



Single-change circular covering designs

John P. McSorley*

Department of Mathematics, Southern Illinois University, Carbondale, IL 62901-4408, USA

Received 9 July 1997; revised 20 November 1997; accepted 3 August 1998

Abstract

A single-change circular covering design (scccd) based on the set $[v] = \{1, \dots, v\}$ with block size k is an ordered collection of b blocks, $\mathcal{B} = \{B_1, \dots, B_b\}$, each $B_i \subset [v]$, which obey: (1) each block differs from the previous block by a single element, as does the last from the first, and, (2) every pair of $[v]$ is covered by some block. The object is to minimize b for a fixed v and k . We present some minimal constructions of scccds for arbitrary v when $k = 2$ and 3, and for arbitrary k when $k + 1 \leq v \leq 2k$. Tight designs are those in which each pair is covered exactly once. Start-Finish arrays are used to construct tight designs when $v > 2k$; there are 2 non-isomorphic tight designs with $(v, k) = (9, 4)$, and 12 with $(v, k) = (10, 4)$. Some non-existence results for tight designs, and standardized, element-regular, perfect, and column-regular designs are considered. © 1999 Elsevier Science B.V. All rights reserved

Keywords: Single-change; Circular; Covering; Design; Tight

1. Definitions; notation; examples

A *single-change circular covering design* based on the set $[v] = \{1, \dots, v\}$ with block size k is an ordered collection of b blocks, $\mathcal{B} = \{B_1, \dots, B_b\}$, each an unordered subset of k distinct elements from $[v]$, which obey:

- (1) each block differs from the previous block by a single element, i.e., $|B_{i-1} \cap B_i| = k - 1$ for $i = 2, \dots, b$; and the last block, B_b , differs from the first, B_1 , by a single element, i.e., $|B_b \cap B_1| = k - 1$;
- (2) every (unordered) pair $\{x, y\}$ of $[v]$, with $x \neq y$, can be written as $\{e_i, z\}$ where $e_i \in B_i \setminus B_{i-1}$ and $z \in B_i$ for some $i = 2, \dots, b$, or as $\{e_1, z\}$ where $e_1 \in B_1 \setminus B_b$ and $z \in B_1$.

* E-mail: jpmcsorl@pfeinsil.math.siu.edu.

blocks	element introduced	pairs covered	
B_i	e_i	$\{e_i, z\}, z \in B_i$	
B_1 6 4 2	6	$\{6, 4\}$ $\{6, 2\}$	
B_2 6 3 2	3	$\{3, 6\}$ $\{3, 2\}$	
B_3 6 3 5	5	$\{5, 6\}$ $\{5, 3\}$	
B_4 6 3 1	1	$\{1, 6\}$ $\{1, 3\}$	1 2 3
B_5 4 3 1	4	$\{4, 3\}$ $\{4, 1\}$	2 3 4
B_6 4 5 1	5	$\{5, 4\}$ $\{5, 1\}$	3 4 5
B_7 2 5 1	2	$\{2, 5\}$ $\{2, 1\}$	4 5 1
B_8 2 4 1	4	$\{4, 2\}$ $\{4, 1\}$	5 1 2
\mathcal{E} , an economical scccd(6, 3, 8)		\mathcal{Y}_3 , a tight scccd(5, 3, 5)	
(a)		(b)	

Fig. 1. Examples: \mathcal{E} and \mathcal{Y}_3 .

For $i = 2, \dots, b$ we say that element e_i is *introduced* in block B_i , and the pairs $\{e_i, z\}$ where $z \in B_i$ are *covered* by B_i . Similarly, e_1 is introduced in B_1 and pairs $\{e_1, z\}$ where $z \in B_1$ are covered by B_1 . We also say that a pair is *covered* by \mathcal{B} if it is covered by some block in \mathcal{B} .

A single-change circular covering design is simply a *single-change covering design* (see [2,4]) in which a ‘single-change’ is also required between B_b and B_1 .

We denote a single-change circular covering design by scccd; and a scccd based on $[v]$ with block size k by scccd(v, k), or by scccd(v, k, b) if we wish to mention that it contains b blocks. For a fixed v and k , where $k \geq 2$ and $v \geq k + 1$, we denote by $b_*(v, k)$ the smallest b for which there exists a scccd(v, k, b), and we call such a scccd($v, k, b_*(v, k)$) *minimal*. We write our designs vertically as in [4].

Our first example is a scccd(6, 3, 8) labelled \mathcal{E} and shown in Fig. 1(a). In \mathcal{E} each pair, except $\{4, 1\}$, is covered once; $\{4, 1\}$ is covered twice, in B_5 and in B_8 . This is an *economical* design, see Section 5.

As the ordering of the elements in a block is immaterial, we often (but not always) leave a block’s unchanged elements in the same columns as in the previous block, see \mathcal{E} in Fig. 1(a); such a representation of \mathcal{E} is called *column-strict*. In block B_4 element 1 is introduced because $1 \notin B_3$ and element 6 is *changed* because $6 \notin B_5$.

Our second example is a scccd(5, 3, 5), see Fig. 1(b); we call this design \mathcal{Y}_3 , see Section 6. Here the 10 pairs from the set $\{1, 2, 3, 4, 5\}$ are each covered exactly once, 2 per block; we call such a design *tight*, see Section 5.

We generally use the notation $\mathcal{B} = \{B_1, \dots, B_b\}$ for an arbitrary scccd(v, k, b) and \mathcal{C} for an arbitrary tight scccd(v, k, b), often with $v > 2k$.

The main object of this paper is to study scccds, with special interest in the function $b_*(v, k)$ and in tight designs. We will see that the structure and construction of these designs are somewhat different from those of the single-change covering designs of [2,4].

2. Requirements for \mathcal{B} to form a single-change circular covering design

Let $\mathcal{B} = \{B_1, \dots, B_b\}$ be an ordered collection of b blocks; each B_i contains k distinct elements from $[v]$.

We say that \mathcal{B} has the *single-change circular property* if $|B_{i-1} \cap B_i| = k - 1$ for $i = 2, \dots, b$, and $|B_b \cap B_1| = k - 1$.

Lemma 2.1. *\mathcal{B} is a $\text{scccd}(v, k)$ if and only if*

- (i) *\mathcal{B} has the single-change circular property, and*
- (ii) *every pair of $[v]$ is in some block of \mathcal{B} , and*
- (iii) *no pair of $[v]$ is in every block of \mathcal{B} .*

Proof. First suppose that \mathcal{B} is a $\text{scccd}(v, k)$. Then (i) and (ii) are true by definition of a scccd . Now suppose that (iii) is false, and that the pair $\{x, y\}$ is in every block of \mathcal{B} . Then neither x nor y is introduced in any block, hence $\{x, y\}$ is not covered by \mathcal{B} , a contradiction because \mathcal{B} covers every pair.

Now suppose that \mathcal{B} satisfies (i), (ii), and (iii). As \mathcal{B} satisfies (i) we need only show that it covers every pair of $[v]$ to conclude that it is a $\text{scccd}(v, k)$. Now, by (ii), every pair $\{x, y\}$ lies in some block of \mathcal{B} and, by (iii), the pair $\{x, y\}$ is not in every block. So let $B_{i_1}, B_{i_2}, \dots, B_{i_t}$, where $t \leq b - 1$, be a sequence of consecutive blocks each containing $\{x, y\}$; the block immediately ‘before’ B_{i_1} , say B , does not contain $\{x, y\}$. So exactly one of x or y lies in B and the other does not. Hence, either y or x (respectively) is introduced in B_{i_1} , and so $\{x, y\}$ is covered there. Hence \mathcal{B} covers every pair and is a $\text{scccd}(v, k)$. \square

3. Designs with $k = 2$ and 3

$k = 2$: If a tight $\text{scccd}(v, 2, b)$ exists, then $b = v(v - 1)/2$, see Section 5. Now, given a tight $\text{scccd}(v, 2)$ with first block $B_1 = (1, 2)$ and last block $B_b = (v, 1)$, we may add on the v blocks as shown in Fig. 2(a) to obtain a tight $\text{scccd}(v + 1, 2, v(v + 1)/2)$ based on $[v + 1]$ with last block $(v + 1, 1)$.

Beginning with the tight $\text{scccd}(3, 2, 3)$ shown in Fig. 2(b) we can use this construction repeatedly to obtain a tight $\text{scccd}(v, 2)$ for any $v \geq 3$.

$k = 3$: If a tight $\text{scccd}(v, 3, b)$ exists, then $b = v(v - 1)/4$, see Section 5; and so $v \equiv 0$ or 1 (mod 4). Let $v \equiv 0$ (mod 4), and suppose that we have a tight $\text{scccd}(v, 3)$ with $B_1 = (1, 2, 3)$ and $B_b = (v, 1, 2)$ in which element 1 is introduced in B_b . We can then construct a tight $\text{scccd}(v + 4, 3, (v + 3)(v + 4)/4)$ based on $[v + 4]$ by altering B_b to $(v, v + 1, 2)$ and adding on the $2v + 3$ blocks as shown in Fig. 2(c). This new design has last block $(v + 4, 1, 2)$ in which 1 is introduced; so we can use this construction repeatedly to obtain a tight $\text{scccd}(v, 3)$ for any $v \equiv 0$ (mod 4) beginning with the tight $\text{scccd}(4, 3, 3)$ shown in Fig. 2(d).

				1 2 3
		1, 2, 3		1 4 3
		· · ·	1 2 3	5 4 3
		· · ·	4 2 3	5 4 2
		· · ·	4 1 2	5 1 2
1, 2	$\underline{v}, \underline{v+1}, \underline{2}$			
· ·	$v, v+2, 2$	a tight		a tight
· ·	$v, v+2, v+4$	scccd(4, 3, 3)		scccd(5, 3, 5)
· ·	$v, v+2, v+3$			
$\underline{v}, \underline{1}$	$v, 1, v+3$	(d)		(e)
$v, v+1$	$v+1, 1, v+3$			
$v-1, v+1$	$v+1, 1, v+2$			1 2 3
· ·	$v+1, v-1, v+2$			1 7 3
· ·	· · ·			1 7 5
· ·	· · ·	1 2 3		1 4 5
1, v+1	· · ·	4 2 3		1 4 6
	$v+1, 3, v+2$	4 5 3		3 4 6
(a)	$v+1, 3, v+4$	4 6 3		3 5 6
	$v+3, 3, v+4$	4 6 1		2 5 6
1 2	· · ·	5 6 1		2 7 6
3 2	· · ·	5 6 2		2 7 4
3 1	· · ·	6 1 2		7 1 2
	$v+3, v-1, v+4$			
a tight	$v+3, 2, v+4$	an economical		an economical
scccd(3, 2, 3)	$v+4, 1, 2$	scccd(6, 3, 8)		scccd(7, 3, 11)
(b)	(c)	(f)		(g)

Fig. 2. Starter designs and additions for $k=2$ and 3.

We can also construct a tight $scccd(v, 3)$ when $v \equiv 1 \pmod{4}$ for any $v \geq 5$ starting with the tight $scccd(5, 3, 5)$ shown in Fig. 2(e); and an economical $scccd(v, 3, \lceil v(v-1)/4 \rceil)$ for $v \equiv 2$ or $3 \pmod{4}$ for any $v \geq 6$ starting with the economical $scccd$ s in (f) or (g) respectively.

- Theorem 3.1.** (i) *A tight $scccd(v, 2)$ exists for all $v \geq 3$;*
 (ii) *a tight $scccd(v, 3)$ exists for all $v \equiv 0$ or $1 \pmod{4}$, $v \geq 4$;*
 (iii) *an economical $scccd(v, 3)$ exists for all $v \equiv 2$ or $3 \pmod{4}$, $v \geq 6$.*

4. Standardized forms; isomorphisms; reverses

- A $scccd(v, k, b)$ is *standardized* or in *standardized form* (see Section 1 of [2]) if:
- (1) the elements of the first block are $1, 2, \dots, k$ in that order;
 - (2) the other elements are introduced initially in the order $k+1, k+2, \dots, v$;
 - (3) the elements of the first block are changed initially in the order $k, k-1, \dots, 2, 1$ (if our $scccd(v, k, b)$ has one element, say element 1, in every block, then the elements of the first block are changed initially in the order $k, k-1, \dots, 2$);

		<i>in ch</i>		<i>in ch</i>						
		e_i	e'_i	e_i	e'_i					
B_1	6 4 2	L_1	1 2 3	1	$\overline{3}$	L_1	1 2 3	2	$\overline{3}$	2 4 1
B_2	6 3 2	L_2	1 2 4	$\overline{4}$	$\overline{2}$	L_2	1 2 4	$\overline{4}$	$\overline{2}$	2 5 1
B_3	6 3 5	L_3	1 5 4	$\overline{5}$	5	L_3	1 5 4	$\overline{5}$	$\overline{1}$	4 5 1
B_4	6 3 1	L_4	1 6 4	$\overline{6}$	$\overline{1}$	L_4	6 5 4	$\overline{6}$	6	4 3 1
B_5	4 3 1	L_5	3 6 4	3	4	L_5	3 5 4	3	4	6 3 1
B_6	4 5 1	L_6	3 6 5	5	3	L_6	3 5 2	2	5	6 3 5
B_7	2 5 1	L_7	2 6 5	2	5	L_7	3 6 2	6	2	6 3 2
B_8	2 4 1	L_8	2 6 3	3	6	L_8	3 6 1	1	6	6 4 2
\mathcal{E}		$\text{sf}(\mathcal{E}) = \text{sf}(\mathcal{E}_1)$		$\text{rsf}(\mathcal{E}) = \text{sf}(\mathcal{E}_8)$		$\text{rev}(\mathcal{E})$				
(a)		(b)		(c)		(d)				

Fig. 3. The economical scccd(6, 3, 8), \mathcal{E} , its standardized form, its representative standardized form, and its reverse.

