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Counting structures in the Möbius ladder

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Abstract

The Möbius ladder, M_n, is a simple cubic graph on $2n$ vertices. We present a technique which enables us to count exactly many different structures of M_n, and somewhat unifies counting in M_n. We also provide new combinatorial interpretations of some sequences, and ask some questions concerning extremal properties of cubic graphs. © 1998 Elsevier Science B.V. All rights reserved

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Introduction

The *Möbius ladder*, M_n, is a simple cubic graph on $2n$ vertices. It is shown in Fig. 1(a), and the representation of it used in this paper appears in Fig. 1(b).

This class of graphs is interesting both for mathematicians, see Biggs [1], where M_n is often used as an example; and chemists, see Hosoya and Harary [7], and the references therein.

A *spanning edge-subgraph* of a simple graph is a subset of the edges of the graph, together with all of its vertices. All structures considered here, except those in Sections 19 and 20, are spanning, labelled edge-subgraphs; henceforth referred to simply as edge-subgraphs.

We present a counting technique which enables us to count exactly many different structures of M_n.

0. Preliminaries

Consider Fig. 1(b). The edges ii' for $i = 1, \ldots, n$ are *diagonals*, all other edges are *outside edges*. The graph M_n contains n diagonals and $2n$ outside edges. Unless otherwise stated, we will always assume that $n \geqslant 2$ and $1 \leqslant k \leqslant n$.

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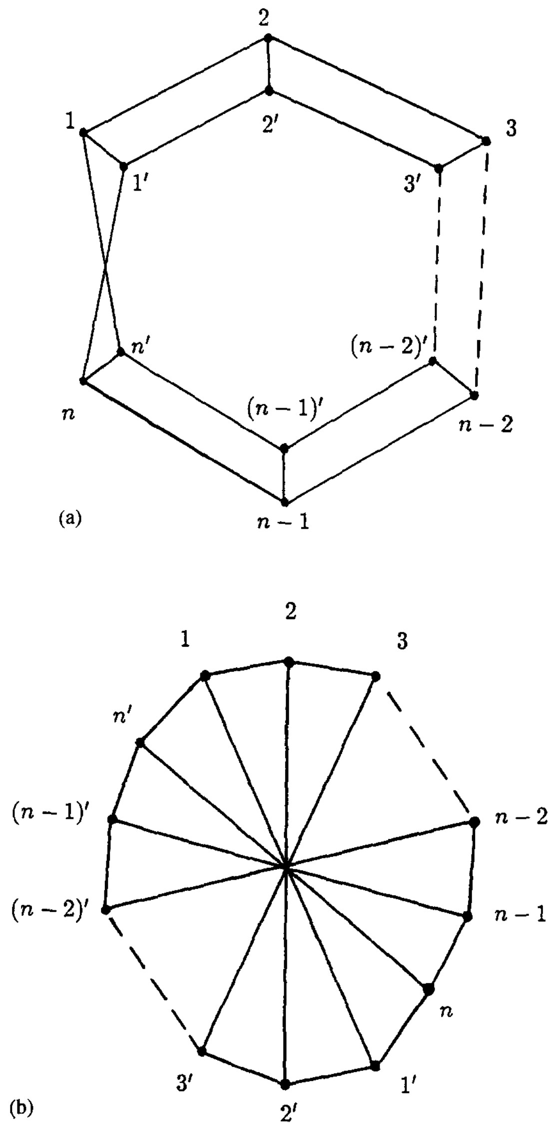

Fig. 1.

Now consider the $2n$ vertices labelled $1,2,3,\ldots,n,1',2',3',\ldots,n'$ arranged clockwise around a circle, see Fig. 2(a). A *(n,k)-graph*, G, is a subgraph of M_n with vertex set these $2n$ vertices, and edge set any k diagonals and any number of outside edges.

