Bounds for Overfull Sets of One-Factors

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Abstract

An *overfull set* of one-factors of K_{2n} is a set of one-factors that between them cover all the edges of K_{2n}, but contain no one-factorization of K_{2n}. We ask how many members such a set can contain, and obtain upper and lower bounds.

1 Introduction

If G is any graph, then a *one-factor* of G is a subgraph with vertex-set $V(G)$ that is a regular graph of degree 1. In other words, a one-factor is a set of pairwise disjoint edges of G that between them contain every vertex. A *one-factorization* of G is a decomposition of the edge-set of G into edge-disjoint one-factors. Clearly any graph must have an even number of vertices to possess a one-factor or one-factorization.

In particular, we consider one-factors and one-factorizations of complete graphs. We denote the complete graph on $2n$ vertices by K_{2n}. We label the vertices of K_{2n} as $\infty, 0, 1, \cdots, 2n-2$; the labels are treated as integers modulo $2n-1$ with the proviso that $\infty + 1 = \infty$.

We shall denote the set of all one-factors of K_{2n} by $\mathcal{A}(K_{2n})$. It is clear that $\mathcal{A}(K_{2n})$ has $(2n-1)!!$ elements, where $(2n-1)!!$ denotes the semifactorial

$$(2n-1)!! = (2n-1) \times (2n-3) \times \ldots \times 3 \times 1 = \frac{(2n)!}{n!2^n}.$$

If (x, y) is any edge of K_{2n}, then there are $(2n-3)!!$ factors containing the edge (x, y). (Essentially, count the one-factors of the K_{2n} produced by deleting x and y.)

We write $b(2n)$ for the number of one-factorizations of K_{2n}. No closed formula for $b(2n)$ is known.

Lemma 1 *If F is any one-factor of K_{2n}, let $r(F)$ denote the number of one-factorizations of K_{2n} of which F is a member. Then*

$$r(F) = \frac{b(2n)}{(2n-3)!!}. \tag{1}$$

Proof. We first observe that $r(F)$ equals the number of one-factorizations of the graph $K_{2n} - F$. But this graph (the cocktail-party graph on $2n$ vertices) is unique, independent of the choice of F. So $r(F)$ depends only on $2n$.

Now we count all the ordered pairs $(F, \mathcal{F})$, where F is a factor in the factorization $\mathcal{F}$, in two ways, and obtain

$$(2n-1)!!r(F) = (2n-1)b(2n),$$

which gives the result. □

For convenience we denote this common value by r_{2n}.

2 Overfull sets

Bonisoli [1] introduced the concept of an *excessive set* of one-factors of K_{2n}. This is a set $\mathcal{S}$ of $2n$ one-factors that covers all edges of K_{2n}, in which no $2n-1$ factors form a one-factorization. Such a set exists for all even orders $2n$ greater than 4 (see [2, 1]), while orders 2 and 4 are easily seen to be impossible.

We address the more general question: for what values $t = |S|$ does there exist a set $\mathcal{S}$ of one-factors of K_{2n} with the following properties:

(1) the members of $\mathcal{S}$ cover all edges of K_{2n};

(2) no $2n-1$ members of $\mathcal{S}$ form a one-factorization of K_{2n}?

Such a set is called an *overfull set* of one-factors of K_{2n} of order t.

In particular we ask, what is the greatest order M_{2n} of an overfull set of one-factors of K_{2n}?

3 A lower bound

In [3] there was constructed a set $\mathcal{T}_n$ of one-factors that covers K_{2n} and contains no one-factorization, when $2n \geq 8$. If $\mathcal{F}_i$ is the set of all one-factors of K_{2n} containing edge (∞, i). Then

$$\mathcal{T}_n = \{P_0\} \cup \mathcal{D} \cup \bigcup_{i=2}^{2n-2} \mathcal{F}_i$$

where

$$P_0 = \{(\infty, 0), (1, 2n-2), \ldots, (k, 2n-k-1), \ldots, (n-1, n)\},$$

and $\mathcal{D}$ is the set of all one-factors that contain both $(\infty, 1)$ and one or more edges from P_0.

This set has order

$$1 + (2n-3)(2n-3)!! + \sum_{k=1}^{n-2} (-1)^{k+1} \binom{n-2}{k} (2n-2k-3)!!.$$

Now $M_6 = 11$, and an overfull set of 11 one-factors of K_6 was presented in [3], so we have

Theorem 1 *For all $n \geq 3$,*

$$M_{2n} \geq 1 + (2n-3)(2n-3)!! + \sum_{k=1}^{n-2} (-1)^{k+1} \binom{n-2}{k} (2n-2k-3)!!.$$

4 An upper bound

Suppose $\mathcal{R}$ is a set of one-factors of K_{2n} whose *complement* $\mathcal{A}(K_{2n}) \backslash \mathcal{R}$ is overfull. Then $\mathcal{R}$ must contain at least one member of every one-factorization of K_{2n}. Each factor in $\mathcal{R}$ belongs to exactly r_{2n} one-factorizations, so $\mathcal{R}$ can have non-empty intersection with at most $r_{2n}|\mathcal{R}|$ one-factorizations. So

$$r_{2n}|\mathcal{R}| \geq b(2n). \tag{2}$$

Combining (1) and (2) we have

Lemma 2 *If $\mathcal{R}$ is the complement of an overfull set of one-factors of K_{2n} then*

$$|\mathcal{R}| \geq (2n-3)!!$$

Theorem 2 $M_{2n} \leq 2(n-1)((2n-3)!!)$

Proof. Suppose $\mathcal{S}$ is an overfull set with complement $\mathcal{R}$. Then

$$\begin{aligned} |\mathcal{S}| &= |\mathcal{A}(K_{2n})| - |\mathcal{R}| \\ &= (2n-1)!! - |\mathcal{R}| \\ &\leq (2n-1)!! - (2n-3)!! \\ &= 2(n-1)((2n-3)!!) \end{aligned}$$

□

Combining Theorems 1 and 2, we obtain, for example, $11 \leq M_6 \leq 12$ and $81 \leq M_8 \leq 90$.

One case that achieves the order $|\mathcal{R}| = (2n-3)!!$ is when $\mathcal{R}$ consists of all the one-factors containing a given edge (x, y). In this case the complement of $\mathcal{R}$ does not cover (x, y), so it is not overfull. In particular, when $2n = 6$, these sets are the only possible sets $\mathcal{R}$ with $(2n-3)!!$ elements, so $M_6 < 2(3-1)((6-3)!!) = 12$. Combined with Theorem 1, this shows that $M_6 = 11$. Is it true that $|\mathcal{R}| = (2n-3)!!$ can only be achieved when $\mathcal{R}$ consists of all the one-factors containing a given edge?

References

[1] A. Bonisoli, Excessive factorizations of complete graphs, *preprint.*

[2] D. R. Stinson and W. D. Wallis, An even-side analogue of Room squares. *Aeq. Math.* **27** (1984), 201–213.

[3] W. D. Wallis, Overfull sets of one-factors. *Congressus Num.* (*to appear*).