial\Omega$, and the arguments in the proof of Theorem 4.1 in [19] show that $\det \nabla\mathbf{u}^e > 0$ a.e. Since $\mathbf{u}^{\mathrm{h}}$ is a diffeomorphism, it follows that $\mathbf{u}^{\mathrm{h}}(\overline{\Omega})$ is closed, and, since we may assume $\mathbf{u}_{\epsilon_n} \to \mathbf{u}$ a.e. in Ω, we conclude that $\mathbf{u}(\mathbf{x}) \in \mathbf{u}^{\mathrm{h}}(\overline{\Omega})$ for a.e. $\mathbf{x} \in \Omega$ and consequently

$$\mathbf{u}^e(\mathbf{x}) \in \mathbf{u}^{\mathrm{h}}(\Omega^{\mathrm{e}}) \quad \text{for a.e. } \mathbf{x} \in \Omega^{\mathrm{e}}. \tag{4.5}$$

We next prove that if $\mathbf{u}^e$ satisfies condition (INV) on $\Omega^{\mathrm{e}}\backslash\{\mathbf{x}_0\}$, then $\mathbf{u}^e$ satisfies condition (INV) on Ω^{e}. Following this, we then prove in Lemma 4.5 that the distributional Jacobian of $\mathbf{u}^e$ has the appropriate form, which will complete the proof that $\mathbf{u}^e$ lies in $\mathcal{A}_0(\mathbf{x}_0)$.

THEOREM 4.3. *Let $p > m-1$ and suppose that $\mathbf{u}_{\epsilon_n} \in \mathcal{A}_{\epsilon_n}(\mathbf{x}_0)$ is the sequence of minimizers given in Proposition* 3.1. *Suppose further that there exists $\mathbf{u}^{\mathrm{e}} \in W^{1,p}(\Omega^{\mathrm{e}};\mathbb{R}^m)$ such that $\det \nabla\mathbf{u}^{\mathrm{e}} > 0$ a.e. and that for any fixed $N \in \mathbb{N}$*

$$\mathbf{u}^{\mathrm{e}}_{\epsilon_n} \rightharpoonup \mathbf{u}^{\mathrm{e}} \quad \text{in} \quad W^{1,p}(\Omega^{\mathrm{e}}_{\epsilon_N};\mathbb{R}^m). \tag{4.6}$$

Then $(\mathbf{u}^{\mathrm{e}})^*$ satisfies (INV) *on Ω^{e}.*

Proof. Without loss of generality we take $\mathbf{u}^e_{\epsilon_n} = (\mathbf{u}^e_{\epsilon_n})^*$, $\mathbf{u}^e = (\mathbf{u}^e)^*$, and fix $\mathbf{x}_1 \in \Omega^{\mathrm{e}}$. Then we must show that for $\mathcal{L}^1$ a.e. $r \in (0, \mathrm{dist}(\mathbf{x}_1, \partial\Omega^{\mathrm{e}}))$

$$\text{(4.7)} \quad \begin{aligned} &\text{(i)} \quad \mathbf{u}^e(\mathbf{x}) \in \mathrm{im}_{\mathrm{T}}(\mathbf{u}^e, B_r(\mathbf{x}_1)) \cup \mathbf{u}^e(\partial B_r(\mathbf{x}_1)) \text{ for a.e. } \mathbf{x} \in \overline{B_r(\mathbf{x}_1)};\\ &\text{(ii)} \quad \mathbf{u}^e(\mathbf{x}) \in \mathbb{R}^m \backslash \mathrm{im}_{\mathrm{T}}(\mathbf{u}^e, B_r(\mathbf{x}_1)) \text{ for a.e. } \mathbf{x} \in \Omega^{\mathrm{e}} \backslash \overline{B_r(\mathbf{x}_1)}, \end{aligned}$$

where $B_r(\mathbf{x}_1) := B(\mathbf{x}_1, r)$.

Let N be fixed and suppose that $n > N$. By definition of $\mathcal{A}_{\epsilon_n}(\mathbf{x}_0)$ the minimizers $\mathbf{u}^e_{\epsilon_n}$ satisfy condition (INV) on $\Omega^{\mathrm{e}}_{\epsilon_N}$ and $\det \nabla \mathbf{u}^e_{\epsilon_n} > 0$ a.e. Consequently, by (4.6) and Proposition 2.4, $\mathbf{u}^e$ satisfies condition (INV) on $\Omega^{\mathrm{e}}_{\epsilon_N}$. Since N is arbitrary, it follows that $\mathbf{u}^e$ satisfies condition (INV) on $\Omega^{\mathrm{e}} \backslash \{\mathbf{x}_0\}$. Next, fix $r_1 > 0$ such that $B(\mathbf{x}_1, r_1) \subset \Omega^{\mathrm{e}}$ and $\mathbf{x}_0 \notin \partial B(\mathbf{x}_1, r_1)$. Then for all r sufficiently close to r_1 either $\mathbf{x}_0 \in B(\mathbf{x}_1, r)$ or $\mathbf{x}_0 \in \Omega^{\mathrm{e}} \backslash \overline{B(\mathbf{x}_1, r)}$. In the former case define

$$\text{(4.8)} \quad U_r := \Omega^{\mathrm{e}} \backslash \overline{B(\mathbf{x}_1, r)} \quad \text{and} \quad V_r := B(\mathbf{x}_1, r)$$

and in the latter case define

$$U_r := B(\mathbf{x}_1, r) \quad \text{and} \quad V_r := \Omega^{\mathrm{e}} \backslash \overline{B(\mathbf{x}_1, r)}.$$

Since $\partial\Omega^{\mathrm{e}}$ is smooth, U_r and V_r are each open sets with C^1 boundary.