Call vertices which lie on a diagonal *diagonal vertices*. Moving clockwise, call the edge-subgraph of G between two consecutive diagonal vertices, u and v, a *join*. Denote it by $[u,v)$, see Fig. 2(b); clearly G has $2k$ joins. The join $[u,v)$ does not contain vertex v, hence every vertex and every outside edge of G lie in exactly one join. Call the join

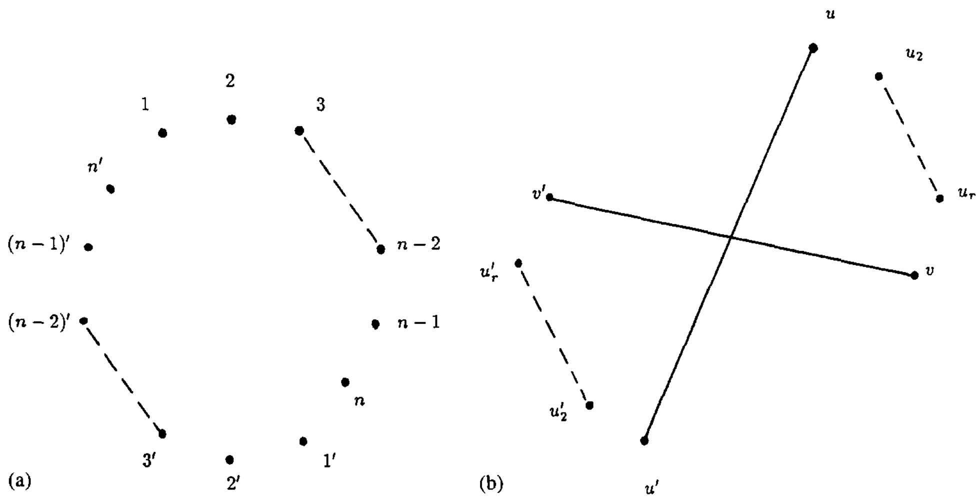

Fig. 2.

$[u, v)$ *complete* if all possible outside edges are present in it, *incomplete* otherwise. Thus, a complete join is a path with its last vertex removed.

Call the join $[u, v)$ *empty* if it contains no edges, and G *empty* if all of its joins are empty. We will often use K to denote an empty (n, k)-graph. The joins $[u, v)$ and $[u', v')$ (see Fig. 2(b)) are *opposite* and comprise a *join-pair*. Such a join-pair is *incomplete* if at least one of $[u, v)$ or $[u', v')$ is incomplete. The graph G is *join-pair incomplete* if each of its k join-pairs is incomplete.

Let the non-diagonal vertices between u and v be labelled $u_2, \dots, u_r$. We define the *size* of the join $[u, v) = [u, u_2, \dots, u_r, v)$ to be r, which is the maximum possible number of edges in the join. A join of size 1 is an edge or a non-edge; a join is even/odd if its size is even/odd. It is straightforward to prove the following theorem.

Theorem A. *Let G be a (n, k)-graph*:

(i) *G is acyclic if and only if G is join-pair incomplete and has $\leqslant k-1$ complete joins*;

(ii) *if G is connected then G has $\geqslant k-1$ complete joins*;

(iii) *if G is acyclic and connected then G is join-pair incomplete and has exactly $k-1$ complete joins.*

The following is a key definition and theorem.

A *k-composition* of n, $x_1 + \cdots + x_k$, is an expression of n as an ordered sum of k positive integers, each $x_j \geqslant 1$. Throughout this paper we let $\underline{x} = x_1 \cdots x_k = x_1 + \cdots + x_k$ be an arbitrary k-composition of n, there are $\binom{n-1}{k-1}$ such compositions.

Let $\theta(n, k)$ be the number of empty (n, k)-graphs.

Consider the vertices $[n] = \{1, 2, \dots, n\}$ of a (n, k)-graph. Now the diagonal vertices without the $'$ form a k-set of $[n]$. Conversely, a k-set of $[n]$ determines k diagonals by choosing each vertex in the k-set to be a diagonal vertex. Thus $\theta(n,k)=\binom{n}{k}$. However, we need the following theorem, central to the main idea of this paper.