(4) beginning at the second block, a block’s unchanged elements are in the same columns as in the previous block (i.e., it is column-strict).

Given any scccd(v, k, b), \mathcal{B} , that satisfies (4) above, in order to change it to its standardized form we need to apply a permutation of $[v]$ to it, followed by a permutation of its columns. For example, if we apply the permutation (1, 6)(3, 4) to $\mathcal{E} = \{B_1, B_2, \dots, B_8\}$ shown in Fig. 3(a), and then permute its 2nd and 3rd columns, we arrive at its standardized form shown in (b), with blocks labelled L_i .

A *cyclic shift* of the ordered blocks $\mathcal{B} = \{B_1, B_2, \dots, B_b\}$ is one of the following rearrangements:

$$\begin{aligned}
 \mathcal{B} = \mathcal{B}_1 &= \{B_1, B_2, \dots, B_b\} \\
 \mathcal{B}_2 &= \{B_2, B_3, \dots, B_b, B_1\}, \\
 &\vdots \\
 \mathcal{B}_i &= \{B_i, \dots, B_b, B_1, \dots, B_{i-1}\}, \\
 &\vdots \\
 \mathcal{B}_b &= \{B_b, B_1, \dots, B_{b-1}\}.
 \end{aligned}$$

The block arrangement \mathcal{B}_i is called the *ith cyclic shift* of \mathcal{B} . For each i the first block in \mathcal{B}_i is B_i .

Each scccd $\mathcal{B} = \{B_1, B_2, \dots, B_b\}$ has b standardized forms, one for each cyclic shift \mathcal{B}_i . Let $\text{sf}(\mathcal{B}_i)$ denote the standardized form of \mathcal{B}_i for each $i = 1, \dots, b$.

The scccd(5, 3, 5), \mathcal{Y}_3 , of Fig. 1(b) has each of its 5 standardized forms identical (shown in Fig. 15(d)); but \mathcal{E} in Fig. 3(a) has each of its 8 standardized forms different. If a design \mathcal{B} does not have all of its standardized forms identical, one of these forms can usefully be chosen as the *representative standardized form*, $\text{rsf}(\mathcal{B})$. In order to do this for an arbitrary $\mathcal{B} = \{B_1, B_2, \dots, B_b\}$, we will presently define, for each i , four

finite sequences associated with $\text{sf}(\mathcal{B}_i)$, namely S_1, S_2, S_3 , and S_4 . (In general, the sequences S_1, S_2, S_3 , and S_4 will be different for each i , but we choose this notation for simplicity.) We then consider the ordered set $\{S_1, S_2, S_3, S_4\}$. So each $\text{sf}(\mathcal{B}_i)$ gives us an ordered set of four sequences. Next, we order these ordered sets of four sequences according to the criteria below, and choose the ‘least’ in this ordering. Suppose this least ordered set comes from $\text{sf}(\mathcal{B}_i)$; then $\text{sf}(\mathcal{B}_i)$ is taken as $\text{rsf}(\mathcal{B})$.

Again for simplicity, we use L_1, L_2, \dots, L_b , to denote the b blocks of $\text{sf}(\mathcal{B}_i)$, for every $i = 1, \dots, b$, even though generally the blocks differ for each i . So $\text{sf}(\mathcal{B}_i) = \{L_1, L_2, \dots, L_b\}$ where $L_1 = (1, 2, \dots, k)$ and $L_2 = (1, 2, \dots, k + 1)$.

The sequences S_1 and S_2 are sequences of distinct blocks from $\{L_1, L_2, \dots, L_b\}$; and S_3 and S_4 are sequences of not necessarily distinct elements from $[v]$. For a fixed $\text{sf}(\mathcal{B}_i)$ we define S_1 with reference to (2) above. For $t = 1, \dots, v - k$, the t th member of S_1 is the block into which element $k + t$ is initially introduced. Thus S_1 begins with L_2 . Sequence S_2 (see (3) above) is a sequence of blocks beginning at L_1 . For $t = 1, \dots, k$, the t th member of S_2 is the first block from which element $k - t + 1$ is initially changed (t stops at $k - 1$ if \mathcal{B} has element 1 in every block).

Just as for $i = 1, \dots, b$, the element e_i is introduced into block L_i , let e'_i be the element changed from L_i . So we have $e'_i \in L_i \setminus L_{i+1}$ for $i = 1, \dots, b - 1$, and $e'_b \in L_b \setminus L_1$. Now define sequence $S_3 = \{e_i: L_i \notin S_1\}$, arranged with increasing i ; so, e_1 is its first member. Similarly, we define $S_4 = \{e'_i: L_i \notin S_2\}$, arranged with increasing i .

For an example see $\text{sf}(\mathcal{E}_1)$ in Fig. 3(b), where the column of introduced elements, e_i , is labelled ‘in’, and the column of changed elements, e'_i , is labelled ‘ch’. We have $S_1 = \{L_2, L_3, L_4\}$, $S_2 = \{L_1, L_2, L_4\}$, $S_3 = \{1, 3, 5, 2, 3\}$, and $S_4 = \{5, 4, 3, 5, 6\}$.

Thus each $\text{sf}(\mathcal{B}_i)$ gives us an ordered set of four sequences. We now order these ordered sets of four sequences by, first of all, lexicographically ordering their first elements, the S_1 sequences, according to the rule: $L_r < L_s$ if and only if $r < s$, and choosing the set(s) whose S_1 sequence is the first (i.e., the least) in this list. If two (or more) sets have identical S_1 sequences, we choose the one with the least S_2 sequence using the same ordering. If two (or more) sets have identical S_1 and S_2 sequences then we compare their S_3 sequences and order them lexicographically using the natural $<$ ordering on $[v]$, and choose the least. If still identical, we compare their S_4 sequences, with the $<$ ordering, and choose the least. Two such sets with identical S_1, S_2, S_3 , and S_4 sequences can easily be shown to correspond to standardized forms that are identical.

By this process we arrive at the particular $\text{sf}(\mathcal{B}_i)$ with the least set of sequences according to our lexicographic orderings; we take this particular standardized form as the representative standardized form of \mathcal{B} , $\text{rsf}(\mathcal{B})$. For example, $\text{rsf}(\mathcal{E})$ shown in Fig. 3(c), is $\text{sf}(\mathcal{E}_8)$; and $\text{rsf}(\mathcal{Y}_3)$ is shown in Fig. 15(d).

Two $\text{scccd}(v, k, b)$ s \mathcal{B} and \mathcal{B}' are *isomorphic*, ($\mathcal{B} \cong \mathcal{B}'$), if we can apply a permutation of $[v]$ combined with a cyclic shift of the blocks of \mathcal{B} to obtain \mathcal{B}' . Similarly, an *automorphism* of a $\text{scccd}(v, k, b)$ \mathcal{B} is a permutation of $[v]$ which, when applied to \mathcal{B} , produces a cyclic shift of \mathcal{B} . For example, the permutation $(1, 2, 3, 4, 5)$ is an automorphism of \mathcal{Y}_3 .

Theorem 4.1. *Let \mathcal{B} and \mathcal{B}' be two $\text{scccd}(v, k, b)$ s. Then $\mathcal{B} \cong \mathcal{B}'$ if and only if $\text{rsf}(\mathcal{B}) = \text{rsf}(\mathcal{B}')$.*

Proof. Suppose $\mathcal{B} \cong \mathcal{B}'$, then, for any $i \in \{1, \dots, b\}$, there exists a $j \in \{1, \dots, b\}$ such that \mathcal{B}_i , the i th cyclic shift of \mathcal{B} , can be changed into \mathcal{B}'_j using only a permutation on v , i.e., with no cyclic shift of the blocks of \mathcal{B}_i . So $\text{sf}(\mathcal{B}_i) = \text{sf}(\mathcal{B}'_j)$, and so $\{\text{sf}(\mathcal{B}_1), \dots, \text{sf}(\mathcal{B}_b)\} \subseteq \{\text{sf}(\mathcal{B}'_1), \dots, \text{sf}(\mathcal{B}'_b)\}$. Similarly, $\{\text{sf}(\mathcal{B}'_1), \dots, \text{sf}(\mathcal{B}'_b)\} \subseteq \{\text{sf}(\mathcal{B}_1), \dots, \text{sf}(\mathcal{B}_b)\}$ and this gives: $\{\text{sf}(\mathcal{B}_1), \dots, \text{sf}(\mathcal{B}_b)\} = \{\text{sf}(\mathcal{B}'_1), \dots, \text{sf}(\mathcal{B}'_b)\}$. Hence $\text{rsf}(\mathcal{B}) = \text{rsf}(\mathcal{B}')$.

Now suppose that $\text{rsf}(\mathcal{B}) = \text{rsf}(\mathcal{B}')$, then $\mathcal{B} \cong \text{rsf}(\mathcal{B}) = \text{rsf}(\mathcal{B}') \cong \mathcal{B}'$, as required. \square

The reverse of the $\text{scccd } \mathcal{B} = \{B_1, B_2, \dots, B_b\}$ is the scccd obtained by reversing the order of the blocks of \mathcal{B} ; we denote this design by $\text{rev}(\mathcal{B}) = \{B_b, B_{b-1}, \dots, B_1\}$. The reverse of \mathcal{E} is shown in Fig. 3(d). If $\mathcal{B} \cong \text{rev}(\mathcal{B})$ we say that \mathcal{B} is *self-reverse*.

5. Lower bounds on $b_*(v, k)$; constructions of designs for $k + 1 \leq v \leq 2k$

Lemma 5.1. *For $v \geq 4$ and $k \geq 3$, the value of $b_*(v, k)$, the minimum number of blocks in a $\text{scccd}(v, k)$, satisfies*

$$b_*(v, k) \geq \max \left\{ v - 1, \left\lceil \frac{v(v - 1)}{2(k - 1)} \right\rceil \right\}.$$

Proof. In a $\text{scccd}(v, k)$ exactly one element is introduced per block, so, if $b_*(v, k) < v - 1$, then at most $v - 2$ distinct elements are introduced. Hence at least 2 distinct elements are not introduced, and the pair containing them is not covered, a contradiction. So $b_*(v, k) \geq v - 1$.

A $\text{scccd}(v, k)$ must cover all $v(v - 1)/2$ pairs of $[v]$, and $k - 1$ pairs are covered per block. Thus $b_*(v, k) \cdot (k - 1) \geq v(v - 1)/2$, and so the result. \square

Corollary 5.2. *For $v \geq 4$ and $k \geq 3$ we have*

$$b_*(v, k) \geq \begin{cases} v - 1 & \text{for } k + 1 \leq v \leq 2k - 2, \\ \left\lceil \frac{v(v - 1)}{2(k - 1)} \right\rceil & \text{for } v > 2k - 2. \end{cases} \tag{1}$$

If $b_*(v, k)$ satisfies equation (1) with equality, then we say that the corresponding $\text{scccd}(v, k)$ is *economical*; except that if $b_*(v, k) = v(v - 1)/[2(k - 1)]$ then the corresponding $\text{scccd}(v, k)$ is *tight*. An economical and a tight scccd are shown in Fig. 1.

