There Are No Non-Trivially Uniformly 
(t, r)-Regular Graphs for t > 2.

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Abstract

A finite simple graph is uniformly (t, r)-regular if it has at least t 
vertices and the open neighbor set of each set of t of its vertices is of 
cardinality r. If t > 1, such a graph is trivially uniformly (t, r)-regular 
if either it is a matching (t = r) or r is the number of non-isolated 
vertices in the graph. We prove the result stated in the title.

1 Uniform (t, r)-regularity

All graphs will be finite and simple, in this paper, and notation will largely 
be as in [10]. If G and H are graphs, V(G) is the vertex set of G, G + H is the 
disjoint union of G and H, and for a positive integer m, mG = G + ··· + G (m 
summands). If u ∈ V(G), N_G(u) = {v ∈ V(G) | u and v are adjacent in G}, 
and if S ⊆ V(G), N_G(S) = \bigcup_{u \in S} N_G(u), the open neighbor set of S in G. 
The order of G will be denoted by n(G)(= |V(G)|), or just n, if G is the only 
graph in the discussion.

G is uniformly (t, r)-regular if 1 ≤ t ≤ n and for each S ⊆ V(G) with 
|S| = t, |N_G(S)| = r. This property of graphs was introduced in [4] as 
“(t, r)-regularity”; the problem with that terminology is that it is also used 
for a seemingly similar but rather less exigent property, introduced in [3] and 
written on in [2], [5], and [7]. In [6] the word “strong” plays the role we assign 
to “uniform” here; we abandon that terminology because it misleadingly 
suggests an analogy with strong regularity of graphs. There is a powerful 
connection between the two when t = 2 (see [9]), but the analogy at the 
definitional level is distant.
Uniform \((1, r)\)-regularity is just plain \(r\)-regularity. When \(t > 1\) there are two easily found classes of uniformly \((t, r)\)-regular graphs:

(i) \(G = mK_2\) for some \(m \geq t/2\), a matching. In this case, \(t = r\).

(ii) \(r = n(G_1)\), where \(G_1\) is the subgraph of \(G\) induced by the non-isolated vertices of \(G\), and \(t\) is “sufficiently large”. Indeed, as noted in [6], if \(r = n(G_1) > 0\) and \(n(G) - \delta(G_1) + 1 \leq t \leq n(G)\) then \(G\) is uniformly \((t, r)\)-regular, but \(G\) is not uniformly \((n(G) - \delta(G_1), r)\)-regular. And if \(r = n(G_1) = 0\) then \(G = nK_1\) and is uniformly \((t, 0)\)-regular for all \(t = 1, \ldots, n\).

For \(t > 1\), uniform \((t, r)\)-regularity due to either condition (i) or (ii) will be called \textit{trivial}, and the big question (raised in [6]) is: are there non-trivially uniformly \((t, r)\)-regular graphs, and, if so, what are they?

This question has been satisfactorily answered for \(t = 2\). Any “strongly regular graph with \(\lambda = \mu > 0\)”, that is, a regular graph \(G\), say with degree \(d > 0\), not complete, for which there exists \(\mu\) such that for any two distinct \(u, v \in V(G)\), \(|N_G(u) \cap N_G(v)| = \mu\), is non-trivially uniformly \((2, 2d - \mu)\)-regular. There are infinitely many such graphs (see, e.g., [8]), and it has recently been shown [9] that there are no other non-trivially uniformly \((2, r)\)-regular graphs besides these. Here we settle the question for \(t > 2\). The proof of the following theorem is postponed until section 3.

\textbf{Theorem 1} If \(t > 2\) then for no \(r\) does there exist a non-trivially uniformly \((t, r)\)-regular graph.

\section*{2 An excursion into designs}

If \(n \geq t > 0\), an \((n, t, \lambda)\)-design is a pair \((V, B)\) where \(V\) is a set with \(n\) elements (“points”) and \(B = [B(i) | i \in I]\) is an indexed collection of subsets of \(V\) (“blocks”) such that for each \(T \subseteq V\) with \(|T| = t\), \(|\{i \in I | T \subseteq B(i)\}| = \lambda\). (That is, any \(t\) points of \(V\) lie together in exactly \(\lambda\) blocks.) We require \(B\) to be an indexed collection because we want to allow “repeated blocks”; that is, it may be that \(B(i) = B(j)\) even though \(i \neq j\). Also note that there is no requirement that the blocks be of the same size. Given such a design, let \(b = |I|\), the number of blocks.