We prove the result in the first case, $\mathbf{x}_0 \in B(\mathbf{x}_1, r)$ and (4.8), and note that the proof in the second case is similar. Let $\delta > 0$ be sufficiently small. Then by [16, Theorem 9.1] (see the appendix) there exists an $\epsilon_0 > 0$ such that for a.e. $r \in (r_1 - \epsilon_0, r_1 + \epsilon_0)$

$$\text{(4.9)} \quad \begin{aligned} &\text{(o)} \quad \mathbf{u}^e|_{\partial U_r} \in W^{1,p}(\partial U_r, \mathbb{R}^m) \cap C(\partial U_r, \mathbb{R}^m);\\ &\text{(i)} \quad \mathbf{u}^e(\mathbf{x}) \in \mathrm{im}_{\mathrm{T}}(\mathbf{u}^e, U_r) \cup \mathbf{u}^e(\partial U_r) \quad \text{for a.e. } \mathbf{x} \in \overline{U_r};\\ &\text{(ii)} \quad \mathbf{u}^e(\mathbf{x}) \in \mathbb{R}^m \backslash \mathrm{im}_{\mathrm{T}}(\mathbf{u}^e, U_r) \quad \text{for a.e. } \mathbf{x} \in \Omega^{\mathrm{e}} \backslash (\overline{U_r} \cup \overline{B(\mathbf{x}_0, \delta)}\,). \end{aligned}$$

Fix one such r. For each $n \in \mathbb{N}$ we successively take $\delta = \epsilon_n$ in $(4.9)_3$. Then since the countable union of null sets is a null set, as is the set $\{\mathbf{x}_0\}$, we conclude that

$$\text{(4.10)} \quad \begin{aligned} &\text{(i)}' \quad \mathbf{u}^e(\mathbf{x}) \in \mathrm{im}_{\mathrm{T}}(\mathbf{u}^e, U_r) \cup \mathbf{u}^e(\partial U_r) \quad \text{for a.e. } \mathbf{x} \in \overline{U_r} = \overline{\Omega^{\mathrm{e}}} \backslash B_r(\mathbf{x}_1);\\ &\text{(ii)}' \quad \mathbf{u}^e(\mathbf{x}) \in \mathbb{R}^m \backslash \mathrm{im}_{\mathrm{T}}(\mathbf{u}^e, U_r) \quad \text{for a.e. } \mathbf{x} \in \Omega^{\mathrm{e}} \backslash \overline{U_r} = B_r(\mathbf{x}_1). \end{aligned}$$

In order to obtain (4.7) we first note that V_r, U_r, and ∂V_r are pairwise disjoint sets with V_r and U_r open, ∂V_r compact, and $\Omega^e = V_r \cup U_r \cup \partial V_r$. Thus standard properties of degree[10] imply

$$\text{(4.11)} \quad \deg(\mathbf{u}^e, \Omega^e, \mathbf{y}) = \deg(\mathbf{u}^e, U_r, \mathbf{y}) + \deg(\mathbf{u}^e, V_r, \mathbf{y})$$

for all $\mathbf{y} \notin \mathbf{u}^e(\partial\Omega^{\mathrm{e}} \cup \partial V_r)$ (note that $\partial\Omega^{\mathrm{e}} \cup \partial V_r = \partial U_r$).

Next, $\mathbf{u}^{\mathrm{h}}$ is an orientation-preserving diffeomorphism whose degree satisfies $\deg(\mathbf{u}^{\mathrm{h}}, \Omega^{\mathrm{e}}, \mathbf{y}) = 1$ if $\mathbf{y} \in \mathrm{im}_{\mathrm{T}}(\mathbf{u}^{\mathrm{h}}, \Omega^{\mathrm{e}}) = \mathbf{u}^{\mathrm{h}}(\Omega^e)$ and $\deg(\mathbf{u}^{\mathrm{h}}, \Omega^{\mathrm{e}}, \mathbf{y}) = 0$ if $\mathbf{y} \in \mathbb{R}^m \backslash \mathbf{u}^{\mathrm{h}}(\overline{\Omega}^{\mathrm{e}})$. Since $\mathbf{u}^{\mathrm{h}}$ and $\mathbf{u}^e$ assume the same boundary values on $\partial\Omega^{\mathrm{e}}$ and since the degree depends only on the boundary values, it follows that their degrees are equal:

$$\deg(\mathbf{u}^e, \Omega^{\mathrm{e}}, \mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{y} \in \mathbf{u}^{\mathrm{h}}(\Omega^{\mathrm{e}}),\\ 0 & \text{if } \mathbf{y} \in \mathbb{R}^m \backslash \mathbf{u}^{\mathrm{h}}(\overline{\Omega}^{\mathrm{e}}). \end{cases}$$

[10]These are the domain decomposition and excision properties; see, e.g., [7] or [22].

Consequently, in view of (4.11), if $\mathbf{y} \in \mathbf{u}^{\mathrm{h}}(\Omega^{\mathrm{e}})$ and $\mathbf{y} \notin \mathbf{u}^e(\partial U_r)$, then

$$\deg(\mathbf{u}^e, U_r, \mathbf{y}) + \deg(\mathbf{u}^e, V_r, \mathbf{y}) = 1. \tag{4.12}$$

We next recall that $\mathbf{x}_0 \in B_r(\mathbf{x}_1) = V_r$ and that $\mathbf{u}^e$ satisfies (INV) on $U_r = \Omega^{\mathrm{e}}\backslash\overline{V_r} \subset \Omega^{\mathrm{e}}\backslash\{\mathbf{x}_0\}$. Therefore (4.10) and the proof of [16, Theorem 9.1] yield

$$\deg(\mathbf{u}^e, U_r, \mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{y} \in \mathrm{im_T}(\mathbf{u}^e, U_r), \\ 0 & \text{if } \mathbf{y} \in \mathbb{R}^m\backslash(\mathrm{im_T}(\mathbf{u}^e, U_r) \cup \mathbf{u}^e(\partial U_r)), \end{cases}$$

which, together with (4.12), allows us to conclude that if $\mathbf{y} \in \mathbf{u}^{\mathrm{h}}(\Omega^{\mathrm{e}})$, then

$$\deg(\mathbf{u}^e, V_r, \mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{y} \in \mathrm{im_T}(\mathbf{u}^e, U_r), \\ 1 & \text{if } \mathbf{y} \in \mathbb{R}^m\backslash(\mathrm{im_T}(\mathbf{u}^e, U_r) \cup \mathbf{u}^e(\partial U_r)). \end{cases} \tag{4.13}$$

We are now ready to prove $(4.7)_1$: by (4.5) we may assume that $\mathbf{u}^e(\mathbf{x}) \in \mathbf{u}^{\mathrm{h}}(\Omega^{\mathrm{e}})$ for a.e. $\mathbf{x} \in \overline{B_r(\mathbf{x}_1)}$. By $(4.10)_2$ and $(4.13)_2$ it follows that for a.e. $\mathbf{x} \in B_r(\mathbf{x}_1)$

$$\mathbf{u}^e(\mathbf{x}) \in \mathbf{u}^e(\partial U_r) \quad \text{or} \quad \deg(\mathbf{u}^e, V_r, \mathbf{u}^e(\mathbf{x})) = 1 \ .$$