Consider the first case of Corollary 5.2. If a $\text{scccd}(v, k)$ with $k + 1 \leq v \leq 2k - 2$ and with the minimal number of $v - 1$ blocks exists, then a total of $v - 1$ elements are introduced in the design; from the proof of Lemma 5.1 these $v - 1$ elements are distinct. Hence, without loss of generality, in such a design the elements $1, \dots, v - 1$

are each introduced exactly once; the element v is not introduced, and so appears in every block. The following construction satisfies these requirements:

Arrange the elements of $[v-1]$ in a circle and call this arrangement the *circular* $[v-1]$. Now, for $1 \leq i \leq v-1$, let A_i be the block beginning at i and containing $k-1$ consecutive elements taken clockwise from the circle; i.e., $A_i = (i, i+1, \dots, i+k-2)$, where addition is taken modulo $v-1$ with $v-1$ replacing 0. Now let $B_i = A_i \cup \{v\}$ and $\mathcal{B} = \{B_1, \dots, B_{v-1}\}$.

Theorem 5.3. *For $k \geq 3$ and $k+1 \leq v \leq 2k-2$ the blocks $\mathcal{B} = \{B_1, \dots, B_{v-1}\}$ where $B_i = A_i \cup \{v\}$ form an economical scccd($v, k, v-1$).*

Proof. We need only show that \mathcal{B} is a scccd(v, k). It then follows, since $|\mathcal{B}| = v-1$, that it is an economical scccd($v, k, v-1$). We show that \mathcal{B} satisfies (i)–(iii) of Lemma 2.1.

(i) This is clear from the definition of \mathcal{B} .

(ii) Now $B_i = (i, i+1, \dots, i+k-2) \cup \{v\}$, where addition is taken modulo $v-1$ with $v-1$ replacing 0. So element $i+k-2$ is introduced in B_i , i.e., element i is introduced in B_{i-k+2} . Hence, for $1 \leq i \leq v-1$, pair $\{i, v\}$ is covered by B_{i-k+2} . This deals with pairs that contain v .

Now consider the pair $\{i, j\}$ where $1 \leq i < j \leq v-1$ and let $v-1$ be odd. The pair $\{i, j\}$ can be covered by a ‘run’ of elements starting either at i or at j and containing $v/2$ consecutive elements of the circular $[v-1]$. But $v \leq 2k-2$ and so $v/2 \leq k-1$, i.e., such a run is contained in A_i or A_j , so in B_i or B_j . Hence $\{i, j\}$ is covered by \mathcal{B} . A similar argument works when $v-1$ is even.

(iii) Let $\{i, j\}$ be in every block; then, without loss of generality, we have $1 \leq i \leq v-1$. But i is introduced in B_{i-k+2} , and so cannot be in the previous block, a contradiction. Hence no pair is in every block.

Thus \mathcal{B} is an economical scccd($v, k, v-1$). \square

An economical scccd(5, 4, 4) constructed using Theorem 5.3 is shown in Fig. 4(a).

For a fixed $k \geq 3$ the only tight designs amongst the economical scccd($v, k, v-1$) with $k+1 \leq v \leq 2k-2$ occur when $v(v-1)/2 = (k-1)(v-1)$, i.e., when $v = 2k-2$. So Theorem 5.3 with $v = 2k-2$ yields tight designs:

Theorem 5.4. *For $k \geq 3$ and $v = 2k-2$ the blocks $\mathcal{B} = \{B_1, \dots, B_{2k-3}\}$ where $B_i = A_i \cup \{2k-2\}$ form a tight scccd($2k-2, k, 2k-3$).*

See Fig. 4(b) for a tight scccd(6, 4, 5) constructed using Theorem 5.4.

So far, for a fixed $k \geq 3$, we have constructed economical scccds when $k+1 \leq v \leq 2k-2$. We now consider $v = 2k-1$ and $v = 2k$, so we are in the second case of Corollary 5.2 in which a scccd has $b \geq \lceil v(v-1)/[2(k-1)] \rceil$ blocks.

$v = 2k-1$: Here $b \geq 2k-1$. Consider the circular $[2k-1]$. For $1 \leq i \leq 2k-1$ let $B_i = (i, i+1, \dots, i+k-1)$, where addition is taken modulo $2k-1$ with $2k-1$ replacing

		B_1 1 2 3 4
		B_2 2 3 4 5
B_1 1 2 3 5	B_1 1 2 3 6	B_3 3 4 5 6
B_2 2 3 4 5	B_2 2 3 4 6	B_4 4 5 6 7
B_3 3 4 1 5	B_3 3 4 5 6	B_5 5 6 7 1
B_4 4 1 2 5	B_4 4 5 1 6	B_6 6 7 1 2
	B_5 5 1 2 6	B_7 7 1 2 3
an economical scccd(5, 4, 4)	a tight scccd(6, 4, 5)	a tight scccd(7, 4, 7)
(a)	(b)	(c)

Fig. 4. Designs with $k = 4$ constructed using Theorems 5.3–5.5.

B_1	1,	2,	3,	...	k		
B_2	2,	3,	4,	...	$k + 1$		
⋮	⋮	⋮	⋮	⋮	⋮		
⋮	⋮	⋮	⋮	⋮	⋮		
B_{k-1}	$k - 1,$	$k,$	$k + 1,$...	$2k - 2$	B_1 1 2 3 4	
C	$2k,$	$k,$	$k + 1,$...	$2k - 2$	B_2 2 3 4 5	
B_k	$k,$	$k + 1,$	$k + 2,$...	$2k - 1$	B_3 3 4 5 6	
⋮	⋮	⋮	⋮	⋮	⋮	C 8 4 5 6	
⋮	⋮	⋮	⋮	⋮	⋮	B_4 4 5 6 7	
⋮	⋮	⋮	⋮	⋮	⋮	B_5 5 6 7 1	
B_{2k-1}	$2k - 1,$	1,	2,	...	$k - 1$	B_6 6 7 1 2	
C'	$2k - 1,$	$2k,$	2,	...	$k - 1$	B_7 7 1 2 3	
C''	$2k,$	1,	2,	...	$k - 1$	C' 7 8 2 3	
	scccd($2k, k, 2k + 2$)					C'' 8 1 2 3	
	(a)					scccd(8, 4, 10)	(b)

Fig. 5. The economical scccd($2k, k, 2k + 2$) of Theorem 5.6 (tight only for $k = 2$), this design for $k = 4$.

0. Then we have the following result. This construction and the next also work for $k = 2$.

Theorem 5.5. For $k \geq 2$ and $v = 2k - 1$ the blocks $\mathcal{B} = \{B_1, \dots, B_{2k-1}\}$ where $B_i = (i, i + 1, \dots, i + k - 1)$ form a tight scccd($2k - 1, k, 2k - 1$).

See Fig. 4(c) for a tight scccd(7, 4, 7) constructed using Theorem 5.5.

$v = 2k$: Here $b \geq 2k + 2$. To construct an economical scccd($2k, k, 2k + 2$) based on [2k] we take the blocks $\{B_1, \dots, B_{2k-1}\}$ of the scccd($2k - 1, k, 2k - 1$) in Theorem 5.5 above and add on finally 3 new blocks: C between B_{k-1} and B_k , then C' after B_{2k-1} , and C'' after C' , i.e., between C' and B_1 ; see Fig. 5(a).

As the elements introduced in B_1 and B_k are unchanged, the pairs covered by these blocks are unchanged, and pairs containing the element $2k$ are covered in the 3 new blocks. Using Lemma 2.1 it is clear this is an economical $\text{scccd}(2k, k, 2k + 2)$. This economical design is tight only when $2k + 2 = 2k(2k - 1)/[2(k - 1)]$, i.e., when $k = 2$.

Theorem 5.6. For $k \geq 2$ and $v = 2k$ the blocks $\mathcal{B} = \{B_1, B_2, \dots, B_{k-1}, C, B_k, \dots, B_{2k-1}, C', C''\}$ form an economical $\text{scccd}(2k, k, 2k + 2)$, which is tight when $k = 2$. \square

See Fig. 5(b) for an economical $\text{scccd}(8, 4, 10)$ constructed using Theorem 5.6. The theorem below summarizes this section, cf., Corollary 5.1 of [4].

Theorem 5.7. For $k \geq 3$ we have

$$b_*(v, k) = \begin{cases} v - 1 & \text{for } k + 1 \leq v \leq 2k - 2; \\ v & \text{for } v = 2k - 1; \\ v + 2 & \text{for } v = 2k. \end{cases}$$

So, in this section, we have constructed scccd s with a minimal number of blocks for $k \geq 3$ and $k + 1 \leq v \leq 2k$, and have given examples for $k = 4$. From now on we concentrate mainly on tight designs.

6. Some families of tight designs

Tight scccd s are of special interest; they are analogous to *tight* single-change covering designs, see [2].

So far we have three infinite families of tight scccd s with a fixed k , see Theorem 3.1:

- (i) $\text{scccd}(v, 2)$ for all $v \geq 3$;
- (ii) $\text{scccd}(v, 3)$ for all $v \equiv 0 \pmod{4}$, $v \geq 4$;
- (iii) $\text{scccd}(v, 3)$ for all $v \equiv 1 \pmod{4}$, $v \geq 5$.

For the tight designs of [2] infinite families are not known to exist with k variable, except in the case $v = k$, here we have two infinite families with k variable:

- (iv) $\mathfrak{F} = \{\mathcal{X}_k: \mathcal{X}_k \text{ is the tight } \text{scccd}(2k - 2, k, 2k - 3) \text{ from Theorem 5.4, } k \geq 3\}$;
- (v) $\mathfrak{G} = \{\mathcal{Y}_k: \mathcal{Y}_k \text{ is the tight } \text{scccd}(2k - 1, k, 2k - 1) \text{ from Theorem 5.5, } k \geq 2\}$.

(Note that \mathcal{Y}_3 is shown in Fig. 1(b).)

If a scccd has the same parameters as a member of one of these two families it is isomorphic to that member:

Theorem 6.1. For a fixed $k \geq 3$ let \mathcal{X} be a $\text{scccd}(2k - 2, k, 2k - 3)$. Then $\mathcal{X} \cong \mathcal{X}_k \in \mathfrak{F}$.

Proof. The parameters of \mathcal{X} indicate that it is tight. Let \mathcal{X} be based on $[v]$ where $v = 2k - 2$.

Now \mathcal{X} has $v - 1$ blocks, so, from the comments preceding Theorem 5.3, it contains some element, say v , in every block. Any other element $1, \dots, v - 1$ is introduced exactly

once and remains in $k - 1$ successive blocks because it must appear in $v - 1 = 2k - 3$ covered pairs; it appears in $k - 1$ covered pairs in its first block and in one covered pair in each of its $k - 2$ successive blocks.

Hence, up to a permutation of $[v]$, we may construct \mathcal{X} as follows: first, put v in every block; then, for $i = 1, \dots, v - 1$, introduce i in B_i , and leave it there for $k - 1$ successive blocks. Then block B_{k-1} will be then block B_{k-1} will be $(1, 2, \dots, k - 1, v)$ and a cyclic shift of the blocks making this the first block will produce \mathcal{X}_k . Hence $\mathcal{X} \cong \mathcal{X}_k$. \square

Similarly for the family \mathfrak{G} .

Theorem 6.2. *For a fixed $k \geq 2$ let \mathcal{Y} be a $\text{scccd}(2k - 1, k, 2k - 1)$. Then $\mathcal{Y} \cong \mathcal{Y}_k \in \mathfrak{G}$.*

7. The numbers t_j and f_j for a tight design

Now we consider constructions of tight scccd s for $v > 2k$. First we need some preparatory material, much of which is similar to that of Section 4 in [2].

In an arbitrary tight $\text{scccd}(v, k)$ let $T_j \subseteq [v]$ denote the set of elements which are introduced j times, $j \geq 0$, and let $t_j = |T_j|$.

Now consider t_0 , the number of elements not introduced. From the proof of Lemma 5.1 we must have $t_0 = 0$ or 1. Let \mathcal{Z} be a tight $\text{scccd}(v, k)$ in which $t_0 = 1$, and call v the element not introduced. Any other element $z = 1, \dots, v - 1$ is introduced exactly once, for, if some z is introduced twice or more, then the pair $\{z, v\}$ is covered twice or more; a contradiction because \mathcal{Z} is tight. Hence, each of $1, \dots, v - 1$ is introduced exactly once, and $b = v - 1 = v(v - 1)/[2(k - 1)]$, i.e., $v = 2k - 2$. Thus \mathcal{Z} is a tight $\text{scccd}(2k - 2, k, 2k - 3)$ and, by Theorem 6.1, lies in \mathfrak{F} .

Thus, all tight scccd s with $t_0 = 1$ are known; they are members of \mathfrak{F} with $v = 2k - 2$. As we are interested in tight designs with $v > 2k$, we assume that $t_0 = 0$ and restrict ourselves to $j \geq 1$.

Let \mathcal{C} be an arbitrary tight $\text{scccd}(v, k, b)$, and, for any $x \in [v]$, let $f_{\{x\}}$ denote the number of blocks that contain x .

