Constructing $t$-designs from $t$-wise balanced designs

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Abstract

We give a construction to obtain a $t$-design from a $t$-wise balanced design. More precisely, given a positive integer $k$ and a $t$-$(v, \{k_1, k_2, \ldots, k_s\}, \lambda)$ design $\mathcal{D}$, with with all block-sizes $k_i$ occurring in $\mathcal{D}$ and $1 \leq t \leq k \leq k_1 < k_2 < \cdots < k_s$, the construction produces a $t$-$(v, k, n\lambda)$ design $\mathcal{D}^*$, with $n = \text{lcm}(\binom{k_1-1}{k-t}, \ldots, \binom{k_s-1}{k-t})$. We prove that $\text{Aut}(\mathcal{D})$ is a subgroup of $\text{Aut}(\mathcal{D}^*)$, with equality when both $\lambda = 1$ and $t < k$. We employ our construction in another construction, which, given a $t$-$(v, k, \lambda)$ design with $1 \leq t < k < v$, and a point of this design, yields a $t$-$(v-1, k-1, (k-t)\lambda)$ design. Many of the $t$-designs coming from our constructions appear to be new.

1 Introduction

For $t$ a positive integer, a $t$-wise balanced design $\mathcal{D}$ is an ordered pair $(X, \mathcal{B})$, where $X$ is a finite non-empty set (of points) and $\mathcal{B}$ is a finite non-empty multiset of subsets of $X$ (called blocks), such that every $t$-subset of $X$ is contained in a constant number $\lambda > 0$ of blocks. If $v = |X|$ and $K$ is the set of sizes of the blocks, then we call $\mathcal{D}$ a $t$-$(v, K, \lambda)$ design. If all blocks of $\mathcal{D}$ have the same size $k$ (i.e. $K = \{k\}$), then $\mathcal{D}$ is called a $t$-design or a $t$-$(v, k, \lambda)$ design.

In this note we give a construction (the $\ast$-construction) to obtain a $t$-design from a $t$-wise balanced design. More precisely, given a positive integer $k$ and a $t$-$(v, \{k_1, k_2, \ldots, k_s\}, \lambda)$ design $\mathcal{D}$, with with all block-sizes $k_i$ occurring in $\mathcal{D}$ and $1 \leq t \leq k \leq k_1 < k_2 < \cdots < k_s$, the $\ast$-construction produces a $t$-$(v, k, n\lambda)$ design $\mathcal{D}^*$, with $n = \text{lcm}(\binom{k_1-1}{k-t}, \ldots, \binom{k_s-1}{k-t})$. We prove that $\text{Aut}(\mathcal{D})$ is a subgroup of $\text{Aut}(\mathcal{D}^*)$, with equality when both $\lambda = 1$ and $t < k.$
We employ the $\ast$-construction in another construction (the $\#$-construction), which, given a $t$-$(v, k, \lambda)$ design with $1 \leq t < k < v$, and a point of this design, yields a $t$-$(v - 1, k - 1, (k - t)\lambda)$ design. Many of the $t$-designs coming from our constructions appear to be new, and although they usually have repeated blocks, they often, via their constructions, have quite large automorphism groups.

2 The $\ast$-construction

The input to the $\ast$-construction consists of positive integers $t$ and $k$, and a $t$-$(v, \{k_1, k_2, \ldots, k_s\}, \lambda)$ design $D$, with all block-sizes $k_i$ occurring in $D$ and $1 \leq t \leq k \leq k_1 < k_2 < \cdots < k_s$. Now for $i = 1, 2, \ldots, s$ define

\[ n_i = \binom{k_i - t}{k - t}, \quad n = \text{lcm}(n_1, n_2, \ldots, n_s), \quad m_i = \frac{n}{n_i}. \quad (1) \]

The output of the $\ast$-construction is a block design $D^\ast$, which we prove below to be a $t$-$(v, k, n\lambda)$ design.

The point-set of $D^\ast$ is that of $D$, and to construct the block-multiset $B^\ast$ of $D^\ast$ we proceed as follows:

- start by setting $B^\ast$ to be the empty multiset;
- for each $i = 1, 2, \ldots, s$ and for each block $B \in \mathcal{B}$ of size $k_i$ (including repeats) do:
  - insert $m_i$ copies of every $k$-subset of $B$ into $B^\ast$.

Clearly, $D^\ast$ depends on the choice of $k$ as well as on $D$. Less obviously, since the $t$-wise balanced design $D$ may be $t'$-wise balanced for some $t' \neq t$, $D^\ast$ may depend on the choice of $t$. When we wish to make these dependencies explicit, we shall use the notation $D^\ast(t, k)$ instead of $D^\ast$.