However, $\mathbf{u}^e(\partial U_r) = \mathbf{u}^e(\partial V_r) \cup \mathbf{u}^e(\partial\Omega^{\mathrm{e}})$, $\mathbf{u}^e(\partial\Omega^{\mathrm{e}}) \cap \mathbf{u}^{\mathrm{h}}(\Omega^{\mathrm{e}}) = \emptyset$, and by (4.5) we have $\mathbf{u}^e(\mathbf{x}) \in \mathbf{u}^{\mathrm{h}}(\Omega^{\mathrm{e}})$ for a.e. $\mathbf{x} \in B_r(\mathbf{x}_1)$. Therefore $(4.7)_1$ follows from the definition of the topological image (2.2). Similarly, by (4.5), $(4.10)_1$, and $(4.13)_1$ it follows that for a.e. $\mathbf{x} \in U_r = \Omega^e\backslash\overline{B_r(\mathbf{x}_1)}$

$$\mathbf{u}^e(\mathbf{x}) \in \mathbf{u}^e(\partial V_r) \quad \text{or} \quad \deg(\mathbf{u}^e, V_r, \mathbf{u}^e(\mathbf{x})) = 0,$$

and so $(4.7)_2$ follows from (2.2). ∎

Remark 4.4. The containment condition $\mathbf{u}^e(\mathbf{x}) \in \mathbf{u}^{\mathrm{h}}(\Omega^{\mathrm{e}})$ for a.e. $\mathbf{x} \in \Omega^{\mathrm{e}}$, which follows from Lemma 2.7, is crucial to the argument in the last proof.

The following lemma combined with the last two subsections will allow us to conclude that $\mathbf{u}^e$ lies in $\mathcal{A}(\mathbf{x}_0)$.

LEMMA 4.5. *Let $m = 3$, let $p > 2 = m - 1$, and suppose that $\mathbf{u}_{\epsilon_n} \in \mathcal{A}_{\epsilon_n}(\mathbf{x}_0)$ is the sequence of minimizers of the mixed displacement/traction problem. Let $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^m)$ be the weak limit of the sequence as given in Theorem* 4.2*, so that the precise representative of its homogeneous extension $(\mathbf{u}^e)^* \in W^{1,p}(\Omega^{\mathrm{e}}; \mathbb{R}^m)$ satisfies* (INV) *on Ω^{e} and*

$$\mathbf{u}_{\epsilon_n} \rightharpoonup \mathbf{u}^e \quad \text{in} \quad W^{1,p}(\Omega^{\mathrm{e}}_{\epsilon_N}; \mathbb{R}^m)$$

for any $N \in \mathbb{N}$. Then there exists $\alpha_{\mathbf{u}} \geq 0$ such that

$$\mathrm{Det}\, \nabla\mathbf{u}^e = (\det \nabla\mathbf{u}^e)\mathcal{L}^m + \alpha_{\mathbf{u}}\delta_{\mathbf{x}_0},$$

where $\delta_{\mathbf{x}_0}$ denotes the Dirac measure supported at $\mathbf{x}_0$.

Proof. We first note that $\mathbf{u}^e \in W^{1,p}(\Omega^{\mathrm{e}}; \mathbb{R}^m)$, $\det \nabla\mathbf{u}^e > 0$ a.e., and $(\mathbf{u}^e)^*$ satisfies condition (INV) on Ω^{e}. Therefore, $\mathrm{Det}\, \nabla\mathbf{u}^e$ is a Radon measure and

$$\mathrm{Det}\, \nabla\mathbf{u}^e = (\det \nabla\mathbf{u}^e)\mathcal{L}^m + \mu^s, \tag{4.14}$$

where μ^s is a (nonnegative) Radon measure that is singular with respect to $\mathcal{L}^m$ (see section 2.4).

Next, $\mathbf{u}_{\epsilon_n} \rightharpoonup \mathbf{u}^e$ in $W^{1,p}(\Omega^{\rm e}_{\epsilon_N};\mathbb{R}^m)$ for any N and $\mathbf{u}_{\epsilon_n}$ satisfies (INV) on $\Omega^{\rm e}_{\epsilon_N}$. Therefore, by Lemma 3.3 in [19], there exists a subsequence (still labeled $\mathbf{u}_{\epsilon_n}$) that satisfies

$$\mathbf{u}_{\epsilon_n} \to \mathbf{u}^e \text{ in } L^q_{\rm loc}(\Omega^{\rm e}_{\epsilon_N};\mathbb{R}^m) \tag{4.15}$$

for every $1 < q < \infty$. Moreover, since $p > m-1$,

$$\operatorname{adj}\nabla\mathbf{u}_{\epsilon_n} \rightharpoonup \operatorname{adj}\nabla\mathbf{u}^e \quad \text{in } L^{\frac{p}{m-1}}(\Omega^{\rm e}_{\epsilon_N};\mathbb{R}^m) \tag{4.16}$$

(see [2, Theorem 3.4]). Consequently, by (4.15) and (4.16) (see [19, Lemma 6.7]),

$$(\operatorname{adj}\nabla\mathbf{u}_{\epsilon_n})\mathbf{u}_{\epsilon_n} \rightharpoonup (\operatorname{adj}\nabla\mathbf{u}^e)\mathbf{u}^e \quad \text{in } L^1_{\rm loc}(\Omega^{\rm e}_{\epsilon_N};\mathbb{R}^m),$$

and hence, in view of (2.6),

$$(\operatorname{Det}\nabla\mathbf{u}_{\epsilon_n})(\phi) \to (\operatorname{Det}\nabla\mathbf{u}^e)(\phi) \tag{4.17}$$

for every $\phi \in C_0^\infty(\Omega^{\rm e}_{\epsilon_N})$. Since $\mathbf{u}_{\epsilon_n} \in \mathcal{A}_{\epsilon_n}(\mathbf{x}_0)$, the sequence $\mathbf{u}_{\epsilon_n}$ satisfies

$$\operatorname{Det}\nabla\mathbf{u}_{\epsilon_n} = (\det\nabla\mathbf{u}_{\epsilon_n})\mathcal{L}^m \ \text{ on } \ \Omega^{\rm e}_{\epsilon_N}. \tag{4.18}$$