Let $x \in T_j$. Each time x is introduced $k - 1$ pairs containing x are covered. There are $v - 1$ pairs containing x to be covered, so we may let $j \leq \lfloor (v - 1)/(k - 1) \rfloor = A$ because $t_j = 0$ for $j > A$.

Now $x \in T_j$, so there are j blocks in which x is introduced; $(k - 1)$ pairs containing x are covered in each of these blocks. There are $f_{\{x\}} - j$ blocks that contain x but in which it is not introduced; only 1 pair containing x is covered in each of these blocks. This gives $v - 1 = j(k - 1) + (f_{\{x\}} - j)1$, i.e., $f_{\{x\}} = (v - 1) - j(k - 2)$. Hence $f_{\{x\}}$ is constant on T_j , so we let $f_j = (v - 1) - j(k - 2)$ be the number of blocks that contain any fixed element from T_j . In particular, $f_1 = v - k + 1$.

We have, for $1 \leq j \leq A$,

$$A = \left\lfloor \frac{v-1}{k-1} \right\rfloor, \quad \sum_{j=1}^A t_j = v, \quad \sum_{j=1}^A j t_j = b, \quad f_j = (v-1) - j(k-2). \quad (2)$$

Some further properties of the numbers t_j and f_j are given below.

Lemma 7.1. For $k > 2$ and any j satisfying $1 \leq j \leq A$, we have

- (i) $f_A < f_{A-1} < \dots < f_2 < f_1$;
- (ii) $f_j \geq j$, and, if $f_j < j$, then $t_l = 0$ for all $l \geq j$;
- (iii) if $f_j = j$, then $t_j = 0$ or 1;
- (iv) if $f_j = j + 1$, then $t_j = 0, 1$, or 2.

Proof. (i) Clear because $f_j = (v-1) - j(k-2)$ and $k > 2$.

(ii) For each of the j times when $x \in T_j$ is introduced it appears in at least 1 block, so $f_j \geq j$. So, clearly, if $f_j < j$ then $t_j = 0$. Also, for any $l > j$, we have $f_l < f_j < j < l$ by (i), hence, $t_l = 0$.

(iii) For a fixed j , suppose $f_j = j$ but $t_j = |T_j| \geq 2$, and let x and $y \in T_j$. Now, because x is introduced j times and appears in j blocks, each time it is introduced it must be immediately removed; similarly for y . But pair $\{x, y\}$ must appear in some block, hence both x and y must be introduced in this block, a contradiction.

(iv) Now suppose $f_j = j + 1$ but $t_j \geq 3$, and let x, y , and $z \in T_j$. By the pigeonhole principle, for one of the j times when x is introduced it must stay for 2 successive blocks; similarly for y and z . So the configurations $\begin{smallmatrix} x \\ x \end{smallmatrix}$, $\begin{smallmatrix} y \\ y \end{smallmatrix}$, and $\begin{smallmatrix} z \\ z \end{smallmatrix}$ occur once each. In order to cover the pairs $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$ we must have the arrangement $\begin{smallmatrix} xz \\ xy \\ zy \end{smallmatrix}$ in the design, i.e., $b = 3$. But there are only 2 tight scccds with $b = 3$: one is $\mathcal{X}_3 \in \mathfrak{F}$, which we have excluded; the other is $\mathcal{Y}_2 \in \mathfrak{G}$, which is also excluded because this design has $k = 2$ and we are restricted to $k > 2$. \square

8. Start–Finish arrays for a tight design; Criteria for a tight design with $v > 2k$

This section is mainly concerned with the subset T_1 of elements introduced exactly once in \mathcal{C} , an arbitrary tight scccd(v, k).

Suppose $T_i \neq \emptyset$ and let $x \in T_1$, and consider the $f_{\{x\}} = f_1 = v - k + 1$ successive blocks in \mathcal{C} which contain x ; call these blocks $\mathcal{B}_x = \{B_{x,1}, \dots, B_{x,f_1}\}$, see Fig. 6(a). We may write x as the leftmost element in each of these blocks. We say that x *starts*, S , in $B_{x,1}$ (i.e., is introduced there), and *finishes*, F , in B_{x,f_1} , see Fig. 6(a). Call this occurrence of x in f_1 successive blocks the *run* containing x . Now let y be some other element in T_1 . The pair $\{x, y\}$ must be covered in \mathcal{B}_x and so, because $f_{\{y\}} = f_1$ also, either the S or the F of y must appear in \mathcal{B}_x . Similarly for all the other elements in T_1 .

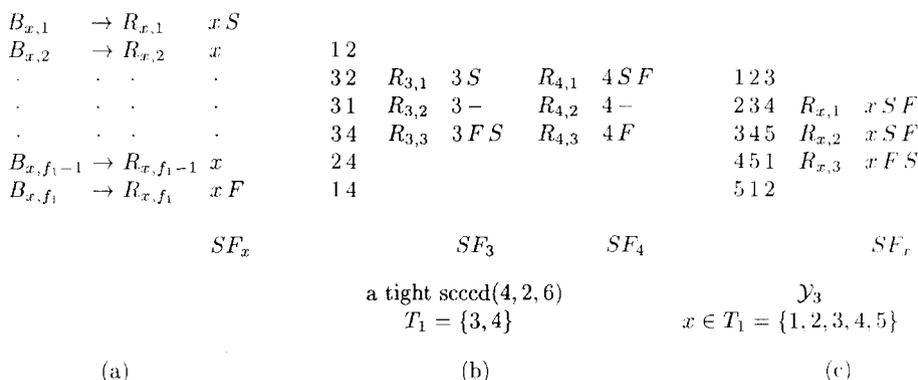


Fig. 6. SF -arrays and examples.

Thus, as we run through the elements in T_1 , each adds its S or its F to the array in the final column of (a). We call this the *Start–Finish array*, or SF -array, for element x , and denote it by SF_x . Call the S s and F s *symbols*.

In Fig. 6(a), block $B_{x,i}$ gives rise to the i th row, $R_{x,i}$, of SF_x , which contains x and, perhaps, some symbols we write $B_{x,i} \rightarrow R_{x,i}$. If a row contains no symbols it is *empty* ($-$).

Fig. 6(b) shows a tight scccd(4, 2, 6) with elements 3 and $4 \in T_1$. The SF -array for element 3 , SF_3 , is shown first; here the S in $R_{3,3}$ appears because element 4 starts in $B_{3,3}$. The array SF_4 is shown next; here $F \in R_{4,1}$ because element 3 finishes in $B_{4,1}$. Fig. 6(c) shows the tight scccd(5, 3, 5), \mathcal{Y}_3 ; it has $T_1 = \{1, 2, 3, 4, 5\}$ and, for all $x \in T_1$, the arrays SF_x are identical.

The main idea of this section is to place restrictions on the structure of an SF -array of a tight scccd(v, k) when $v > 2k$. In the following section we ‘extend’ these SF -arrays to tight scccds for $(v, k) = (9, 4)$ and $(10, 4)$.

So, let us assume that a tight scccd(v, k, b), \mathcal{C} , exists whose set of elements introduced exactly once is T_1 , and let $x \in T_1$. Then, using the following 10 Observations, we will establish 10 Criteria that SF_x must satisfy. In the Observations, R denotes an arbitrary row of SF_x , with corresponding block in \mathcal{B}_x denoted by B_R .

Observations

(1) Each row R of SF_x contains at most one S and at most one F . For suppose R contains two or more S s say, then two or more elements are introduced in B_R , a contradiction. Similarly for the F s because the reverse of a scccd is again a scccd. Clearly the order of the symbols in a row does not matter.

(2) The number of empty rows between any row containing F and the next (different) row containing S as we go down SF_x is $\geq b - 2f_1 + 1$. To see this let $y \in T_1$ finish in any row of SF_x and let $z \in T_1$ start in a later one, with α empty rows between them. Now, the pair $\{y, z\}$ is not covered in \mathcal{B}_x , so the runs containing y and z must meet outside \mathcal{B}_x . That is, $\alpha + f_{\{y\}} + f_{\{z\}} \geq b + 1$, so $\alpha \geq b - 2f_1 + 1$.

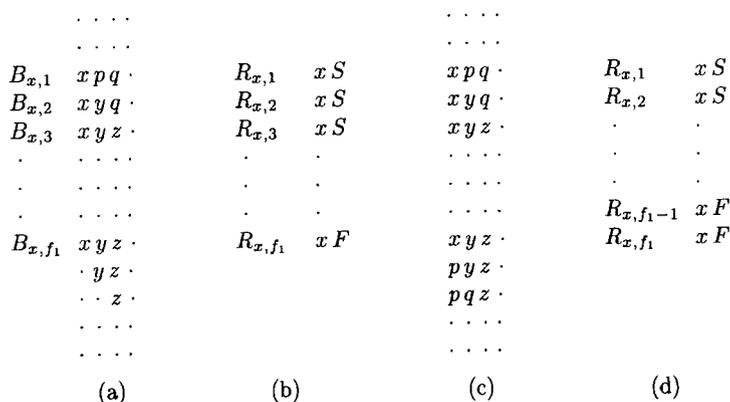


Fig. 7. Persistent triples and their corresponding forbidden configurations.

Furthermore, let $FS(v, k) = b - 2f_1 + 1$. Now $FS(v, k) \geq 0$ with equality if and only if $v = 2k - 2$ or $2k - 1$. Because we are interested only in $v > 2k$, we may assume that $FS(v, k) \geq 1$, i.e., that there is always at least 1 empty row between any F and the next S .

(3) Suppose the three elements x, y , and $z \in T_1$ (where, without loss of generality, the first is the x of our SF_x) are introduced in three successive blocks in \mathcal{C} . See Fig. 7(a) where y starts in $B_{x,2}$ ‘changing’ p , and z starts in $B_{x,3}$ changing q ; see (b) for the SF -array so formed. To cover the pairs $\{p, y\}$ and $\{q, z\}$ the elements p and q must occur in the two blocks immediately succeeding B_{x,f_1} , as shown in (c). Hence, pair $\{p, q\}$ must be in all remaining blocks outside \mathcal{B}_x , for, if not, it will be covered more than once. So both p and $q \in T_1$, which forces $b = 2f_1 - 1$, i.e., $v = 2k - 2$ or $2k - 1$. So, with our restriction of $v > 2k$, we may assume that the configurations $\begin{smallmatrix} S \\ S \end{smallmatrix}$ and $\begin{smallmatrix} F \\ F \\ F \end{smallmatrix}$ do not appear (in \mathcal{C} and so) in SF_x . (Such a triple $\{x, y, z\}$ is called a *persistent triple*, it persists through $v - k - 1$ blocks, see Phillips and Wallis [1]. Here we have shown that if a tight scccd contains a persistent triple then it must belong to one of the families \mathfrak{F} or \mathfrak{G} ; see the constructions in Section 5 and in the proof of Theorem 6.1; all designs in both these families contain persistent triples.)

Similarly, the configuration shown in Fig. 7(d) cannot occur in SF_x ; for, if it did, then the element which starts in $B_{x,2}$ must finish in the block succeeding B_{x,f_1} , thus producing the forbidden configuration $\begin{smallmatrix} F \\ F \\ F \end{smallmatrix}$ (in \mathcal{C}).

(4) The S of x lies in $R_{x,1}$ and the F in R_{x,f_1} ; and then each of the remaining $t_1 - 1$ elements in T_1 have either their S or their F present in SF_x , (but not both, for, if element y , say, has both its S and its F present, then its S must appear after its F and so pair $\{x, y\}$ is covered twice). This gives a total of $t_1 + 1$ symbols.

(5) Consider R , an arbitrary row of SF_x ; for each S in or above R there will be an element from T_1 in B_R ; similarly for each F in or below R , except that the S and F of

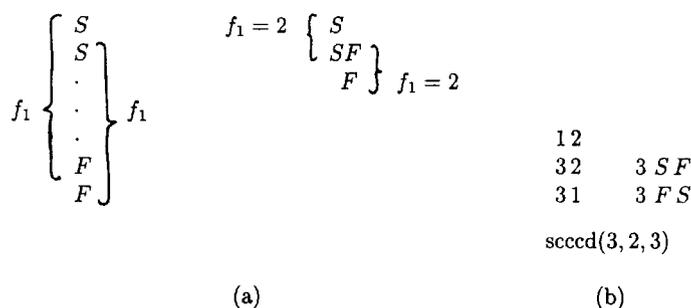


Fig. 8. Configurations corresponding to a persistent pair.

x contribute only one element (x itself) to B_R . Now \mathcal{C} has block size k , so we must have

$$\left\{ \begin{array}{l} \text{the number of } Ss \\ \text{in or above } R \end{array} \right\} + \left\{ \begin{array}{l} \text{the number of } Fs \\ \text{in or below } R \end{array} \right\} - 1 \leq k.$$

Call the left-hand side of the above equation the *weight* of R , $wt(R)$; it is the number of elements from T_1 in B_R .