**Theorem 2.1** Let $k$ be a positive integer and let $D = (X, \mathcal{B})$ be a $t$-$(v, \{k_1, k_2, \ldots, k_s\}, \lambda)$ design, with all block-sizes $k_i$ occurring in $D$ and $1 \leq t \leq k \leq k_1 < k_2 < \cdots < k_s$. Then $D^\ast = D^\ast(t, k) = (X, \mathcal{B}^\ast)$ is a $t$-$(v, k, n\lambda)$ design, where $n = \text{lcm}(n_1, n_2, \ldots, n_s)$ and $n_i = \binom{k_i - t}{k - t}$.
Proof. Let $T$ be any $t$-subset of $X$. Suppose that $B$ is a block of $B$ of size $k_i$ containing $T$. Then the number of $k$-subsets of $B$ which contain $T$ is $n_i = \binom{k_i - t}{k_i}$. Each of these $k$-subsets is added to $B^*$ exactly $m_i = n/n_i$ times. Hence $B$ contributes exactly $n_i m_i = n$ blocks containing $T$ to $B^*$. Now $T$ is contained in exactly $\lambda$ blocks in $B$, and so in exactly $n \lambda$ blocks in $B^*$. 

We have defined $n$ to be $\text{lcm}(n_1, n_2, \ldots, n_s)$. We could have chosen $n$ to be any common multiple of $\{n_1, n_2, \ldots, n_s\}$, but, in order to keep $n \lambda$ as small as possible, we choose the least common multiple. We also remark that the $*$-construction works perfectly well when $s = 1$, that is, when $D$ is a $t$-design.

Example 1 Let $D$ be the 2-(11, $\{3, 5\}$, 1) design with point-set $X = \{1, 2, \ldots, 9, T, E\}$ (here $T = 10$ and $E = 11$), and block-multiset $B =$

\[ [167, 18E, 19T, 268, 279, 2TE, 369, 37E, 38T, 46T, 478, 49E, 56E, 57T, 589, 12345] \]

(see [1, p.187]).

(a) Suppose $t = k = 2$. Here $k_1 = 3$, $k_2 = 5$, and each $n_i = n = m_i = 1$. So $D^*(2, 2)$ is the 2-(11, 2, 1) design consisting of all the 2-subsets of $X$.

(b) The case $t = 2$, $k = 3$ is more interesting. Here $k = k_1 = 3$, $k_2 = 5$, $n_1 = 1$, $n_2 = 3$, $n = 3$, $m_1 = 3$, and $m_2 = 1$. So $D^* = D^*(2, 3)$ is a 2-(11, 3, 3) design, an $(11, 55, 15, 3, 3)$-BIBD. The block-multiset of $D^*$ consists of three copies of each block of $D$ of size 3, together with all the 3-subsets of $\{1, 2, 3, 4, 5\}$.

The $*$-construction was found as a result of looking for 2-designs with repeated blocks to help fill up Preece’s catalogue [4]. Many new examples coming from this construction have since gone into the catalogue.

3 The $\#$-construction

Let $T = (X, B)$ be a $t$-$(v, k, \lambda)$ design with $1 \leq t < k < v$, and let $x \in X$. We employ the $*$-construction in a new construction (the $\#$-construction) which produces a $t$-$(v - 1, k - 1, (k - t)\lambda)$ design when given input $T$ and $x$. The $\#$-construction proceeds as follows:

Let $X' = X \setminus \{x\}$, and let $B'$ be the multiset consisting of all $B \setminus \{x\}$ with $B \in B$ (counting repeats). Denote the resulting block design $(X', B')$
by \( T \setminus x \), which is a \( t-(v-1, \{k-1,k\}, \lambda) \) design (whose isomorphism class may depend on the choice of \( x \)). Next, apply the \(*)\)-construction with input \( t, k-1 \) and \( T \setminus x \) to obtain \( (T \setminus x)^*(t,k-1) \), a \( t-(v-1,k-1,(k-t)\lambda) \) design. We denote this output of the \#-construction by \( T^#(t,x) \).

**Example 2** Start with the large Witt design \( W \), the unique (up to isomorphism) \( 5-(24,8,1) \) design; see [3, Chapter 8], where \( W \) is called the Mathieu design \( M_{24} \). Now \( W \) is also a \( 4-(24,8,5) \) design, a \( 3-(24,8,21) \) design, and a \( 2-(24,8,77) \) design. Let \( x \) be a point of \( W \) (it matters not which one, since the automorphism group \( M_{24} \) of \( W \) acts transitively (in fact 5-transitively) on the point-set of \( W \)). Then \( W^#(5,x) \) is a \( 5-(23,7,3) \) design, \( W^#(4,x) \) is a \( 4-(23,7,20) \) design, \( W^#(3,x) \) is a \( 3-(23,7,105) \) design, and \( W^#(2,x) \) is a \( 2-(23,7,462) \) design.