Now, by (4.1), (4.3), (4.4), and hypothesis (H3), for any $N \in \mathbb{N}$ and all $n > N$

$$\int_{\Omega_{\epsilon_N}} \Gamma(\det\nabla\mathbf{u}_{\epsilon_n})\,d\mathbf{x} \le \int_\Omega W(\mathbf{x},\mathbf{A})\,d\mathbf{x} < \infty,$$

where Γ is the convex, superlinear function given in (H3). Thus, by the de la Vallée Poussin and Dunford–Pettis criteria (see, e.g., [19, Theorem 4.1]), there is a $\theta \in L^1(\Omega_{\epsilon_N})$ such that (for a subsequence) $\det\nabla\mathbf{u}_{\epsilon_n} \rightharpoonup \theta$ in $L^1(\Omega^{\rm e}_{\epsilon_N})$. Moreover, Lemma 3.2 in [19] implies that $\theta = \det\nabla\mathbf{u}^e$ a.e. in $\Omega^{\rm e}_{\epsilon_N}$. Therefore,

$$\det\nabla\mathbf{u}_{\epsilon_n} \rightharpoonup \det\nabla\mathbf{u}^e \quad \text{in } L^1(\Omega^{\rm e}_{\epsilon_N};\mathbb{R}^m), \tag{4.19}$$

and in view of (4.17)–(4.19) we find that, for every $\phi \in C_0^\infty(\Omega^{\rm e}_{\epsilon_N})$,

$$(\operatorname{Det}\nabla\mathbf{u}^e)(\phi) = \int_{\Omega^{\rm e}_{\epsilon_N}} \phi(\mathbf{x})\det\nabla\mathbf{u}^e(\mathbf{x})\,d\mathbf{x}$$

and consequently that

$$(\operatorname{Det}\nabla\mathbf{u}^e)(\Omega^{\rm e}_{\epsilon_N}) = \int_{\Omega^{\rm e}_{\epsilon_N}} \det\nabla\mathbf{u}^e(\mathbf{x})\,d\mathbf{x}. \tag{4.20}$$

Finally, (4.14) and (4.20) imply that for every $N \in \mathbb{N}$

$$\mu^s(\Omega^{\rm e}_{\epsilon_N}) = 0.$$

Thus, since

$$\Omega^{\rm e} \subset \{\mathbf{x}_0\} \cup \bigcup_{N=1}^\infty \Omega^{\rm e}_{\epsilon_N},$$

we find that

$$\mu^s(\Omega^{\rm e}) \le \mu^s(\{\mathbf{x}_0\}) + \sum_{N=1}^\infty \mu^s(\Omega^{\rm e}_{\epsilon_N}) = \mu^s(\{\mathbf{x}_0\}),$$

which yields the desired result (set $\alpha_{\mathbf{u}} = \mu^s(\{\mathbf{x}_0\})$). ∎

4.3. u is a minimizer for the pure displacement problem. Thus far we have shown that the weak limit $\mathbf{u}$ of a subsequence of the minimizers $\mathbf{u}_{\epsilon_n}$ of the mixed displacement/traction boundary value problem lies in $\mathcal{A}(\mathbf{x}_0)$. Next, we will prove that $\mathbf{u}$ is a minimizer of the pure displacement boundary value problem considered in Proposition 3.1. Throughout this section we will use $\mathbf{u}_{\epsilon_n}$ to denote this convergent subsequence identified in sections 4.1 and 4.2.

THEOREM 4.6. *Suppose that* E_0 *and* $\mathcal{A}_0(\mathbf{x}_0)$ *are as defined in section* 3 *and that* $\mathbf{u} \in \mathcal{A}(\mathbf{x}_0)$ *is the weak limit of the sequence of minimizers* $(\mathbf{u}_{\epsilon_n})$ *of the mixed displacement/traction problems. Then* $E_0(\mathbf{u}) \leq E_0(\tilde{\mathbf{u}})$ *for all* $\tilde{\mathbf{u}} \in \mathcal{A}_0(\mathbf{x}_0)$.

Proof. Define

$$\lambda := \liminf_{n\to\infty} \int_{\Omega_{\epsilon_n}} W(\mathbf{x}, \nabla\mathbf{u}_{\epsilon_n}(\mathbf{x}))\, d\mathbf{x}$$

and let N be fixed. Then, since W is nonnegative, for any $n > N$

$$\int_{\Omega_{\epsilon_N}} W(\mathbf{x}, \nabla\mathbf{u}_{\epsilon_n}(\mathbf{x}))\, d\mathbf{x} \leq \int_{\Omega_{\epsilon_n}} W(\mathbf{x}, \nabla\mathbf{u}_{\epsilon_n}(\mathbf{x}))\, d\mathbf{x}. \tag{4.21}$$

Next, by sequential weak lower semicontinuity of E_{ϵ_N} (see [3, Theorem 7.1] and [19, pp. 93–100]) and since $\mathbf{u}_{\epsilon_n} \rightharpoonup \mathbf{u}$ in $W^{1,p}(\Omega_{\epsilon_N}; \mathbb{R}^m)$ as $n \to \infty$, we have

$$\int_{\Omega_{\epsilon_N}} W(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x} \leq \liminf_{n\to\infty} \int_{\Omega_{\epsilon_N}} W(\mathbf{x}, \nabla\mathbf{u}_{\epsilon_n}(\mathbf{x}))\, d\mathbf{x}. \tag{4.22}$$

If we combine (4.21) and (4.22) we find that

$$\int_{\Omega_{\epsilon_N}} W(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x} \leq \lambda$$

and hence by the monotone convergence theorem that

$$\int_{\Omega} W(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x} \leq \lambda. \tag{4.23}$$

Now, suppose that $\tilde{\mathbf{u}} \in \mathcal{A}_0(\mathbf{x}_0)$. Then $\tilde{\mathbf{u}}|_{\Omega_\epsilon} \in \mathcal{A}_\epsilon(\mathbf{x}_0)$ for every sufficiently small ϵ, and thus

$$\int_{\Omega_{\epsilon_n}} W(\mathbf{x}, \nabla\mathbf{u}_{\epsilon_n}(\mathbf{x}))\, d\mathbf{x} \leq \int_{\Omega_{\epsilon_n}} W(\mathbf{x}, \nabla\tilde{\mathbf{u}}(\mathbf{x}))\, d\mathbf{x}.$$

Then since W is nonnegative,

$$\int_{\Omega_{\epsilon_n}} W(\mathbf{x}, \nabla\mathbf{u}_{\epsilon_n}(\mathbf{x}))\, d\mathbf{x} \leq \int_{\Omega} W(\mathbf{x}, \nabla\tilde{\mathbf{u}}(\mathbf{x}))\, d\mathbf{x},$$

and if we take the liminf we conclude that

$$\lambda = \liminf_{n\to\infty} \int_{\Omega_{\epsilon_n}} W(\mathbf{x}, \nabla\mathbf{u}_{\epsilon_n}(\mathbf{x}))\, d\mathbf{x} \leq \int_{\Omega} W(\mathbf{x}, \nabla\tilde{\mathbf{u}}(\mathbf{x}))\, d\mathbf{x},$$

which together with (4.23) yields the desired result. ∎

We conclude from the results in section 4 that minimizers of E_{ϵ_n} on $\mathcal{A}_{\epsilon_n}(\mathbf{x}_0)$ converge to a minimizer of E_0 on $\mathcal{A}_0(\mathbf{x}_0)$ as $\epsilon_n \to 0$, i.e., that (a subsequence of) the minimizers of the mixed displacement/traction boundary value problem converge weakly to a minimizer of the pure displacement boundary value problem. This completes the proof of Theorem 4.1.