Theorem B. *Let $\theta(n,k)$ denote the number of empty (n,k)-graphs, then*

$$\theta(n,k)=\frac{n}{k}\sum_{\underline{x}}1=\frac{n}{k}\sum_{\underline{x}}\prod_{j=1}^{k}1^2.$$

Proof. Let K be an empty (n,k)-graph which contains the diagonal $11'$, see Fig. 3. Let us start at vertex 1 and move clockwise, as we move we record the size of the joins which we pass through. Let the jth join be X_j, and let $|X_j|=x_j$ for $1\leqslant j\leqslant k$. After k joins we obtain $x_1\cdots x_k$, a k-composition of n. Conversely, any k-composition of n, $y_1\cdots y_k$, uniquely determines a (n,k)-graph which contains $11'$, by starting at 1 and (moving clockwise) letting the size of the jth join be y_j for $1\leqslant j\leqslant k$. Hence, the number of (n,k)-graphs containing $11'$ is $\sum_{\underline{x}}1$.

Similarly, for any fixed i, $2\leqslant i\leqslant n$, the number of (n,k)-graphs containing ii' is $\sum_{\underline{x}}1$.

Hence

$$\sum_{i=1}^{n}\left|\left\{\begin{matrix}(n,k)-\text{graphs}\\ \text{containing } ii'\end{matrix}\right\}\right|=n\sum_{\underline{x}}1.$$

But each (n,k)-graph is counted k times in the left-hand side of this equation. Hence $k\theta(n,k)=n\sum_{\underline{x}}1$, i.e., $\theta(n,k)=(n/k)\sum_{\underline{x}}1$.

Now, an empty join can be made into an empty join in 1 way, and an empty join-pair into an empty join-pair in 1^2 way(s). So the product $\prod_{j=1}^{k}1^2$ gives the number of ways the k join-pairs of K can be extended to an empty (n,k)-graph. Hence the result. □

Let $[x^n]H(x)$ be the coefficient of x^n in $H(x)$. We have the following result, where $'$ denotes differentiation.

Theorem C.

(i) $\sum_{\underline{x}}\prod_{j=1}^{k}\sigma(x_j)=[x^n]\{\sigma(1)x+\sigma(2)x^2+\cdots\}^k$.

(ii) *Let $H(x)=xJ(x)$, where $J(0)$ and $J'(0)$ are defined. Then* $[x^n]\sum_{k=1}^{n}\frac{n}{k}H(x)^k=[x^n]\frac{xH'(x)}{1-H(x)}$.

Now let the elements of $[n]$ be arranged in a circle. A *non-consecutive*, *cyclic* (ncc) set of $[n]$ is a subset of elements of $[n]$ which are non-consecutive when chosen from this circular arrangement. Let $\psi(n,t)$ denote the number of ncc t-sets of $[n]$, where