\textbf{Fisher’s Inequality} [1, Theorem 2.6, p.66] If \((V, B)\) is an \((n, 2, \lambda)\)-design with \(\lambda > 0\) and \(V\) not appearing as a block, then \(b \geq n\).
**Theorem 2** If \( t > 2, \lambda > 0, \) and \((V,B)\) is an \((n,t,\lambda)\)-design with \( V \) not appearing as a block, then \( b \geq n \) with equality if and only if \( B \) can be re-indexed to be \([V \setminus \{v\}]|v \in V\).

**Proof.** We go by induction on \( t \), starting with \( t = 3 \). For each \( v \in V \), let \( I(v) = \{i \in I|v \in B(i)\} \) and consider the derived design \((V \setminus \{v\}, B'(v))\), where \( B'(v) = [B(i) \setminus \{v\}|i \in I(v)] \). Each derived design is an \((n-1,2,\lambda)\)-design (because \( t = 3 \)) and \( V \setminus \{v\} \) does not appear in \( B'(v) \) because \( V \) does not appear in \( B \). By Fisher’s inequality, \( b'(v) = |I(v)| \geq n-1 \). On the other hand, \( b'(v) \leq b \).

If \( b = n - 1 \), then \( b'(v) = n - 1 = b \), for every \( v \in V \), so \( I(v) = I \) for every \( v \). But then \( v \in B(i) \) for every \( i \in I \), and every \( v \), so, not only does \( V \) appear in \( B \), it is equal to \( B(i) \) for each \( i \), wildly contrary to hypothesis. So \( b \geq n \), as asserted. Suppose that \( b = n \). Then \( b'(v) = |I(v)| = n \) or \( n - 1 \) for each \( v \in V \)–i.e., \( v \) is in every block of \( B \) or in every block but one.

On the other hand, each block of \( B \) is missing some element of \( V \). Think of a bipartite graph with bipartition \( V,I \), with \( v \in V \) adjacent to \( i \in I \) if and only if \( v \notin B(i) \). Then each \( v \in V \) has degree \( \leq 1 \) in this graph, and each \( i \in I \) has degree \( \geq 1 \), and \( |V| = n = b = |I| \). Thus the bipartite graph is a matching, and \( B \), possibly after renaming, is \([V \setminus \{v\}]|v \in V\).

Now suppose that \( t > 3 \). With \( I(v) \) and \( B'(v), v \in V \), defined as above, each derived design \((V \setminus \{v\}, B'(v))\) is an \((n-1,t-1,\lambda)\)-design, with \( V \setminus \{v\} \) not among the blocks in \( B'(v) \). By the induction hypothesis, \( b \geq b'(v) = |B'(v)| \geq n - 1 \) for each \( v \in V \). From here the proof proceeds as in the case \( t = 3 \).

### 3 Proof of Theorem 1

**Lemma 1** If \( t > 1 \) and \( G \) is non-trivially uniformly \((t,r)\)-regular, then \( G \) has no isolated vertices.

**Proof.** Suppose that \( u \) is an isolated vertex of \( G \). Let \( G_1 \) be the subgraph of \( G \) induced by the non-isolated vertices of \( G \). Since \( G \) is non-trivial, \( 0 < r < n(G_1) \), and, therefore, \( t < n(G_1) \). Let \( S \) be a \((t-1)\)-subset of \( V(G_1) \), and \( T = S \cup \{u\} \); then \( |N_G(T)| = |N_G(S)| = r \). Since \( r < n(G_1) \), there is some \( w \in V(G_1) \setminus N_G(S) \), and, by the definition of \( G_1 \), some \( v \in V(G_1) \) adjacent to \( w \). But then \( |S \cup \{v\}| = t \) while \( |N_G(S \cup \{v\})| \geq r+1 \), contradicting the assumption that \( G \) is uniformly \((t,r)\)-regular.
The main idea that starts the proof of Theorem 1 is due to Khodkar and Leach [8]. Suppose that $G$ is non-trivially $(t, r)$-regular, $t \geq 3$. For $v \in V(G)$, let $B(v) = V(G) \setminus N_G(v)$, and $\mathcal{B} = \{B(v) | v \in V(G)\}$. By the Lemma, no $v \in V(G)$ is isolated, so $B(v) \neq V(G)$. Further, $r < n$ (non-triviality of $G$) and $(V(G), \mathcal{B})$ is an $(n, t, n-r)$-design, with $b = |V(G)| = n$. Since $t \geq 3$, by Theorem 2, for each $v \in V(G)$ there is a $u \in V(G)$ such that $B(v) = V(G) \setminus \{u\}$. Thus $G$ is a matching, and is thus trivially uniformly $(t, r)$-regular, after all.

References


[9] Abdollah Khodkar, David Leach, and David Robinson, Every $(2, r)$-regular graph is strongly regular, submitted.