Remark 4.7. We cannot conclude in general that any sequence of minimizers $(\mathbf{u}_{\epsilon_n})$ of the mixed problem necessarily converges to a minimizer $\mathbf{u}$ of the pure displacement problem; this follows only after passing to a subsequence, since we do not know whether the minimizers given by Theorem 3.1 are unique.

5. Surface energy. In this section we examine the effect of regularizing our pure displacement variational problem by adding a surface energy term which penalizes the formation and growth of cavities. In particular, we consider the case of a typical surface energy term proportional to the surface area[11] of the holes produced. This modification is partly motivated by experimental observations of Gent [8], which indicate that such surface energy effects are relevant in certain situations involving cavitation. In this case, the original energy functional is replaced by the augmented functional:

$$\tilde{E}(\mathbf{u}) = \int_\Omega W(\nabla\mathbf{u}(\mathbf{x}))\,d\mathbf{x} \;+\; \kappa\,\mathrm{Per}(\mathrm{im}(\mathbf{u},\Omega)) = E(\mathbf{u}) + \kappa\,\mathrm{Per}(\mathrm{im}(\mathbf{u},\Omega)), \tag{5.1}$$

where $\kappa > 0$ is a constant and $\mathrm{Per}(\mathrm{im}(\mathbf{u},\Omega))$ corresponds to the perimeter or surface area of the holes produced (see [16] for a precise definition of the perimeter of the measure-theoretic image of Ω under $\mathbf{u}$). Again we consider the displacement boundary value problem in which the admissible deformations are required to satisfy the boundary condition (1.3). If W satisfies the hypotheses of section 3 (with $1 \le p < 3$), then analogous arguments to those given in [21, Theorem 2] show that, for "large $\mathbf{A}$,"[12] the homogeneous deformation $\mathbf{u}^{\mathrm{h}}(\mathbf{x}) \equiv \mathbf{A}\mathbf{x}$ is no longer the global minimizer of the energy (5.1). In particular, the energy can be lowered through the introduction of holes, or cavities, in the material. However, in contrast to the bifurcation diagram (Figure 1) for the radial problem without surface energy, we will show that these discontinuous minimizers do not bifurcate[13] from the trivial homogeneous deformations (i.e., that there do not exist discontinuous minimizers producing holes of arbitrarily small volume).

Our proof will consist of a simple energy estimate; however, for ease of presentation we will restrict our attention to the following class of homogeneous energy functions (but our arguments apply to much more general[14] stored energy functions):

$$W(\mathbf{F}) = c|\mathbf{F}|^p + \Gamma(\det\mathbf{F}),$$

where $2 < p < 3$, $c > 0$, and Γ is convex and differentiable. Let $\mathbf{u} \in \tilde{\mathcal{A}}$,

$$\tilde{\mathcal{A}} = \{\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^m) : \mathbf{u}|_{\partial\Omega} = \mathbf{u}^{\mathrm{h}},\ (\mathbf{u}^{\mathrm{e}})^* \text{ satisfies (INV) on } \Omega^{\mathrm{e}},\\ \det\nabla\mathbf{u} > 0 \text{ a.e.}\},$$

[11] As argued in [16], this may be criticized on the grounds that it assigns the same energy to creating the surface of a new cavity as it does to stretching the surface of a preexisting cavity.

[12] This is the case, in particular, if $\mathbf{A} = t\mathbf{B}$ for large $t > 0$, where $\mathbf{B} \in \mathrm{M}^{3\times 3}_+$ is any fixed matrix.

[13] See also [1, p. 608], which studies the addition of surface energy for radial deformations of a ball of incompressible material.

[14] In particular, our arguments clearly extend to stored energy functions of the form $W(\mathbf{F}) = W_0(\mathbf{F}) + c|\mathbf{F}|^p + \Gamma(\det\mathbf{F})$, $2 < p < 3$, where W_0 is $W^{1,p}$-quasiconvex.

and suppose that $\tilde{E}(\mathbf{u}) < \tilde{E}(\mathbf{u}^{\mathrm{h}})$, where $\mathbf{u}^{\mathrm{h}}(\mathbf{x}) \equiv \mathbf{A}\mathbf{x}$ and $\tilde{E}$ is given by (5.1). From the definition of $\tilde{E}$ it then follows that $E(\mathbf{u}) < E(\mathbf{u}^{\mathrm{h}})$. Next, by the convexity of the mapping $\mathbf{F} \mapsto |\mathbf{F}|^p$ and the boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{u}^{\mathrm{h}}$,

$$c\int_\Omega |\mathbf{A}|^p \, d\mathbf{x} \le c\int_\Omega |\nabla\mathbf{u}|^p \, d\mathbf{x},$$

and hence

$$\begin{aligned} 0 < E(\mathbf{u}^{\mathrm{h}}) - E(\mathbf{u}) &= \int_\Omega c|\mathbf{A}|^p - c|\nabla\mathbf{u}|^p + \Gamma(\det\mathbf{A}) - \Gamma(\det\nabla\mathbf{u})\, d\mathbf{x} \\ &\le \int_\Omega \Gamma(\det\mathbf{A}) - \Gamma(\det\nabla\mathbf{u})\, d\mathbf{x} \\ &\le \int_\Omega \Gamma'(\det\mathbf{A})(\det\mathbf{A} - \det\nabla\mathbf{u})\, d\mathbf{x}, \end{aligned} \tag{5.2}$$

where we have used the convexity of Γ in the last step. The above estimate will allow us to bound the decrease in the bulk energy due to the formation of a cavity by a constant times the volume of the cavity.