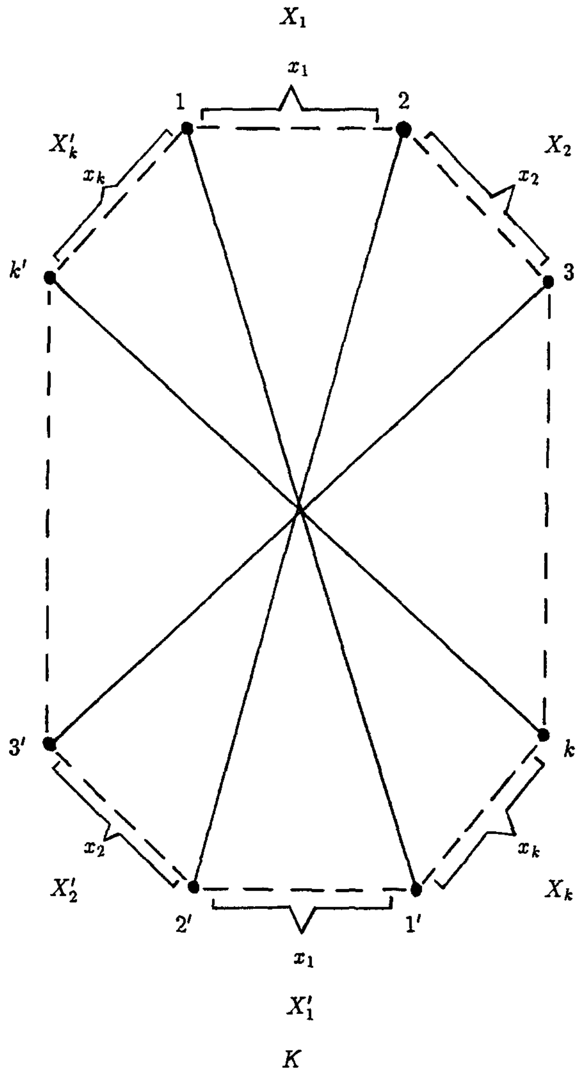

Fig. 3.

$0 \leqslant t \leqslant \lfloor n/2 \rfloor$. So

$$\psi(n,t) = \frac{n}{n-t}\binom{n-t}{t} \quad \text{and}$$

$$\mathscr{L}(n) = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \psi(n,t) = ((1+\sqrt{5})/2)^n + ((1-\sqrt{5})/2)^n,$$

where $\mathscr{L}(n)$ is the nth Lucas number; see pp. 24 and 46 of Comtet [3], and pp. 73 and 246 of Stanley [10].

In the following sections, when counting *Generic Structures* (gs) of M_n, we let $\mathrm{gs}(n,k)$ denote the number of generic structures of M_n with k diagonals, and $\mathrm{gs}(n)$ the total number of generic structures of M_n.

The idea of this paper is to count generic structures in M_n by extending an empty (n,k)-graph, K, to a generic structure by adding edges to its empty joins, whilst keeping k fixed. For example in Section 2, if K is extended to a forest of M_n, then a join of the extended K can be any arbitrary edge-subgraph of the complete join; for a tree (Section 8), a join must be either complete or complete less one edge. So, we first find $\mathrm{gs}(n,k)$ by summing over $\underline{x}$, and then sum over k to determine $g(n)$.

Even though some of these structures have been counted before, and thus some counts appear in the literature, it seems that this technique is new, and somewhat unifies the approach to counting structures in M_n; it also works for the cubic prism. For a unified approach to counting structures in K_n, the complete graph on n vertices, see Harary and Palmer [6].

We are now ready to start counting.

1. Edge-subgraphs (e)

Clearly M_n has $3n$ edges, so $e(n)=2^{3n}=8^n$.

We now illustrate our technique by redetermining $e(n)$.

First consider $k=0$. Let C_{2n} denote the cycle with $2n$ vertices and $2n$ edges; any edge-subgraph of C_{2n} is an edge-subgraph of M_n with no diagonals, hence $e(n,0)=4^n$.

For $k\geqslant 1$, let K be the empty (n,k)-graph that contains diagonal $11'$ and determines the composition $\underline{x}=x_1\cdots x_k$, see Fig. 3, and let $[k]=\{1,2,\ldots,k\}$. Now, if K is to be extended to an edge-subgraph, then a join of size r of the extended K can be any of the 2^r edge-subgraphs of the complete join. From Theorem B, the number of empty (n,k)-graphs is $(n/k)\sum_{\underline{x}}\prod_{j=1}^{k}1^2$, so, using the multiplication principle, we have

$$e(n,k)=\frac{n}{k}\sum_{\underline{x}}\prod_{j=1}^{k}(2^{x_j})^2. \tag{1}$$

So

$$e(n)=e(n,0)+\sum_{k=1}^{n}e(n,k)=4^n+\sum_{k=1}^{n}\left(\frac{n}{k}\sum_{\underline{x}}\prod_{j=1}^{k}4^{x_j}\right)=8^n.$$

2. Forests (f)

A *forest* is an acyclic edge-subgraph. If a (n,k)-graph, F, is a forest of M_n then an incomplete join of F is an arbitrary edge-subgraph of the complete join, (except the complete join).