(6) Let R be the last row of SF_x ; then $wt(R)$ is the number of Ss in SF_x , which is $\leq k$ by Observation (5). Similarly, the number of Fs in SF_x is $\leq k$. Also, using Observation (4), the total number of symbols, $t_1 + 1$, is $\leq 2k$. Thus, $t_1 \leq 2k - 1$. There is a tight scccd(4, 2, 6) with $(t_1, t_2, t_3) = (3, 0, 1)$ for which this inequality is sharp; it is also sharp for any design in the family \mathfrak{G} . (Cf. Section 4 of [2], where $t_1 \leq k$.)

(7) Suppose that two rows of weight k are adjacent, and the single-change between their corresponding two blocks is caused by y finishing in the first block and z starting in the successive block. Then there are no empty rows between the row containing the F of y and the row containing the S of z , a contradiction to Observation (2). Hence, two rows of weight k cannot be adjacent.

(8) The configuration $\begin{smallmatrix} S \\ S \end{smallmatrix}$ does not occur in SF_x so the number of configurations $\begin{smallmatrix} S \\ S \end{smallmatrix}$ in SF_x is $\leq \lfloor k/2 \rfloor$, otherwise the first inequality of Observation (6) is violated. Similarly for the configuration $\begin{smallmatrix} F \\ F \end{smallmatrix}$.

(9) A *persistent pair* in \mathcal{C} , see [1] and Section 4 of [2], is a pair of elements from T_1 which start in successive blocks; thus they persist together through $v - k$ blocks. Each persistent pair has a configuration $\begin{smallmatrix} S \\ S \end{smallmatrix}$ and $\begin{smallmatrix} F \\ F \end{smallmatrix}$. We claim that our SF_x contains exactly one of the configurations $\begin{smallmatrix} S \\ S \end{smallmatrix}$ or $\begin{smallmatrix} F \\ F \end{smallmatrix}$ for each persistent pair of \mathcal{C} , except if $\mathcal{C} \in \mathfrak{F}$ or \mathfrak{G} .

For any persistent pair $\{y, z\}$ of \mathcal{C} its configurations $\begin{smallmatrix} S \\ S \end{smallmatrix}$ and $\begin{smallmatrix} F \\ F \end{smallmatrix}$ can be arranged in one of the two ways shown in Fig. 8(a), where the upper S belongs to y . Our claim is clearly true if $x = y$ or z , so assume $x \neq y, z$. Now, the f_1 rows of SF_x must

include either the S or the F of y , and either the S or the F of z . That is, they must include either the upper S or the upper F in (a), and either the lower S or the lower F , not both in each case. If we choose our f_1 rows of SF_x such that this is true and that neither the whole of the $\overset{S}{S}$ nor the whole of the $\overset{F}{F}$ is included, then, without loss of generality, the first row must be the row containing the lower F and, as it proceeds downwards and cycles around, its last row must be the row containing the upper S . Hence, $b = 2f_1 - 1$, i.e., $v = 2k - 2$ or $2k - 1$, so $\mathcal{C} \in \mathfrak{F}$ or \mathfrak{G} , as in Observation (3). Hence, because $v > 2k$, we may assume that, for any persistent pair of \mathcal{C} , our SF_x contains either the whole of the pair's $\overset{S}{S}$ or the whole of the pair's $\overset{F}{F}$, but not both.

Now, because each persistent pair in \mathcal{C} contains 2 elements from T_1 and different persistent pairs contain distinct elements and $t_1 \leq 2k - 1$ from Observation (6), the total number of appearances of $\overset{S}{S}$ and $\overset{F}{F}$ in SF_x is $\leq \lfloor t_1/2 \rfloor \leq \lfloor (2k - 1)/2 \rfloor = k - 1$. For even k this upper bound is 1 smaller than the upper bound of $2\lfloor k/2 \rfloor$ obtained by adding the upper bounds for the number of appearances of $\overset{S}{S}$ and $\overset{F}{F}$ in Observation (8); for odd k they are the same.

(10) From Observation (4) SF_x contains a total of $t_1 + 1$ symbols, of which at least 1 is F . Hence, the number of appearances of S is $\leq t_1$. So, via Observation (6), the number of appearances of S is $\leq \min\{t_1, k\}$. Hence, the number of appearances of F is $\geq t_1 + 1 - \min\{t_1, k\}$; similarly for the number of appearances of S . So, finally, we have: $t_1 + 1 - \min\{t_1, k\} \leq |S| \leq \min\{t_1, k\}$, and similarly for $|F|$.

So, to summarize our 10 Observations, let \mathcal{C} be an arbitrary tight scccd(v, k, b) with $v > 2k$, and let $x \in T_1$ and let R be an arbitrary row in SF_x . Then, corresponding to the 10 Observations above, SF_x must satisfy Criteria (1)–(10) below, where $|C|$ denotes the number of appearances of configuration C .

Criteria

- (1) R contains at most one S and at most one F .
- (2) Between any F and the next S there are $\geq FS(v, k) = b - 2f_1 + 1 \geq 1$ empty rows.
- (3) The configurations $\overset{S}{S}, \overset{F}{F}$, and the configuration of Fig. 7(d) do not appear.
- (4) $S \in R_{x,1}, F \in R_{x,f_1}, |S| + |F| = t_1 + 1$.
- (5) $wt(R) \leq k$.
- (6) $1 \leq t_1 \leq 2k - 1$.
- (7) Two rows of $wt(k)$ cannot be adjacent.
- (8) $\left| \overset{S}{S} \right| \leq \lfloor k/2 \rfloor, \left| \overset{F}{F} \right| \leq \lfloor k/2 \rfloor$.
- (9) $\left| \overset{S}{S} \right| + \left| \overset{F}{F} \right| \leq k - 1$.
- (10) $t_1 + 1 - \min\{t_1, k\} \leq |S|, |F| \leq \min\{t_1, k\}$.

Finally, some comments relevant to Observation (9).

To see that our claim fails in a design from \mathfrak{F} or \mathfrak{G} , consider the tight scccd(3, 2, 3) $\mathscr{Y}_2 \in \mathfrak{G}$ shown in Fig. 8(b). (This corresponds to the second arrangement in (a) where

$f_1 = 2$, so $b = 3$.) All 3 pairs $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ are persistent pairs; however $k - 1 = 1$. The array SF_3 contains the $\frac{S}{S}$ of persistent pair $\{1, 3\}$ and the $\frac{F}{F}$ of persistent pair $\{2, 3\}$, but neither the $\frac{S}{S}$ nor the $\frac{F}{F}$ of persistent pair $\{1, 2\}$. Similarly for all other designs in \mathfrak{F} or \mathfrak{G} , where the number of persistent pairs is equal to the number of blocks.

When $v > 2k$, because SF_x must contain either the $\frac{S}{S}$ or the $\frac{F}{F}$ of every persistent pair in \mathcal{C} , the total number of persistent pairs in \mathcal{C} is also $\leq k - 1$; this upper bound is sharp for the tight $\text{scccd}(9, 4, 12)$, \mathcal{M}_2 , which contains 3 persistent pairs, see Section 9. (Cf. Section 4 of [2], where the number of persistent pairs is $\leq k/2$.)

9. Constructions of tight $\text{scccd}(9, 4, 12)$ s and $\text{scccd}(10, 4, 15)$ s using SF-arrays

We now illustrate the method of constructing tight designs with $v > 2k$ using SF-arrays.

$(v, k) = (9, 4)$ First we construct all non-isomorphic tight $\text{scccd}(9, 4, 12)$ s. To start, we must find all SF-arrays for $(v, k) = (9, 4)$ that satisfy Criteria (1)–(10) of Section 8.

Eq. (2) of Section 7 gives $A = 2$, $t_1 + t_2 = 9$, and $t_1 + 2t_2 = 12$; hence $t_1 = 6$ and $t_2 = 3$. We also have $f_1 = 6$ and $f_2 = 4$. From Criterion (2), $FS(9, 4) = 1$, i.e., there must be at least 1 empty row between any F and the next S in our SF-arrays. Let $T_1 = \{1, 2, 3, 4, 5, 6\}$ and $T_2 = \{7, 8, 9\}$, and let $x \in T_1$.

There are exactly 8 SF-arrays that satisfy Criteria (1)–(10). Of these, 6 are shown in Fig. 9(a)–(f), and the remaining 2 in (a) and (b) of Fig. 12.

In an arbitrary $\text{scccd}\mathbb{C}$ each of the t_1 elements of T_1 has 2 symbols, an S and an F; this gives $2t_1$ symbols of which $t_1 + 1$ appear in SF_x . We now consider the $t_1 - 1$ ‘missing’ symbols.

Now consider the SF_x in Fig. 10(a). Let us enlarge this SF_x from $f_1 = 6$ rows to $b = 12$ rows by including the $t_1 - 1 = 5$ ‘missing’ symbols and dropping every x , see Fig. 10(b). For example, the element which starts in row $R_{x,4}$ of (a), i.e., in R_4 of (b), must finish $f_1 = 6$ rows later in R_9 . Call this new array with b rows a SF-skeleton; note that the SF-skeleton (b) is uniquely determined from the SF-array (a).

Now, in Fig. 10(b), let the $S \in R_1$ correspond to element 1, the $S \in R_4$ correspond to 2, the $S \in R_6$ to 3, the $S \in R_8$ to 4, the $S \in R_9$ to 5, and, finally, the $S \in R_{11}$ to element 6.

In the SF-skeleton of Fig. 10(b) the SF-array SF_1 (i.e., Fig. 9(a) with $x = 1$) appears as rows R_1 – R_6 ; also, SF_2 (Fig. 9(b) with $x = 2$) appears as rows R_4 – R_9 ; SF_3 as rows R_6 – R_{11} ; SF_4 as rows R_8 – R_1 ; SF_5 as rows R_9 – R_2 ; and, finally, SF_6 (which is Fig. 9(f) with $x = 6$) appears as rows R_{11} – R_4 . Thus all 6 of the SF-arrays in Fig. 9 occur in the SF-skeleton of Fig. 10(b). We say that these 6 SF-arrays are equivalent (\sim) to one another.

In order to begin extending Fig. 10(b) to a tight $\text{scccd}(9, 4, 12)$, consider Fig. 10(c), which is a potential tight $\text{scccd}(9, 4, 12)$ with all elements from $T_1 = \{1, 2, 3, 4, 5, 6\}$

$R_{x,1}$	xSF	xSF	xSF	xS	xSF	xSF
$R_{x,2}$	xF	$x-$	$x-$	xSF	$x-$	$x-$
$R_{x,3}$	$x-$	xSF	xS	$x-$	xSF	xSF
$R_{x,4}$	xSF	$x-$	xSF	xSF	$x-$	xF
$R_{x,5}$	$x-$	xS	$x-$	$x-$	xSF	$x-$
$R_{x,6}$	$xF S$	$xF S$	$xF S$	$xF S$	xF	$xF S$
	(a)	(b)	(c)	(d)	(e)	(f)

Fig. 9. 6 of the 8 SF -arrays which satisfy criteria (1)–(10) when $(v,k) = (9,4)$. These 6 form an equivalence class of SF -arrays.

$R_{x,1}$	xSF	R_1	SF	B_1	1 4 6 5	R_1	–
$R_{x,2}$	xF	R_2	F	B_2	1 * 6 5	R_2	SF
$R_{x,3}$	$x-$	R_3	–	B_3	1 * 6 *	R_3	–
$R_{x,4}$	xSF	R_4	SF	B_4	1 2 6 *	R_4	SF
$R_{x,5}$	$x-$	R_5	–	B_5	1 2 * *	R_5	F
$R_{x,6}$	$xF S$	R_6	SF	B_6	1 2 3 *	R_6	–
		R_7	–	B_7	* 2 3 *	R_7	SF
		R_8	S	B_8	4 2 3 *	R_8	–
		R_9	SF	B_9	4 2 3 5	R_9	SF
		R_{10}	–	B_{10}	4 * 3 5	R_{10}	–
		R_{11}	SF	B_{11}	4 6 3 5	R_{11}	S
		R_{12}	–	B_{12}	4 6 * 5	R_{12}	SF
SF_x		SF -skeleton of SF_x				reverse of (b)	
(a)		(b)		(c)		(d)	

Fig. 10. An SF -array, its SF -skeleton, and the corresponding potential tight $scccd(9,4)$; the reverse of the SF -skeleton of SF_x .

present. (For example, element 2 starts in B_4 and finishes in B_9 because SF_2 appears as rows R_4 – R_9 in (b), etc.)