Suppose for a contradiction that there exists a sequence of matrices $\mathbf{A}_n \to \mathbf{A}$ and a corresponding sequence of deformations $\mathbf{u}_n \in \tilde{\mathcal{A}}$, with $\mathbf{u}_n(\mathbf{x}) = \mathbf{A}_n\mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$ and

$$\operatorname{Det}\nabla\mathbf{u}_n = (\det\nabla\mathbf{u}_n)\mathcal{L}^3 + \mu_n, \tag{5.3}$$

such that $\tilde{E}(\mathbf{u}_n) < \tilde{E}(\mathbf{u}_n^{\mathrm{h}})$ for all n and $\mu_n(\Omega) \to 0$ as $n \to \infty$, where $\mathbf{u}_n^{\mathrm{h}}(\mathbf{x}) \equiv \mathbf{A}_n\mathbf{x}$. Then by (5.2), (5.1), and (5.3)

$$\begin{aligned} 0 < \tilde{E}(\mathbf{u}_n^{\mathrm{h}}) - \tilde{E}(\mathbf{u}_n) &\le \Gamma'(\det\mathbf{A}_n)\int_\Omega (\det\mathbf{A} - \det\nabla\mathbf{u}_n)\, d\mathbf{x} - \kappa\operatorname{Per}(\operatorname{im}(\mathbf{u}_n,\Omega)) \\ &= \Gamma'(\det\mathbf{A}_n)\mu_n(\Omega) - \kappa\operatorname{Per}(\operatorname{im}(\mathbf{u}_n,\Omega)). \end{aligned} \tag{5.4}$$

The intuitive idea now is that $\mu_n(\Omega)$ represents the volume of the holes formed and $\operatorname{Per}(\operatorname{im}(\mathbf{u}_n,\Omega))$ represents the surface areas produced. Since by assumption $\mu_n(\Omega) \to 0$, the above inequality yields a contradiction for large n. Mathematically, this argument is made rigorous through the use of an isoperimetric inequality. By the definition of $\tilde{E}$ (see (5.1)) it follows that if $\tilde{E}(\mathbf{u}) < \infty$, then $\operatorname{Per}(\operatorname{im}(\mathbf{u},\Omega)) < \infty$, and so by [16, Theorem 8.4]

$$\operatorname{Det}\nabla\mathbf{u}_n = (\det\nabla\mathbf{u}_n)\mathcal{L}^3 + \sum_{i=1}^\infty \alpha_i^{(n)}\delta_{\mathbf{x}_i}, \tag{5.5}$$

where $\alpha_i^{(n)} \ge 0$ for all i and n. Moreover,

$$\sum_{i=1}^\infty \left(\alpha_i^{(n)}\right)^{2/3} \le \bar{c}\operatorname{Per}(\operatorname{im}(\mathbf{u}_n,\Omega)),$$

where $\bar{c} > 0$ is the isoperimetric constant.

Without loss of generality we may assume that $0 \le \Gamma'(\det\mathbf{A}_n) \le K$ for all n (the first inequality follows from the main result in [23] and the second by the assumed convergence of $(\mathbf{A}_n)$ to $\mathbf{A}$). Hence, by (5.4) and the above expression,

$$0 < K\sum_{i=1}^\infty \alpha_i^{(n)} - \frac{\kappa}{\bar{c}}\sum_{i=1}^\infty \left(\alpha_i^{(n)}\right)^{2/3}. \tag{5.6}$$

Now, in view of (5.5), $\mu_n(\Omega) = \sum_{i=1}^{\infty} \alpha_i^{(n)} \to 0$ as $n \to \infty$. Therefore, given $l \in \mathbb{N}$, let $N_l \in \mathbb{N}$ be such that $\alpha_i^{(n)} < \frac{1}{2^{3l}}$ for all $n > N_l$ and $i \in \mathbb{N}$. Then, by (5.6),

$$0 < \left(\frac{K}{2^l} - \frac{\kappa}{\bar{c}} \right) \sum_{i=1}^{\infty} \left(\alpha_i^{(n)} \right)^{2/3} \quad \forall n > N_l \, ,$$

which yields a contradiction for l sufficiently large. □

Remark 5.1. We note that the arguments used in section 4 can be adapted to show that if $\kappa_n \to 0$ and $(\mathbf{u}_n)$ is a corresponding sequence of minimizers of $\tilde{E}_n$ (given by (5.1) with $\kappa = \kappa_n$) on $\mathcal{A}_0(\mathbf{x}_0)$ (given by (3.1)), then, passing to a subsequence if necessary, $(\mathbf{u}_n)$ converges weakly in $W^{1,p}(\Omega)$ to a minimizer of E on $\mathcal{A}_0(\mathbf{x}_0)$.

6. Concluding remarks. It is interesting to note that the convergence result given in Theorem 4.1 could form the rigorous basis for a numerical method to compute approximations to the singular minimizers whose existence is given in Proposition 3.1. In particular, it is sufficient to compute regular minimizers on Ω_ϵ whose existence is given by Proposition 3.1 for small ϵ.

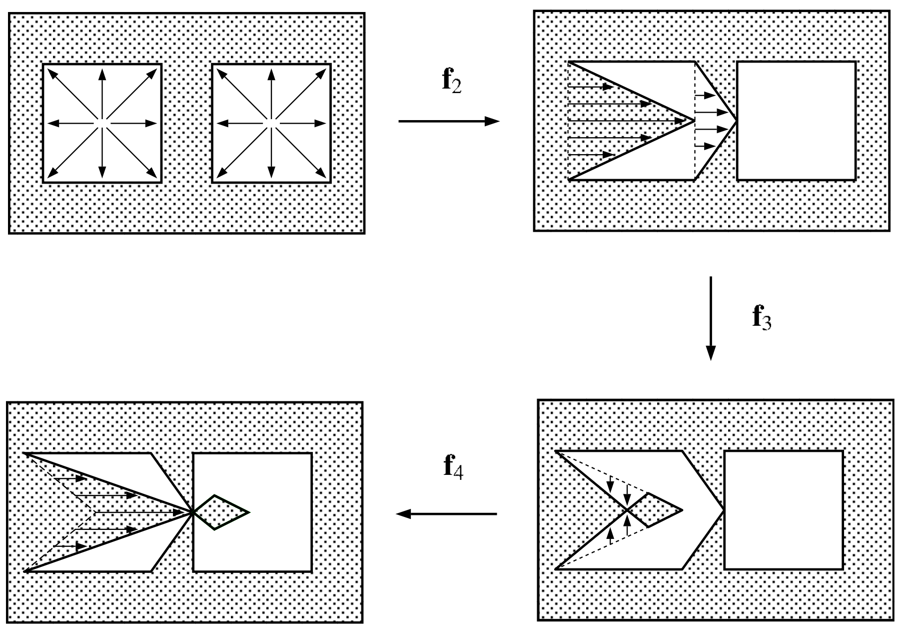

FIG. 3. *Leakage between holes.*

We next mention the main difficulty encountered in trying to extend the convergence result of Theorem 4.1 to the situation where we have, say, two (or more) flaw points located at $\{\mathbf{x}_0, \mathbf{x}_1\}$. Most of the arguments extend to this case; however, a map $\mathbf{u}$ satisfying (INV) on $\Omega \backslash \{\mathbf{x}_0, \mathbf{x}_1\}$ need not satisfy (INV) on Ω. The difficulty is related to the counterexample given in [17]: it is possible to construct a mapping $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^m)$ with $p > m - 1$ which satisfies (INV) on $\Omega^e \backslash \{\mathbf{x}_0, \mathbf{x}_1\}$ but which does not satisfy (INV) on Ω; consider a map which forms two adjacent cavities and which allows leakage from one cavity into the other (see Figure 3). Such a map can be produced, for example, as a composition of a two-hole cavitating map $\mathbf{f}_1$ with three Lipschitz maps $\mathbf{f}_2, \mathbf{f}_3$, and $\mathbf{f}_4$. Explicit formulae for some of the mappings can be found in [16].