Using Theorem A(i), we have:

Lemma 2.1. *Let F be a (n,k)-graph. Then F is a forest of M_n if and only if*
(i) *F is join-pair incomplete;*
(ii) *F has $\leqslant k-1$ complete joins; and*
(iii) *each incomplete join of F is an arbitrary incomplete join.*

If an incomplete join has size r then it can be any of $2^r - 1$ edge-subgraphs.

Any edge-subgraph of C_{2n}, except C_{2n} itself, is a forest of M_n; hence, $f(n,0) = 4^n - 1$.

Now for $k \geqslant 1$. Let t be an arbitrary variable; for $\underline{x} = x_1 \cdots x_k$, with n and k fixed, define

$$\pi(\underline{x}, t) = \pi(x_1 \cdots x_k, t) = \prod_{j=1}^{k} (t^{x_j} - 1),$$

and

$$\alpha(n,k,t) = \sum_{\underline{x}} \pi(\underline{x}, t).$$

Then, using Theorem C(i)

$$\alpha(n,k,t) = [x^n]\{(t-1)x + (t^2-1)x^2 + \ldots\}^k$$

$$= [x^n]\left\{\frac{(t-1)x}{(1-x)(1-tx)}\right\}^k = [x^n]A_{[t]}(x)^k, \quad \text{say.} \tag{2}$$

Again, let K be as in Fig. 3. Consider the (n,k)-graph $\overline{F}_s$ that has s complete joins on the right-hand side of $11'$ for some s, where $0 \leqslant s \leqslant k-1$, and no complete joins on the left-hand side.

By the above lemma $\overline{F}_s$ is a forest and the number of such forests is

$$\pi(\underline{x}, 2) \cdot \sum_{S \subset [k]} \left\{\frac{\pi(\underline{x}, 2)}{\prod_{j \in S}(2^{x_j} - 1)}\right\},$$

where S is a s-set of $[k]$, and $\prod_{j \in \emptyset}(2^{x_j} - 1) = 1$.

The first factor in the expression above gives the number of possible edge-subgraphs on the left-hand side of $11'$ in $\overline{F}_s$, and the second factor the number on the right-hand side.

Now $\overline{F}_s$ is join-pair incomplete and so the s complete joins on the right-hand side of $11'$ can be arranged in 2^s ways: for each complete join $[u,v)$, switch it with its incomplete opposite join $[u',v')$, (if $a(a+1)$ is an edge in $[u,v)$ then, after switching, $a'(a+1)'$ is an edge in $[u',v')$, and vice versa). Conversely, if a forest contains s complete joins then $0 \leqslant s \leqslant k-1$ and it is join-pair incomplete, so we may form such a graph $\overline{F}_s$ by switching all complete joins from the left-hand side of $11'$ to the right-hand side.

Hence, incorporating the factor of 2^s into the above expression and simplifying, the total number of forests which can be formed from K is

$$\pi(\underline{x},2)^2 \sum_{S\subset[k]} \prod_{j\in S}\left(\frac{2}{2^{x_j}-1}\right) \tag{3}$$

$$= \pi(\underline{x},2)^2 \left\{\prod_{j=1}^{k}\left(1+\frac{2}{2^{x_j}-1}\right) - \prod_{j=1}^{k}\left(\frac{2}{2^{x_j}-1}\right)\right\}$$

$$= \pi(\underline{x},2)\left\{\prod_{j=1}^{k}(2^{x_j}+1) - 2^k\right\} \tag{4}$$

$$= \pi(\underline{x},4) - 2^k\pi(\underline{x},2).$$

Now, using Theorem B, the number of empty (n,k)-graphs is $(n/k)\sum_{\underline{x}} 1$. Hence,

$$f(n,k) = \frac{n}{k}\sum_{\underline{x}}\{\pi(\underline{x},4) - 2^k\pi(\underline{x},2)\} = \frac{n}{k}\{\alpha(n,k,4) - 2^k\alpha(n,k,2)\}. \tag{5}$$