**Example 3** Start with a projective plane \( P = (X,B) \) of order \( m \geq 2 \), a \( 2-(m^2 + m + 1, m + 1, 1) \) design. Now, given any \( x \in X \), construct \( P^#(2,x) \), which is a \( 2-(m^2 + m, m, m - 1) \) design.

### 4 Automorphism groups

The automorphism group of a \( t\)-wise balanced design \( D = (X,B) \), denoted \( \text{Aut}(D) \), is the group consisting of all the permutations of \( X \) which leave the block-multiset \( B \) invariant. We now investigate the relationship of the automorphism groups of \( D \) and \( D^*(t,k) \). For a block \( B \in B \), we let \( \text{mult}(B) \) denote its multiplicity in \( B \).

**Theorem 4.1** Let \( k \) be a positive integer, let \( D = (X,B) \) be a \( t-(v,\{k_1,k_2,\ldots,k_s\},\lambda) \) design, with all block-sizes \( k_i \) occurring in \( D \) and \( 1 \leq t \leq k \leq k_1 < k_2 < \cdots < k_s \), and let \( D^* = D^*(t,k) = (X,B^*) \) be the \( t\)-design obtained from the \(*\)-construction. Then

(i) \( \text{Aut}(D) \subseteq \text{Aut}(D^*) \);

(ii) if \( \lambda = 1 \) and \( t < k \), then \( \text{Aut}(D) = \text{Aut}(D^*) \).

**Proof.** (i) Let \( \alpha \in \text{Aut}(D) \). Let \( B^* \) be an arbitrary block in \( B^* \), hence there is a block \( B \in B \) which contains \( B^* \) as a \( k\)-subset. Suppose that \( \alpha(B) = C \) for some block \( C \in B \), and that \( \alpha(B^*) = C^* \). Then clearly \( C^* \) is a \( k\)-subset of \( C \), a block of \( B \), hence \( C^* \in B^* \). Now we must show that \( \text{mult}(C^*) = \text{mult}(B^*) \) (in \( B^* \)) to conclude that \( \alpha \in \text{Aut}(D^*) \).
Fix $i$. Let $B_1, B_2, \ldots, B_d$ be the distinct blocks of $B$ of size $k_i$ which contain $B^*$, and let $C_1, C_2, \ldots, C_e$ be the distinct blocks of $B$ of size $k_i$ which contain $C^*$. Now, because $\alpha \in \text{Aut}(D)$, we must have $d = e$ and for every $j$ with $1 \leq j \leq d$ there must exist a unique $j'$ with $1 \leq j' \leq d$ for which $\alpha(B_j) = C_{j'}$. Hence $\text{mult}(B_j) = \text{mult}(C_{j'})$ since $\alpha$ preserves block multiplicities.

Now let $f_i$ be the number of blocks (counting multiplicities) of $B$ of size $k_i$ which contain $B^*$, and let $g_i$ be the number of blocks (counting multiplicities) of $B$ of size $k_i$ which contain $C^*$. Then $g_i = \sum_{j'=1}^d \text{mult}(C_{j'}) = \sum_{j=1}^d \text{mult}(B_j) = f_i$, and so, in $B^*$, we have $\text{mult}(C^*) = \sum_{i=1}^s g_i m_i = \sum_{i=1}^s f_i m_i = \text{mult}(B^*)$ ($m_i$ defined in (1)), as required. Hence $\alpha \in \text{Aut}(D^*)$.

(ii) We first note that, because $\lambda = 1$, then $\text{mult}(B) = 1$ for every block $B \in B$. Secondly, if $R^*$ is an arbitrary block in $B^*$ then, again because $\lambda = 1$, there is a unique block $R \in B$, with $R^* \subseteq R$.

Now let $\gamma \in \text{Aut}(D^*)$. We must show that, for every block $B \in B$, we have $\gamma(B) \in B$. Then, from above, $\text{mult}(\gamma(B)) = 1 = \text{mult}(B)$, so $\gamma \in \text{Aut}(D)$. This will show that $\text{Aut}(D^*) \subseteq \text{Aut}(D)$; part (i) then gives the result.

Fix $i$. Let $B$ be an arbitrary block of $B$ of size $k_i$, and let $B^*$ be an arbitrary $k$-subset of $B$, and let $\gamma(B^*) = C^*$. Now, because $\gamma \in \text{Aut}(D^*)$, then $C^* \in B^*$. So, from above, there is a unique block $C \in B$, with $C^* \subseteq C$. We will show that $\gamma(B) = C$.