We must now add on the elements in $T_2 = \{7, 8, 9\}$; each is introduced twice, and appears in $f_2 = 4$ blocks.

See Fig. 11(a). Without loss of generality start element 7 in B_2 , then $7 \in B_3$; for, if not, then both 7 and 5 finish in B_2 , a contradiction. Without loss of generality start 8 in B_3 ; this produces (a).

See Fig. 11(b). Now not both 7 and 8 can finish in B_3 , so we must have $7 \in B_4$ or $8 \in B_4$. If $7 \in B_4$ then (because $7 \in T_2$ and $f_2 = 4$) 7 must start once more in a block containing elements 3, 4, and 9 because pairs $\{7, 3\}$, $\{7, 4\}$, and $\{7, 9\}$ will not have been covered, but this is impossible; hence $8 \in B_4$. Further, we have (b) by similar reasoning to the above.

See Fig. 11(c). Now if $8 \in B_7$, then pair $\{8, 9\}$ will be covered twice; so $7 \in B_7$ and 7 must occur for 2 successive blocks because $f_2 = 4$ and it has already occurred in 2 blocks. To finish we must have $8 \in B_{10}$ and $9 \in B_{12}$, producing Fig. 11(c).

B_1 1 4 6 5	B_1 1 4 6 5	B_1 1 4 6 5
B_2 1 7 6 5	B_2 1 7 6 5	B_2 1 7 6 5
B_3 1 7 6 8	B_3 1 7 6 8	B_3 1 7 6 8
B_4 1 2 6 *	B_4 1 2 6 8	B_4 1 2 6 8
B_5 1 2 * *	B_5 1 2 9 8	B_5 1 2 9 8
B_6 1 2 3 *	B_6 1 2 3 9	B_6 1 2 3 9
B_7 * 2 3 *	B_7 * 2 3 9	B_7 7 2 3 9
B_8 4 2 3 *	B_8 4 2 3 *	B_8 4 2 3 7
B_9 4 2 3 5	B_9 4 2 3 5	B_9 4 2 3 5
B_{10} 4 * 3 5	B_{10} 4 * 3 5	B_{10} 4 8 3 5
B_{11} 4 6 3 5	B_{11} 4 6 3 5	B_{11} 4 6 3 5
B_{12} 4 6 * 5	B_{12} 4 6 * 5	B_{12} 4 6 9 5
		\mathcal{U}_1 , scccd(9, 4, 12)
(a)	(b)	(c)

Fig. 11. Extending a potential tight scccd(9, 4, 12) to a tight scccd(9, 4, 12).

Let \mathcal{U}_1 denote this tight scccd(9, 4, 12); clearly, by its construction, it is unique up to labelling.

In general, for a given (v, k) with $v > 2k$, let \mathcal{S} denote the set of SF -arrays that satisfy Criteria (1)–(10) and let SF_x and SF'_x be two arbitrary SF -arrays in \mathcal{S} . We define an equivalence relation \sim on \mathcal{S} as follows:

$$SF_x \sim SF'_x \quad \text{if and only if} \quad \begin{array}{l} SF_x \text{ appears as consecutive rows} \\ \text{in the } SF\text{-skeleton determined by } SF'_x. \end{array}$$

But, because the skeleton determined by an arbitrary SF -array in \mathcal{S} is unique, we can redefine \sim as

$$SF_x \sim SF'_x \quad \text{if and only if} \quad \begin{array}{l} \text{the } SF\text{-skeleton determined by } SF_x \\ \text{is a cyclic shift of the } SF\text{-skeleton} \\ \text{determined by } SF'_x. \end{array}$$

It is straightforward to prove that \sim is an equivalence relation.

Each equivalence class of SF -arrays gives rise to one SF -skeleton; we say that this SF -skeleton *represents* the class. Instead of attempting to extend all SF -arrays (or, rather, their SF -skeletons) from a particular equivalence class to tight designs, we need only attempt to extend the SF -skeleton that represents this class. Also, if two SF -skeletons represent different classes, and both can be extended to tight designs, then these designs are non-isomorphic. However, a single SF -skeleton can some times be extended to Non-isomorphic designs, see Fig. 14.

The remaining 2 SF -arrays for $(v, k) = (9, 4)$ are shown in (a) and (b) of Fig. 12; they form another equivalence class. The SF -skeleton of (b) is (c) which extends uniquely (up to labelling) to the tight scccd(9, 4, 12) shown in (d); call it \mathcal{U}_2 . The SF -arrays (a) and (b) each occur 3 times each amongst the $t_1 = 6$ SF -arrays of \mathcal{U}_2 .

$R_{x,1}$	$x S F$	$x S$		R_1	S	B_1	$1\ 9\ 5\ 6$
$R_{x,2}$	$x F$	$x S F$		R_2	$S F$	B_2	$1\ 2\ 5\ 6$
$R_{x,3}$	$x -$	$x F$	→	R_3	F	B_3	$1\ 2\ 7\ 6$
$R_{x,4}$	$x S$	$x -$		R_4	$-$	B_4	$1\ 2\ 7\ 8$
$R_{x,5}$	$x S F$	$x S$		R_5	S	B_5	$1\ 2\ 3\ 8$
$R_{x,6}$	$x F$	$x F S$		R_6	$S F$	B_6	$1\ 2\ 3\ 4$
				R_7	F	B_7	$9\ 2\ 3\ 4$
				R_8	$-$	B_8	$9\ 7\ 3\ 4$
				R_9	S	B_9	$5\ 7\ 3\ 4$
				R_{10}	$S F$	B_{10}	$5\ 6\ 3\ 4$
				R_{11}	F	B_{11}	$5\ 6\ 8\ 4$
				R_{12}	$-$	B_{12}	$5\ 6\ 8\ 9$
					SF -skeleton of (b)		\mathcal{U}_2 , scccd(9, 4, 12)
(a)	(b)	(c)		(d)			

Fig. 12. The remaining 2 SF -arrays for $(v, k) = (9, 4)$, which form another equivalence class; the SF -skeleton which represents this class and the corresponding tight scccd(9, 4, 12).

This tight design contains $k - 1 = 3$ persistent pairs, which, by the comments at the end of Section 8, is the maximum number allowed in a tight scccd(v, k) with $v > 2k$.

Now, because the SF -skeletons from which \mathcal{U}_1 and \mathcal{U}_2 were formed represent different equivalence classes, we have $\mathcal{U}_1 \not\cong \mathcal{U}_2$. This gives us:

Theorem 9.1. *There are 2 non-isomorphic tight scccd(9, 4, 12)s, namely \mathcal{U}_1 and \mathcal{U}_2 shown above.*

Now, as mentioned at the end of Section 4, the reverse of a scccd(v, k, b), \mathcal{B} , is another scccd(v, k, b), $\text{rev}(\mathcal{B})$; and, if a scccd \mathcal{C} is tight then $\text{rev}(\mathcal{C})$ is also tight. Hence, from Theorem 9.1, $\text{rev}(\mathcal{U}_1) \cong \mathcal{U}_1$ or \mathcal{U}_2 .

To obtain the SF -skeleton of $\text{rev}(\mathcal{C}_1)$ from the SF -skeleton of \mathcal{C} we reverse the order of its rows and switch $S \leftrightarrow F$.

The SF -skeleton of \mathcal{U}_1 is shown in Fig. 10(b) and the SF -skeleton of $\text{rev}(\mathcal{U}_1)$ in Fig. 10(d); it is a cyclic shift of the SF -skeleton of \mathcal{U}_1 . Hence, because the extension of the SF -skeleton of \mathcal{U}_1 to a tight design is unique up to labelling, we have $\text{rev}(\mathcal{U}_1) \cong \mathcal{U}_1$. Thus \mathcal{U}_1 and \mathcal{U}_2 are *self-reverse*.

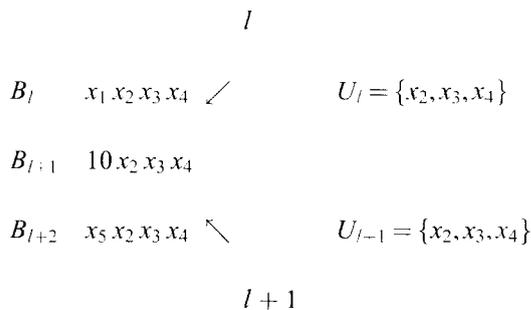
$(v, k) = (10, 4)$. We now construct all non-isomorphic tight scccd(10, 4, 15)s.

Eq. (2) of Section 7 yields the three solutions: $(t_1, t_2, t_3) = (7, 1, 2)$, $(6, 3, 1)$, and $(5, 5, 0)$. Here $f_3 = 3$, so Lemma 7.1(iii) with $j = 3$ disposes of the first solution. For the remaining two let \mathcal{V} be a tight scccd(10, 4, 15).

(i) $(t_1, t_2, t_3) = (6, 3, 1)$. We could use SF -arrays here but, for variety, we prefer the following approach which is justified by the result: there are 2 non-isomorphic tight scccd(10, 4, 15)s, both of which can be constructed by ‘expanding’ \mathcal{U}_2 of Fig. 12(d).

First some definitions, see Section 7 of [2]. For any $\text{scccd}(v, k, b)$, $\mathcal{B} = \{B_1, \dots, B_b\}$, and for any $i = 1, \dots, b - 1$ let U_i be the subset of $k - 1$ elements which survives from B_i to B_{i+1} ; we call U_i the *unchanged subset at location i* . Also, let U_b be the subset of $k - 1$ elements which survives from B_b to B_1 , the unchanged subset at location b .

Again, let $T_1 = \{1, \dots, 6\}$, $T_2 = \{7, 8, 9\}$, and let $T_3 = \{10\}$. Now, because $f_3 = 3$, each time element 10 is introduced into a block it is immediately changed. So the arrangement of blocks shown below must occur 3 times, at the pairs of consecutive locations: $l_1, l_1 + 1$ and $l_2, l_2 + 1$ and $l_3, l_3 + 1$. We have shown the arrangement at the pair of locations $l, l + 1$ for any $l \in \{l_1, l_2, l_3\}$.



Here $U_l = U_{l+1}$, i.e., the two unchanged subsets for this arrangement of blocks are equal. Hence, each of the 3 pairs of locations produces an unchanged subset which survives through the pair of locations. Then, because \mathcal{V} is tight, these 3 unchanged subsets partition the set $\{1, 2, \dots, 9\}$.

If we remove the 3 blocks B_{l_1+1} , B_{l_2+1} , and B_{l_3+1} that contain element 10 from \mathcal{V} we obtain a tight $\text{scccd}(9, 4, 12)$ with 3 unchanged subsets which partition $\{1, 2, \dots, 9\}$, i.e., with an *expansion set of locations*, see Section 7 of [2]. Of \mathcal{U}_1 and \mathcal{U}_2 , only \mathcal{U}_2 has an expansion set of locations, in fact it has two:

- $\{1, 2, 7\}$ at location 3, $\{1, 2, 8\}$ at location 4,
- $\{9, 3, 4\}$ at location 7, and $\{7, 3, 4\}$ at location 8,
- $\{5, 6, 8\}$ at location 11, $\{5, 6, 9\}$ at location 12.

Expanding \mathcal{U}_2 at the first expansion set above with element 10 gives us the tight $\text{scccd}(10, 4, 15)$, \mathcal{V}_1 , shown in Fig. 13(a); similarly, \mathcal{V}_2 in (b) comes from using the second expansion set.

Now we show that \mathcal{V}_1 and \mathcal{V}_2 are non-isomorphic even though their *SF*-skeletons are cyclic shifts of each other.