Using (2) and (5), we have:

Lemma 2.2.

$$f(n,k) = \begin{cases} 4^n - 1, & k=0,\\ [x^n]\frac{n}{k}\{A_{[4]}(x)^k - (2A_{[2]}(x))^k\}, & k\geqslant 1.\end{cases}$$

Then

$$f(n) = 4^n - 1 + [x^n]\left\{\sum_{k=1}^{n}\frac{n}{k}\{A_{[4]}(x)^k - (2A_{[2]}(x))^k\}\right\},$$

now use Theorem C(ii) twice,

$$= 4^n - 1 + [x^n]\left\{\frac{xA'_{[4]}(x)}{1-A_{[4]}(x)} - \frac{2xA'_{[2]}(x)}{1-2A_{[2]}(x)}\right\}$$

$$= 4^n - 1 + [x^n]\frac{x(1+4x-40x^2+16x^3+16x^4)}{(1-2x)(1-4x)(1-5x+2x^2)(1-8x+4x^2)}.$$

Thus, the above rational function is the ordinary generating function for the total number of forests of M_n with $k\geqslant 1$.

Finally, partial fractions give:

Theorem 2.3.

$$f(n) = (4+\sqrt{12})^n + (4-\sqrt{12})^n + 2^n - ((5+\sqrt{17})/2)^n - ((5-\sqrt{17})/2)^n - 1.$$

The numbers $\{f(n)\colon n\geqslant 2\}$ yield the sequence $\{38, 328, 2686, 21224, \ldots\}$ which does not appear in Sloane and Plouffe [9]. The numbers $\{f(n,n)\} = \{3^n - 2^n\}$ appear as sequence M3887, so we have a new combinatorial interpretation of this sequence.

Note that from p. 103 of [1], and [2], the Möbius ladders form a recursive family of graphs, i.e., their Tutte polynomials satisfy a homogeneous linear recurrence relation, and so can be computed. Hence, see [1, p. 104], we may count the number of forests to give the same as above.

So, for large $n, f(n)$ is approximately $(7.4641)^n$. For an arbitrary cubic graph on $2n$ vertices the number of forests is less than 8^n. A natural question is then: Does there exist a family of cubic graphs, G_n, on $2n$ vertices, such that, for some n_0 and all $n > n_0$, the number of forests of G_n is greater than $(7.4641)^n$?

3. Strong edge-subgraphs (se)

A *strong edge-subgraph*, E, of M_n is an edge-subgraph with no isolated vertices. If an edge-subgraph of C_{2n} is strong then its non-edges must be non-consecutive, cyclic. Thus, $\mathrm{se}(n,0) = \mathscr{L}(2n)$.

Similarly, the join $[u, u_2, \ldots, u_r, v)$ of E is strong if it contains no isolated vertices. For this join to be strong we only require that the vertices $u_2, \ldots, u_r$, are not isolated because u is incident to a diagonal.

Let $\beta(r)$ be the number of strong joins for a join of size r. For $k\geqslant 1$, the number of strong edge-subgraphs which come from K of Fig. 3 is $\prod_{j=1}^{k} \beta(x_j)^2$.

Thus, (cf. (1)),

$$\mathrm{se}(n,k) = \frac{n}{k}\sum_{\underline{x}}\prod_{j=1}^{k}\beta(x_j)^2.$$

Now $\beta(1)=2$, $\beta(2)=3$, and $\beta(r)=\beta(r-1)+\beta(r-2)$ for $r\geqslant 3$, see [3, p. 45]. The recurrence for $\beta(r)^2$ is $\beta(r)^2 = 2\beta(r-1)^2 + 2\beta(r-2)^2 - \beta(r-3)^2$.