Appendix. We note that our results depend crucially on Theorem 9.1 in [16], which extends condition (INV) from balls to other regions. However, the original proof of this result contains a small error. We therefore include here a corrected proof.

THEOREM A.1 (see [16, Theorem 9.1]). *Let* $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^m)$ *with* $p > m-1$*. Suppose that* $\det \nabla\mathbf{u} > 0$ *a.e. and that* $\mathbf{u}^*$ *satisfies condition* (INV)*. Assume that* $U \subset\subset \Omega$ *is open with* C^1 *boundary and that there exists an* $\varepsilon_0 > 0$*, an open neighborhood* N *of* ∂U*, and a (surjective) diffeomorphism* $\mathbf{w} : \partial U \times (-\varepsilon_0, \varepsilon_0) \to N$ *that satisfies* $\mathbf{w}(\mathbf{x}, 0) = \mathbf{x}$ *and* $\mathbf{w}_\varepsilon(\mathbf{x}, 0) \cdot \mathbf{n}(\mathbf{x}) > 0$ *for all* $\mathbf{x} \in \partial U$*, where* $\mathbf{n}(\mathbf{x})$ *is the outward unit normal to* U *and* $\mathbf{w}_\varepsilon$ *is the partial derivative of* $\mathbf{w}$ *with respect to its second argument. Define*

$$U_\varepsilon := (U\backslash N) \cup \mathbf{w}\big(\partial U \times (-\varepsilon_0, \varepsilon)\big).$$

Then for a.e. $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

(o) $\mathbf{u}^*|_{\partial U_\varepsilon} \in W^{1,p}(\partial U_\varepsilon;\mathbb{R}^m) \cap C^0(\partial U_\varepsilon;\mathbb{R}^m)$*;*

(i) $\mathbf{u}^*(\mathbf{x}) \in (\mathrm{im}_{\mathrm{T}}(\mathbf{u}^*, U_\varepsilon) \cup \mathbf{u}^*(\partial U_\varepsilon))$ *for a.e.* $\mathbf{x} \in \overline{U}_\varepsilon$*;*

(ii) $\mathbf{u}^*(\mathbf{x}) \in \mathbb{R}^m\backslash \mathrm{im}_{\mathrm{T}}(\mathbf{u}^*, U_\varepsilon)$ *for a.e.* $\mathbf{x} \in \Omega\backslash U_\varepsilon$*.*

Proof. Let $\theta \in C_0^\infty(\mathbb{R}^m)$ satisfy $\theta \geq 0$ and suppose that $\mathbf{g} \in C^\infty(\mathbb{R}^m;\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m;\mathbb{R}^m)$ satisfies $\operatorname{div}\mathbf{g} = \theta$. Then, as in section 8 in [16], the map $\mu_\theta : C_0^\infty(\Omega) \to \mathbb{R}$ given by

$$\mu_\theta(\varphi) := -\int_\Omega \nabla\varphi \cdot (\operatorname{adj}\nabla\mathbf{u})(\mathbf{g}\circ\mathbf{u})\, d\mathbf{x}$$

is a distribution on Ω. If we let φ_t be a standard sequence of (radial) mollifiers, then we find that the computation that leads to equation (8.4) in [16] will now yield

$$(\varphi_t * \mu_\theta)(\mathbf{x}) = -\int_0^t \psi_t'(r) \int_{\mathbb{R}^m} (\operatorname{div}\mathbf{g}) \deg(\mathbf{u}, S(\mathbf{x}, r), \mathbf{y})\, d\mathbf{y}\, dr,$$

where $\varphi(\mathbf{x}) = \psi(|\mathbf{x}|)$ and we have written $\mathbf{u}$ for $\mathbf{u}^*$. Since $\operatorname{div}\mathbf{g} = \theta \geq 0$, the reasoning used in the proof of [16, Lemma 8.1] therefore implies that μ_θ is a Radon measure, for each θ, and that

$$\begin{aligned}\mu_\theta(\overline{B(\mathbf{b}, r)}) &= \int_{\mathbb{R}^m} \theta(\mathbf{y}) \deg(\mathbf{u}, S(\mathbf{b}, r), \mathbf{y})\, d\mathbf{y}\\ &= \int_{\mathbb{R}^m} \theta \chi_{\mathrm{im}_{\mathrm{T}}(\mathbf{u}, B(\mathbf{b}, r))}\, d\mathbf{y} \qquad \text{(A.1)}\end{aligned}$$

for $\mathcal{L}^1$ a.e. $r \in (0, r_{\mathbf{b}})$.

We next show that for $\mathcal{L}^1$ a.e. $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

$$\mu_\theta(U_\varepsilon) = \int_{\mathbb{R}^m} \theta(\mathbf{y}) \deg(\mathbf{u}, \partial U_\varepsilon, \mathbf{y})\, d\mathbf{y}. \qquad \text{(A.2)}$$

Let $\psi \in C^\infty(\mathbb{R})$ satisfy $\psi(s) = 1$ for $s < -\varepsilon_0/2$ and $\psi(s) = 0$ for $s > \varepsilon_0/2$ and define $\varphi := \psi \circ \omega$, where the function $\omega : N \to (-\varepsilon_0, \varepsilon_0)$ denotes the last component of the diffeomorphism $\mathbf{w}^{-1}$. We note that $\nu := \nabla\omega/|\nabla\omega|$ is the outward unit normal to the surfaces ∂U_ε and apply the coarea formula for the C^1 function ω to get

$$\begin{aligned}\mu_\theta(\varphi) &= -\int_\Omega (\psi' \circ \omega)|\nabla\omega|\nu \cdot (\operatorname{adj}\nabla\mathbf{u})(\mathbf{g}\circ\mathbf{u})\, d\mathbf{x}\\ &= \int_{-\varepsilon_0}^{\varepsilon_0} \psi'(s) \int_{\omega=s} (\operatorname{adj}\nabla\mathbf{u})^T\nu \cdot (\mathbf{g}\circ\mathbf{u})\, d\mathcal{H}^{m-1}\, ds.\end{aligned}$$

An application of the area formula together with the fact that $\mathbf{g}$ is bounded shows that the inner integral is an $\mathcal{L}^1$ function in the variable s. If we choose a suitable increasing sequence $\psi_k \nearrow \chi_{(-\infty,\varepsilon)}$, then we find, with the aid of Proposition 2.1 in [16], that (A.2) is satisfied.