See Fig. 13. Consider the *complete SF-skeleton* shown to the right of the *SF*-skeletons. In any block of a scccd one element starts and one finishes. Suppose that an element from T_j starts and that one from $T_{j'}$ finishes, then the corresponding row of the complete *SF*-skeleton is $S_j F_{j'}$. Thus, the complete *SF*-skeleton includes start-finish

B_1	1 9 5 6	S	S_1F_2	B_1	1 9 5 6	S	S_1F_2
B_2	1 2 5 6	SF	S_1F_1	B_2	1 2 5 6	SF	S_1F_1
B_3	1 2 7 6	F	S_2F_1	B_3	1 2 7 6	F	S_2F_1
B_4	1 2 7 10	–	S_3F_3	B_4	1 2 7 8	–	S_2F_2
B_5	1 2 7 8	–	S_2F_2	B_5	1 2 10 8	–	S_3F_3
B_6	1 2 3 8	S	S_1F_2	B_6	1 2 3 8	S	S_1F_2
B_7	1 2 3 4	SF	S_1F_1	B_7	1 2 3 4	SF	S_1F_1
B_8	9 2 3 4	F	S_2F_1	B_8	9 2 3 4	F	S_2F_1
B_9	9 10 3 4	–	S_3F_3	B_9	9 7 3 4	–	S_2F_2
B_{10}	9 7 3 4	–	S_2F_2	B_{10}	10 7 3 4	–	S_3F_3
B_{11}	5 7 3 4	S	S_1F_2	B_{11}	5 7 3 4	S	S_1F_2
B_{12}	5 6 3 4	SF	S_1F_1	B_{12}	5 6 3 4	SF	S_1F_1
B_{13}	5 6 8 4	F	S_2F_1	B_{13}	5 6 8 4	F	S_2F_1
B_{14}	5 6 8 10	–	S_3F_3	B_{14}	5 6 8 9	–	S_2F_2
B_{15}	5 6 8 9	–	S_2F_2	B_{15}	5 6 10 9	–	S_3F_3

$\mathcal{V}_1, \text{scccd}(10, 4, 15)$	$\mathcal{V}_2, \text{scccd}(10, 4, 15)$
(a)	(b)

Fig. 13. The 2 non-isomorphic tight $\text{scccd}(10, 4, 15)$ s with $(t_1, t_2, t_3) = (6, 3, 1)$, their SF -skeletons and complete SF -skeletons. Both of these designs come from expanding \mathcal{U}_2 of Fig. 12(d).

information about all elements in $[v]$, not just those in T_1 . Clearly, if two designs are isomorphic, then their complete SF -skeletons must be cyclic shifts of one another; this is not so for \mathcal{V}_1 and \mathcal{V}_2 , hence $\mathcal{V}_1 \not\cong \mathcal{V}_2$. So there are exactly 2 non-isomorphic tight $\text{scccd}(10, 4, 15)$ s with $(t_1, t_2, t_3) = (6, 3, 1)$, namely \mathcal{V}_1 and \mathcal{V}_2 .

The reverse of a tight $\text{scccd}(v, k)$, \mathcal{C} , is another tight $\text{scccd}(v, k)$, $\text{rev}(\mathcal{C})$. Moreover, for $j = 1, \dots, A$, we have equality amongst the sets T_j for \mathcal{C} and $\text{rev}(\mathcal{C})$, and so equality amongst the numbers t_j for \mathcal{C} and $\text{rev}(\mathcal{C})$.

So $\text{rev}(\mathcal{V}_1)$ also has $(t_1, t_2, t_3) = (6, 3, 1)$, and thus $\text{rev}(\mathcal{V}_1) \cong \mathcal{V}_1$ or \mathcal{V}_2 . Now if $\text{rev}(\mathcal{V}_1) \cong \mathcal{V}_1$, then the complete SF -skeleton of $\text{rev}(\mathcal{V}_1)$ must be a cyclic shift of the complete SF -skeleton of \mathcal{V}_1 , but this is not the case. Hence, $\text{rev}(\mathcal{V}_1) \cong \mathcal{V}_2$. So \mathcal{V}_1 is not isomorphic to its reverse, similarly for \mathcal{V}_2 .

For two scccd s, \mathcal{B} and \mathcal{B}' , we write $\mathcal{B} \text{r} \mathcal{B}'$ if $\mathcal{B} \not\cong \mathcal{B}'$, but $\text{rev}(\mathcal{B}) \cong \mathcal{B}'$ (or, equivalently, $\text{rev}(\mathcal{B}') \cong \mathcal{B}$). Thus $\mathcal{V}_1 \text{r} \mathcal{V}_2$.

(ii) $(t_1, t_2, t_3) = (5, 5, 0)$. There are 32 SF -arrays that satisfy Criteria (1)–(10), and 8 equivalence classes of SF -arrays, 6 of size 5 and 2 of size 1.

The SF -skeletons of the 8 classes are shown in Fig. 14; underneath each is the number of its extensions to non-isomorphic designs, and the names of the designs.

So there are 10 non-isomorphic tight $\text{scccd}(10, 4)$ s with $(t_1, t_2, t_3) = (5, 5, 0)$, namely \mathcal{V}_m for $3 \leq m \leq 12$. We also have: $\mathcal{V}_3 \text{r} \mathcal{V}_5$, $\mathcal{V}_4 \text{r} \mathcal{V}_6$, $\mathcal{V}_7 \text{r} \mathcal{V}_8$, $\mathcal{V}_9 \text{r} \mathcal{V}_{10}$, and $\mathcal{V}_{11} \text{r} \mathcal{V}_{12}$.

The 2 designs from the previous case give us:

Theorem 9.2. *There are 12 non-isomorphic tight $\text{scccd}(10, 4, 15)$ s, namely \mathcal{V}_m for $m = 1, \dots, 12$.*

<i>SF</i>	–	<i>SF</i>	–	<i>SF</i>	<i>F</i>	<i>SF</i>	<i>SF</i>
<i>F</i>	–	<i>F</i>	<i>F</i>	<i>F</i>	–	<i>F</i>	–
–	<i>SF</i>	–	–	–	–	–	–
<i>F</i>	–	–	–	–	–	–	<i>SF</i>
–	<i>F</i>	<i>SF</i>	<i>SF</i>	<i>S</i>	<i>SF</i>	–	–
–	<i>F</i>	–	<i>F</i>	<i>F</i>	<i>F</i>	<i>S</i>	–
<i>SF</i>	–	<i>F</i>	–	<i>F</i>	–	<i>SF</i>	<i>SF</i>
–	–	–	–	–	–	–	–
–	<i>SF</i>	–	<i>S</i>	–	<i>S</i>	–	–
<i>S</i>	–	<i>S</i>	–	<i>S</i>	<i>S</i>	<i>S</i>	<i>SF</i>
<i>S</i>	–	<i>SF</i>	<i>SF</i>	<i>SF</i>	<i>F</i>	<i>S</i>	–
–	<i>S</i>	–	–	–	–	<i>F</i>	–
<i>SF</i>	–	–	–	–	–	<i>F</i>	<i>SF</i>
–	<i>S</i>	<i>S</i>	<i>S</i>	–	<i>S</i>	–	–
–	<i>SF</i>	–	<i>SF</i>	<i>S</i>	<i>SF</i>	–	–
2	2	1	1	1	1	0	2
$\mathcal{V}_3, \mathcal{V}_4$	$\mathcal{V}_5, \mathcal{V}_6$	\mathcal{V}_7	\mathcal{V}_8	\mathcal{V}_9	\mathcal{V}_{10}		$\mathcal{V}_{11}, \mathcal{V}_{12}$

Fig. 14. The 8 *SF*-skeletons which represent the 8 equivalence classes of *SF*-arrays for $(v, k) = (10, 4)$ and $(t_1, t_2, t_3) = (5, 5, 0)$. Underneath each *SF*-skeleton is the number of its extensions to non-isomorphic designs, and the names of the designs.

10. Non-existence of some tight designs

In this section we consider three parameter sets for (v, k) :

- (i) $\{(3k - 3, k) : k \text{ even and } \geq 2\}$;
- (ii) $\{(3k - 2, k) : k \text{ even and } \geq 2\}$;
- (iii) $\{(i + 1)^2/4, (i^2 + 7)/8) : i \text{ odd and } \geq 3\}$.

Every (v, k) in (i)–(iii) above satisfies the division requirement $2(k - 1) | v(v - 1)$ for a tight design to exist, however, for (i) and (ii), tight designs only exist when $k = 2$ or 4; and, for (iii), only when $i = 3$ or 5.

Using the notation of [5], we denote by $SCD(v, k, b)$ a single-change (non-circular) covering design on $[v]$ with b blocks of size k . We let $f(v, k)$ be the smallest b for which there exists a $SCD(v, k, b)$. The function $f(v, k)$ is studied in [3–5].

Now a $scccd(v, k, b)$ is also a $SCD(v, k, b)$. In particular, a minimal $scccd(v, k, b_*(v, k))$ is a $SCD(v, k, b_*(v, k))$, so we have $b_*(v, k) \geq f(v, k)$. (There are many (v, k) s for which equality holds.)

In Sections 5 and 6 we considered \mathfrak{F} and \mathfrak{G} , two families of tight $scccd(v, k)$ s for $v = 2k - 2$ and $2k - 1$ respectively. In the following two theorems we consider $v = 3k - 3$ ((i)) and $3k - 2$ ((ii)) respectively.

Theorem 10.1. *A tight $scccd(3k - 3, k, (9k - 12)/2)$ exists only when $k = 2$ or 4.*

Proof. Here k must be even. Consider the pair $(3k - 3, k)$ for $k \geq 6$, from Theorem 3.3 of [5] we have $f(3k - 3, k) = 5k - 8$. If a tight $scccd(3k - 3, k)$ exists

then $b_*(v, k) = (9k - 12)/2$, in which case the inequality $b_*(3k - 3, k) \geq f(3k - 3, k)$ fails. Thus, a tight $\text{scccd}(3k - 3, k)$ does not exist for $k \geq 6$.

For $k = 2$ we have a tight $\text{scccd}(3, 2, 3)$, \mathcal{U}_2 , and for $k = 4$ a tight $\text{scccd}(9, 4, 12)$, e.g., \mathcal{U}_1 . \square

So we have infinitely many pairs $(v, k) = (3k - 3, k)$ where k is even and ≥ 6 , for which $2(k - 1) | v(v - 1)$ but a tight $\text{scccd}(v, k)$ does not exist, e.g., a tight $\text{scccd}(15, 6, 21)$ does not exist.

Similarly for $v = 3k - 2$:

Theorem 10.2. *A tight $\text{scccd}(3k - 2, k, (9k - 6)/2)$ exists only when $k = 2$ or 4 .*

When $k = 2$ we have a tight $\text{scccd}(4, 2, 6)$ and $k = 4$ a tight $\text{scccd}(10, 4, 15)$, e.g., \mathcal{V}_1 . Theorems 10.1 and 10.2 can also be proved using SF -arrays.

The final result in this section, Theorem 10.4, will, for variety and interest, be proved using the following lemma, although it can also be proved in a similar manner to the above.

Lemma 10.3. *Let \mathcal{C} be a tight $\text{scccd}(v, k)$ with $v > 2k$, $x \in T_1$, $t_1 = |T_1|$, and $f_1 = v - k + 1$. Then*

- (i) *the total number of symbols in any r successive rows of SF_x is $\leq r + 1$;*
- (ii) $t_1 \leq f_1$.

Proof. (i) A straightforward proof by induction on r .

(ii) For any $x \in T_1$, the SF -array SF_x has f_1 rows, hence $\leq f_1 + 1$ symbols. But, by Criterion (4), it has exactly $t_1 + 1$ symbols. Hence $t_1 \leq f_1$. \square

The inequality $t_1 \leq f_1$ is sharp for both of the tight $\text{scccd}(9, 4, 12)$ s \mathcal{U}_1 and \mathcal{U}_2 of Section 9.

Let \mathcal{C} be a tight $\text{scccd}(v, k)$. From Observation (2) we have $2f_1 \leq b + 1$, with equality if and only if $\mathcal{C} \in \mathfrak{F}$ or \mathfrak{G} . So, for tight designs other than those in \mathfrak{F} or \mathfrak{G} , we have $2f_1 \leq b$.

We now classify tight designs with $2f_1 = b$, so $v \geq 2k$.

Theorem 10.4. *A tight $\text{scccd}(v, k, b)$ with $2f_1 = b$ is a tight $\text{scccd}(4, 2, 6)$ or a tight $\text{scccd}(9, 4, 12)$.*

Proof. Let \mathcal{C} be a tight $\text{scccd}(v, k)$ with $2f_1 = b$ and $v \geq 2k$.

We have $2(v - k + 1) = v(v - 1)/[2(k - 1)]$, i.e., $v = (4k - 3 + \sqrt{(8k - 7)})/2$. So let $k = (i^2 + 7)/8$ where i is odd and ≥ 3 . Hence, $(v, k) = ((i + 1)^2/4, (i^2 + 7)/8)$ ((iii)), and $f_1 = (i^2 + 4i + 3)/8$.