Let

$$B(x) = \sum_{r\geqslant 1}\beta(r)^2 x^r = \frac{x(4+x-x^2)}{(1+x)(1-3x+x^2)}.$$

Lemma 3.1.

$$\mathrm{se}(n,k) = \begin{cases} \mathscr{L}(2n), & k=0,\\ [x^n]\frac{n}{k}B(x)^k, & k\geqslant 1. \end{cases}$$

By similar reasoning to that used in Section 2, we have

$$\mathrm{se}(n) = \mathscr{L}(2n) + [x^n]\frac{x(4+2x+3x^2-4x^3+x^4)}{(1+x)(1-3x+x^2)(1-6x-3x^2+2x^3)}.$$

Theorem 3.2.

$$\mathrm{se}(n) \approx (6.4188)^n + (-0.8056)^n + (0.3867)^n - (-1)^n.$$

The first few terms in the sequence $\{\mathrm{se}(n)\colon n \geqslant 2\}$ are $\{41, 265, 1697, 10897, \ldots\}$.

Let H_n be an arbitrary cubic bipartite graph on $2n$ vertices, then, necessarily, the number of vertices in each part is n. Let one part contain the vertices $v_1, \ldots, v_n$. Now each v_i is incident to 3 edges. A strong edge-subgraph of H_n must contain at least one of these edges i.e., it must contain one of the $2^3 - 1 = 7$ non-empty subsets of edges incident to v_i. Hence, the number of strong edge-subgraphs of H_n is at most 7^n.

Now, for n odd, the graph M_n is bipartite, and, for large n, contains approximately $(6.4188)^n$ strong edge-subgraphs. Hence, we may ask the following question: For odd n, does there exist a family of cubic bipartite graphs, G_n, on $2n$ vertices, such that, for some n_0 and all odd $n > n_0$, the number of strong edge-subgraphs of G_n is greater than $(6.4188)^n$?

4. Strong forests (sf)

A *strong forest*, F, of M_n is a forest with no isolated vertices. By comparison with Lemma 2.1, we have:

Lemma 4.1. *Let F be a (n,k)-graph. Then F is a strong forest of M_n if and only if*
(i) *F is join-pair incomplete;*
(ii) *F has $\leqslant k-1$ complete joins; and*
(iii) *each incomplete join of F is a strong incomplete join.*

For $k = 0$, the number of strong forests of M_n equals the number of strong edge-subgraphs of C_{2n} less $1, (C_{2n}$ itself). Thus, $\mathrm{sf}(n,0) = \mathscr{L}(2n) - 1$.

Let $\gamma(r)$ be the number of strong incomplete joins for a join of size r.

By similar reasoning to that used in Section 2, the number of strong forests which come from K of Fig. 3 is given by

$$\prod_{j=1}^{k} \gamma(x_j) \left\{ \prod_{j=1}^{k} (\gamma(x_j) + 2) - 2^k \right\}.$$

Thus, (cf. (5)),

$$\mathrm{sf}(n,k) = \frac{n}{k} \left\{ \sum_{\underline{x}} \prod_{j=1}^{k} (\gamma(x_j)^2 + 2\gamma(x_j)) - \sum_{\underline{x}} \prod_{j=1}^{k} (2\gamma(x_j)) \right\}. \tag{6}$$

Now $\gamma(1) = 1$, $\gamma(2) = 2$, and $\gamma(r) = \gamma(r-1) + \gamma(r-2) + 1$ for $r \geqslant 3$.