We note that it follows from (A.1) and (8.3) in [16] that

$$\mu_\theta(\overline{B(\mathbf{b},r)}) \leq (\sup\theta)(\mathrm{Det}\,\nabla\mathbf{u})(\overline{B(\mathbf{b},r)})$$

for $\mathcal{L}^1$ a.e. $r \in (0,r_{\mathbf{b}})$. This implies that each of the measures μ_θ is absolutely continuous with respect to Det $\nabla\mathbf{u}$.

Next, let $\Theta \subset C_0^\infty(\mathbb{R}^m;\mathbb{R}^{\geq})$ be a countable set that is dense in $L^2(\mathbb{R}^m;\mathbb{R}^{\geq})$ and fix $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ so that (o) is satisfied and (A.2) is satisfied for every $\theta \in \Theta$. Let $N_{\mathbf{b}}$ be the $\mathcal{L}^1$ null set of Lemma 7.3 in [16] and let $\hat{N}_{\mathbf{b}}$ be an $\mathcal{L}^1$ null set such that (A.1) is satisfied for every $\theta \in \Theta$ and every $r \in (0,r_{\mathbf{b}})\backslash\hat{N}_b$. Writing $\tilde{N}_{\mathbf{b}} = N_{\mathbf{b}} \cup \hat{N}_{\mathbf{b}}$ we consider the family of closed balls

$$\mathcal{F} := \{\overline{B(\mathbf{b},r)} : \mathbf{b} \in U_\varepsilon,\ r \in (0,r_{\mathbf{b}})\backslash\tilde{N}_{\mathbf{b}},\ (\mathrm{Det}\ \nabla\mathbf{u})(\partial B(\mathbf{b},r)) = 0\}.$$

Then, since the set of radii for which (Det $\nabla\mathbf{u})(\partial B(\mathbf{b},r)) > 0$ is at most countable, for each $\mathbf{b} \in U_\varepsilon$

$$\inf\{r : \overline{B(\mathbf{b},r)} \in \mathcal{F}\} = 0,$$

and hence we can apply the Besicovitch covering theorem to get a sequence of pairwise disjoint closed balls $\overline{B(\mathbf{b}_k,r_k)} \subset U_\varepsilon$ such that

$$(\mathcal{L}^m + \mathrm{Det}\,\nabla\mathbf{u})\left(U_\varepsilon \setminus \bigcup_{k=1}^\infty \overline{B(\mathbf{b}_k,r_k)}\right) = 0. \tag{A.3}$$

Therefore, since the sets $\overline{B(\mathbf{b}_k,r_k)}$ are pairwise disjoint, (A.1)–(A.3) together with the absolute continuity of each measure μ_θ with respect to Det $\nabla\mathbf{u}$ imply that

$$\int_{\mathbb{R}^m} \theta(\mathbf{y})\deg(\mathbf{u},\partial U_\varepsilon,\mathbf{y})\,d\mathbf{y} = \sum_{k=1}^\infty \int_{\mathbb{R}^m} \theta\chi_{\mathrm{im_T}(\mathbf{u},B(\mathbf{b}_k,r_k))}\,d\mathbf{y}$$

for every $\theta \in \Theta$. Since Θ is dense in $\mathcal{L}^2(\mathbb{R}^m;\mathbb{R}^{\geq})$ and since the sets $\mathrm{im_T}(\mathbf{u},B(\mathbf{b}_k,r_k))$ are pairwise disjoint, the bounded convergence theorem and the last equation yield

$$\deg(\mathbf{u},\partial U_\varepsilon,\cdot\,) = \sum_{k=1}^\infty \chi_{\mathrm{im_T}(\mathbf{u},B(\mathbf{b}_k,r_k))} \text{ a.e.}$$

Consequently, since the sets $\mathrm{im_T}(\mathbf{u},B(\mathbf{b}_k,r_k))$ are pairwise disjoint, $\deg(\mathbf{u},\partial U_\epsilon,\mathbf{y})$ assumes only the values 0 and 1, and hence $\deg(\mathbf{u},\partial U_\epsilon,\cdot\,) = \chi_{\mathrm{im_T}(\mathbf{u},U_\epsilon)}$: thus,

$$\chi_{\mathrm{im_T}(\mathbf{u},U_\varepsilon)} = \sum_{k=1}^\infty \chi_{\mathrm{im_T}(\mathbf{u},B(\mathbf{b}_k,r_k))} \text{ a.e.} \tag{A.4}$$

We note that, by the area formula, the sets $\mathbf{u}(\partial B(\mathbf{b}_k,r_k))$ and $\mathbf{u}(\partial U_\varepsilon)$ are each Lebesgue null sets. Consequently, since the sets $\mathrm{im_T}(\mathbf{u},B(\mathbf{b}_k,r_k))$ are pairwise disjoint, (A.4) implies that there exist Lebesgue null sets M_1 and M_2 such that

$$\bigcup_{k=1}^\infty E(\mathbf{u},B(\mathbf{b}_k,r_k)) \subset M_1 \cup E(\mathbf{u},U_\varepsilon) \tag{A.5}$$

and

$$\mathrm{im_T}(\mathbf{u}, U_\varepsilon) \subset M_2 \cup \bigcup_{k=1}^{\infty} \mathrm{im_T}(\mathbf{u}, B(\mathbf{b}_k, r_k)). \tag{A.6}$$

Finally, since $\det \nabla\mathbf{u} > 0$ a.e., it follows from the area formula (see, e.g., [24, Lemma 2]) that $\mathbf{u}^{-1}(M_1)$ and $\mathbf{u}^{-1}(M_2)$ are Lebesgue null sets. Therefore (i) of this theorem follows from (A.5) and (i) of condition (INV), while (A.6) and (ii) of condition (INV) yield (ii) of this theorem. □

Acknowledgment. J.S. would like to thank Geoffrey Burton for helpful discussions in the course of this work.

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