Eq. (2) from Section 7 then gives $f_3 = (v - 1) - 3(k - 2) = (21 + 4i - i^2)/8$. So, for $i \geq 7$, we have $f_3 \leq 0 < 3$. Thus, from Lemma 7.1(ii) with $j = 3$, we have $t_1 = 0$ for

			1 2 3		
			1 2 4		
			1 5 4		
			1 6 4		
			7 6 4		1 2
			7 8 4		1 3
L_1	1 2 3	1 2 3	7 3 4		4 3
L_2	1 2 4	1 2 4	7 3 5		5 3
L_3	1 5 4	1 5 4	8 3 5		5 2
L_4	1 6 4	1 5 2	6 3 5	1 2 3	5 1
L_5	3 6 4	6 5 2	6 2 5	1 2 4	4 1
L_6	3 6 5	6 5 3	6 2 8	1 5 4	4 5
L_7	2 6 5	6 4 3	1 2 8	3 5 4	4 2
L_8	2 6 3	6 1 3	1 2 7	3 5 2	3 2
	$\text{sf}(\mathcal{E}_1)$	$\text{sf}(\mathcal{E}_5)$	$\text{scccd}(8, 3, 14)$	$\text{rsf}(\mathcal{Y}_3)$	$\text{scccd}(5, 2, 10)$
	$A_1 = [3, 2, 3]$	$A_2 = [1, 4, 3]$			
	(a)	(b)	(c)	(d)	(e)

Fig. 15. Standardized designs and their column-arrays, and other properties. (a) $\text{sf}(\mathcal{E}_1)$, (b) $\text{sf}(\mathcal{E}_5)$, (c) the representative standardized form of a perfect $\text{scccd}(8, 3, 14)$; each A_i is a permutation of $A_1 = [4, 5, 5]$. (d) $\text{rsf}(\mathcal{Y}_3)$; each A_i is a permutation of $A_1 = [2, 1, 2]$, not perfect, element-regular with $\mu = 1$. (e) $\text{scccd}(5, 2, 10)$: column-regular with $\eta = 5$, perfect, element-regular with $\mu = 2$.

$l \geq 3$, i.e., every element in \mathcal{D} is introduced once or twice. So, Eq. (2) yields:

$$t_1 + t_2 = v = \frac{(i + 1)^2}{4}, \quad t_1 + 2t_2 = b = \frac{(i^2 + 4i + 3)}{4}.$$

This gives $t_1 = (i^2 - 1)/4$. Now, for $i \geq 7$ we have $v > 2k$, so, via Lemma 10.3(ii), we must have $t_1 \leq f_1$; but this is false when $i \geq 7$. Thus, a tight $\text{scccd}((i + 1)^2/4, (i^2 + 7)/8)$ does not exist for $i \geq 7$.

For $i = 3$ a tight $\text{scccd}(4, 2, 6)$ exists and for $i = 5$ a tight $\text{scccd}(9, 4, 12)$ exists. \square

11. Perfect designs; column-regular designs; element-regular designs

Again, let $\mathcal{B} = \{B_1, \dots, B_b\}$ be an arbitrary $\text{scccd}(v, k, b)$, and, for each $i = 1, \dots, b$, let $\text{sf}(\mathcal{B}_i)$ be the standardized form of its i th cyclic shift \mathcal{B}_i , see Section 4.

Now consider $\text{sf}(\mathcal{B}_i)$ for any fixed $i = 1, \dots, b$; its first block is $(1, 2, \dots, k)$. For $r = 1, \dots, k$, let its r th column be the column beginning with r , and let $\eta_{i,r}$ be the number of elements introduced into this column. Now let A_i be the ordered k -tuple $[\eta_{i,1}, \dots, \eta_{i,k}]$; call this the *column-array* of $\text{sf}(\mathcal{B}_i)$.

Consider again $\mathcal{E} = \{B_1, \dots, B_8\}$, the $\text{scccd}(6, 3, 8)$ from Fig. 3(a) and its standardized form $\text{sf}(\mathcal{E}_1) = \{L_1, \dots, L_8\}$ from Fig. 3(b) shown again in Fig. 15(a); we have $A_1 = [3, 2, 3]$. Now consider Fig. 15(b), which shows $\text{sf}(\mathcal{E}_5)$, the standardized form of $\mathcal{E}_5 = \{B_5, B_6, B_7, B_8, B_1, B_2, B_3, B_4\}$, this design has $A_5 = [1, 4, 3]$. So, for a fixed \mathcal{B} , we may have different A_i s for different i s.

A standardized $\text{scccd}(v, k, b)$ $\mathcal{B} = \{B_1, \dots, B_b\}$ is *perfect* if each of the unchanged elements between B_b and B_1 is in the same column in B_b as in B_1 . So, the two ends of a perfect standardized scccd can be ‘joined-up’ to give a circular version of requirement (4) in the definition of standardization (Section 4). For any \mathcal{B} , all of its b standardized forms are perfect or none are. Hence, a standardized \mathcal{B} is perfect if and only if $\text{rsf}(\mathcal{B})$ is perfect. The standardized form of \mathcal{Y}_3 (from Fig. 1(b)) is shown in Fig. 15(d); this is also $\text{rsf}(\mathcal{Y}_3)$, it is not perfect. See Fig. 15(c) for the representative standardized form of a perfect $\text{scccd}(8, 3, 14)$.

An interesting property of perfect standardized scccd s is:

Theorem 11.1. *Let A_1, \dots, A_b be the column-arrays of a perfect standardized $\text{scccd}(v, k, b)$. Then each A_i is a permutation of A_1 , for $i = 1, \dots, b$.*

Proof. For any fixed $r = 1, \dots, k$, consider the r th column in a perfect standardized $\text{scccd}(v, k, b)$ $\mathcal{B} = \text{sf}(\mathcal{B}_1) = \{B_1, \dots, B_b\}$. The elements in this column in B_b and B_1 are either the same, or different if the single-change between B_b and B_1 occurs in this column. In either case, we may write the elements of this column in a circle. Then $\eta_{1,r}$, the number of introductions in this column, is counted starting at B_1 ; this number is fixed no matter where on the circle we start. Now let $\text{sf}(\mathcal{B}_2)$ be formed from $\mathcal{B}_2 = \{B_2, B_3, \dots, B_b, B_1\}$ by a permutation of $[v]$ and a permutation ϕ of $[k]$, i.e., of the columns. Then $\eta_{2,r}$, the number of introductions in column r of $\text{sf}(\mathcal{B}_2)$, is equal to the number of introductions in column $\phi^{-1}(r)$ of $\text{sf}(\mathcal{B}_1)$ when starting counting at B_2 , which is the same as starting at B_1 ; this number is $\eta_{1, \phi^{-1}(r)}$. Thus $\eta_{2,r} = \eta_{1, \phi^{-1}(r)}$ for $r = 1, \dots, k$. That is, A_2 is a permutation of A_1 , and so on for A_i , $i = 3, \dots, b$. \square

For example, in the perfect standardized $\text{scccd}(8, 3, 14)$ in Fig. 15(c), each A_i is a permutation of $A_1 = [4, 5, 5]$. The $\text{scccd}(5, 3, 5)$ shown in Fig. 15(d) is $\text{rsf}(\mathcal{Y}_3)$. Each column-array of this design is a permutation of $A_1 = [2, 1, 2]$, even though it is not perfect, so the converse of Theorem 11.1 is not true.

Consider $\text{sf}(\mathcal{B}_i)$ for a fixed i , if the number of introductions into each column is the same, then we say that $\text{sf}(\mathcal{B}_i)$ is *column-regular*, see Section 4 of [2]. So $\eta_{i,r} = \eta = b/k$ for each $r = 1, \dots, k$, and $A_i = [\eta, \dots, \eta]$. Also, \mathcal{B} itself is *column-regular* if $\text{sf}(\mathcal{B}_i)$ is column-regular for each $i = 1, \dots, b$. So a $\text{scccd}(v, k, b)$ is column-regular if each of its b standardized forms is itself column-regular.

Although we have defined a column-array only for a standardized scccd we can also define it for a column-strict scccd . So, the column-array of a column-strict $\text{scccd}(v, k, b)$, \mathcal{B} , is the ordered k -tuple $[\eta_1, \dots, \eta_k]$ where η_r is the number of elements introduced into the r th column of \mathcal{B} , for each $r = 1, \dots, k$.

The definitions of ‘perfect’ and ‘column-regular’ can also be carried over to column-strict scccd s; and a $\text{scccd}(v, k, b)$, $\mathcal{B} = \{B_1, \dots, B_b\}$, is column-regular if the column-strict representation of each \mathcal{B}_i is itself column-regular.

We can now prove:

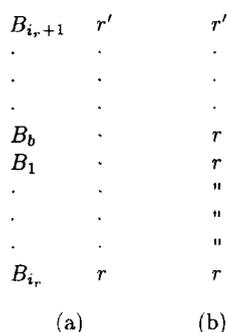


Fig. 16. Figure for Theorem 11.2.

Theorem 11.2. *A standardized column-regular scccd(v, k, b) is perfect.*

Proof. Let $\mathcal{B} = \text{sf}(\mathcal{B}_1) = \{B_1, \dots, B_b\}$ be a standardized column-regular scccd(v, k, b), and let $\eta = b/k$. Now $B_1 = (1, \dots, k)$, without loss of generality let element 1 be introduced in B_1 and let $1'$ be changed from B_b , and suppose that 1 and $1'$ are in different columns; let $1'$ be in the s th column where $s \neq 1$. Now \mathcal{B} is column-regular and so the column-array of the column-strict representation of each \mathcal{B}_i is $A_i = [\eta, \dots, \eta]$.

Now consider the column-strict $\mathcal{B}_2 = \{B_2, B_3, \dots, B_b, B_1\}$, where the elements in B_2 are in the same order as they were in \mathcal{B} , element 1 is now in the same column as $1'$, the s th column; hence the s th element in the column-array of the column-strict \mathcal{B}_2 is $\eta + 1$, a contradiction. So 1 and $1'$ are in the same column in \mathcal{B} .

Now consider element r for any fixed $r \in \{2, \dots, k\} = B_1 \cap B_b$; let it be changed first from B_{i_r} and replaced by r' in B_{i_r+1} . See the column-strict arrangement $\mathcal{B}_{i_r+1} = \{B_{i_r+1}, \dots, B_b, B_1, \dots, B_{i_r}\}$ in Fig. 16(a); elements r' and r lie in the same column by the previous argument. Moreover, $r \in B_b$ and $r \in B_1, \dots, B_{i_r}$, so we must have Fig. 16(b).

Now we can retrieve $\mathcal{B} = \{B_1, \dots, B_b\}$ from (b) without changing the columns in which r and r' appear. Hence, in \mathcal{B} , the $r \in B_1$ lies in the same column as the $r \in B_b$, and, because r was arbitrarily chosen from $B_1 \cap B_b$, so \mathcal{B} is perfect. \square

Combining Theorems 11.1 and 11.2 we have the following theorem in which all designs are assumed to be column-strict.

Theorem 11.3. *Let $\mathcal{B} = \{B_1, \dots, B_b\}$ be an arbitrary scccd(v, k, b). Then*

- (i) *if \mathcal{B} is perfect the column-array of \mathcal{B}_i is a permutation of the column-array of \mathcal{B} for each $i = 1, \dots, b$;*
- (ii) *if \mathcal{B} is column-regular then \mathcal{B} is perfect.*

An arbitrary scccd(v, k, b), \mathcal{B} , is *element-regular* if each of the v elements from $[v]$ is introduced the same number of $\mu = b/v$ times. In the notation of Section 7 we have $t_\mu = v$.

$k \backslash v$	3	4	5	6	7	8	9	10
2	1, (0)	2, (1)	51, (21)					
3		1, (0)	1, (0)	–	–			–
4				1, (0)	1, (0)	–	2, (0)	12, (0)
5						1, (0)	1, (0)	–
6								1, (0)

Fig. 17. Table showing some of the numbers of non-isomorphic tight $scccd(v, k)$ s for $v \geq 2k - 2$ and $v \leq 10$, ($k = 2, v \geq 3$). The number of perfect designs is shown in parenthesis (). For $k \geq 3$ the 1 in column $v = 2k - 2$ corresponds to $\mathcal{X}_k \in \mathfrak{F}$, and for $k \geq 2$ the 1 in column $v = 2k - 1$ to $\mathcal{Y}_k \in \mathfrak{G}$. The symbol – means that $2(k - 1) \nmid v(v - 1)$ and so a tight design with parameters (v, k) cannot exist. The missing numbers are currently being computed.

If \mathcal{C} is tight and element-regular with $\mu = 1$ then $v = b = 2k - 1$, and so \mathcal{C} is a $scccd(2k - 1, k, 2k - 1)$ and, by Theorem 6.2, is isomorphic to $\mathcal{Y}_k \in \mathfrak{G}$.

Our final example is shown in Fig. 15(e). It is the representative standardized form of a tight $scccd(5, 2, 10)$ which is column-regular with $\eta = 5$, and so perfect, and element-regular with $\mu = 2$.

Fig. 17 gives some of the numbers of non-isomorphic tight $scccd(v, k)$ s for $v \leq 10$.

Acknowledgements

Thank you to Donald A. Preece for reading, and commenting upon, an early version of this paper.

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