First we show that $\gamma(B) \subseteq C$. Suppose that $\gamma(B) \not\subseteq C$, then there is an element $x \in B \setminus B^*$ with $\gamma(x) \not\in C$. Let $D$ be a $(k - 1)$-subset of $B^* \subseteq B$, then $D^* = \{x\} \cup D$ is a $k$-subset of $B \in B$, so $D^* \in B^*$. Hence $E^* = \gamma(D^*) \in B^*$, and there is a block $E \in B$ with $E^* \subseteq E$. Now $E \neq C$ because $\gamma(x) \in E$ but $\gamma(x) \not\in C$. Hence $E$ and $C$ are distinct blocks of $B$. However, $\gamma(D) \subseteq E$, and $D \subseteq B^*$ so $\gamma(D) \subseteq \gamma(B^*) = C^* \subseteq C$. Now $t < k$ so $t \leq k - 1 = |\gamma(D)|$. Now let $T$ be any $t$-subset of $\gamma(D)$, then the distinct blocks $E$ and $C$ both contain $T$, a contradiction since $\lambda = 1$. Hence $\gamma(B) \subseteq C$.

To show that $C \subseteq \gamma(B)$ we show that $\gamma^{-1}(C) \subseteq B$ by noting that $\gamma^{-1}(C^*) = B^*$, and so the proof follows as above. Hence $\gamma(B) = C$ and, since $i$ was arbitrary, the result is proved.

Example 4 We take $D$ to be the 2-(11, $\{3, 5\}$, 1) design of Example 1. Then $|\text{Aut}(D)| = 120$; indeed $\text{Aut}(D)$ is isomorphic to $\text{Sym}(5)$, and acts naturally as this group on the subset $\{1, 2, 3, 4, 5\}$ of the point-set (checked using GAP [2] and its DESIGN package [5]).
(a) \( \mathcal{D}^*(2, 2) \) is the complete 2-(11, 2, 1) design. Hence Aut(\( \mathcal{D} \)) \( \subseteq \) Aut(\( \mathcal{D}^*(2, 2) \)) = Sym(11), illustrating Theorem 4.1(i), and also showing that if \( \lambda = 1 \) and \( t = k \) then Aut(\( \mathcal{D} \)) \( \neq \) Aut(\( \mathcal{D}^*(t, k) \)) is possible (see Theorem 4.1(ii)).

(b) \( \mathcal{D}^* = \mathcal{D}^*(2, 3) \) is a 2-(11, 3, 3) design with \(|\text{Aut}(\mathcal{D}^*)| = 120 \) (double checked with the DESIGN package). This illustrates Theorem 4.1(ii).

**Example 5** This example shows that if \( \lambda > 1 \) then Aut(\( \mathcal{D} \)) \( \neq \) Aut(\( \mathcal{D}^*(t, k) \)) is possible, even when \( t < k \). We apply the \#-construction to the projective plane \( \mathcal{P} \) of order 4, to obtain a 2-(20, 4, 3) design \( \mathcal{P}^\# = \mathcal{P}^\#(2, x) = (X, \mathcal{B}) \), which has a point-transitive automorphism group of order 5760. Then, we take \( x \in X \) and obtain a 2-(19, \{3, 4\}, 3) design \( \mathcal{D} = \mathcal{P}^\# \setminus x \) (using the notation of Section 3). (The choice of \( x \) does not affect the isomorphism class of \( \mathcal{D} \) since \( \mathcal{P}^\# \) is point-transitive). Finally, construct a 2-(19, 3, 6) design \( \mathcal{D}^* = \mathcal{D}^*(2, 3) \). It turns out that \(|\text{Aut}(\mathcal{D})| = 288 \), but \(|\text{Aut}(\mathcal{D}^*)| = 576 \). The construction of these designs and the determination of their automorphism groups was done using the DESIGN package.

**Example 6** The DESIGN package shows that, up to isomorphism, there are exactly four 2-(11, \{4, 5\}, 2) designs (not counting the unique 2-(11, 5, 2) design). These designs \( \mathcal{D} \) have automorphism groups of orders 6, 8, 12, and 120, as do the corresponding \( \mathcal{D}^*(2, 4) \), which are (believed to be new) 2-(11, 4, 6) designs. Note that these examples show that the converse of Theorem 4.1(ii) does not hold.

**References**


[4] D. A. Preece, A selection of BIBDs with repeated blocks, $r \leq 20$, \( \gcd(b, r, \lambda) = 1 \), preprint, 2003.