Let

$$C_1(x) = \sum_{r\geqslant 1} (\gamma(r)^2 + 2\gamma(r))x^r = \frac{x(3-x)}{(1-x^2)(1-3x+x^2)}$$

and

$$C_2(x) = \sum_{r\geqslant 1} 2\gamma(r)x^r = \frac{2x}{(1-x)(1-x-x^2)}.$$

Again, using Theorem C(i) and (6), we have:

Lemma 4.2.

$$\mathrm{sf}(n,k) = \begin{cases} \mathscr{L}(2n) - 1, & k=0, \\ [x^n]\frac{n}{k}\{C_1(x)^k - C_2(x)^k\}, & k\geqslant 1. \end{cases}$$

$$\begin{aligned}&\mathrm{sf}(n)\\ &\quad = \mathscr{L}(2n) - 1\\ &\quad + [x^n]\frac{x(1-48x^3+48x^4-4x^5-3x^6+8x^7-8x^8+2x^9)}{(1+x)(1-3x+x^2)(1-x-x^2)(1-4x+x^3)(1-6x+x^2+3x^3-x^4)}.\end{aligned}$$

Theorem 4.3.

$$\begin{aligned}\mathrm{sf}(n) \approx{} & (5.7400)^n + (-0.7340)^n + (0.5953)^n + (0.3986)^n + ((1+\sqrt{5})/2)^n \\ & +((1-\sqrt{5})/2)^n - (-0.4728)^n - (3.9354)^n - (0.5374)^n - (-1)^n - 1.\end{aligned}$$

The first few terms in the sequence $\{\mathrm{sf}(n)\colon n\geqslant 2\}$ are $\{19, 132, 851, 5298, \ldots\}$.

5. Single-component edge-subgraphs (sce)

A *single-component edge-subgraph*, E, of M_n is an edge-subgraph with exactly one non-trivial connected component, i.e., exactly one connected component with one or more edges, all other components being isolated vertices. Clearly $\mathrm{sce}(n,0) = 2n(2n-1)+1$.

If $[u,v)$ has size r then the number of possible incomplete joins is $r(r+1)/2$.

Let E be a single-component edge-subgraph which comes from K of Fig. 3, and let $\overline{E}$ denote the graph obtained from E by switching all complete joins from the left-hand side of $11'$ to the right-hand side. From Theorem A(ii) we see that $\overline{E}$ has $\geqslant k-1$ complete joins to the right-hand side of $11'$.

Let E have $(k-1)+s$ complete joins in total, where $0\leqslant s\leqslant k+1$. Then two cases can occur:

(a) $\overline{E}$ has exactly $k-1$ complete joins to the right-hand side of $11'$. Let the ith join on the right-hand side of $11'$ in $\overline{E}$ be incomplete, for some i, where $1\leqslant i\leqslant k$, then its opposite join is also incomplete, (or otherwise case (b) occurs). So $\overline{E}$ has s complete joins to the left-hand side of $11'$, and E has exactly s complete join-pairs; here $0 \leqslant s \leqslant k-1$.

Let Z be a $(k-1-s)$-subset of $[k]\backslash\{i\}$ and let

$$\prod_{j\in\emptyset} \frac{x_j(x_j+1)}{2} = 1.$$

Then, the number of such E is given by

$$\sum_{i=1}^{k}\left\{\sum_{s=0}^{k-1}\left(\sum_{Z\subseteq[k]\backslash\{i\}}\prod_{j\in Z}\frac{x_j(x_j+1)}{2}\cdot 2^{|Z|}\right)\left(\frac{x_i(x_i+1)}{2}\right)^2\right\}$$
$$=\frac{1}{4}\sum_{i=1}^{k}\left\{\prod_{\substack{j=1\\ j\neq i}}^{k}(x_j^2+x_j+1)(x_i^2+x_i)^2\right\}.$$

(b) $\overline{E}$ has exactly k complete joins to the right-hand side of $11'$, so $\overline{E}$ has $s-1$ complete joins to the left-hand side of $11'$, and E has exactly $s-1$ complete join-pairs; here $1 \leqslant s \leqslant k+1$.

Let Z be a $(k-(s-1))$-subset of $[k]$. Hence, the number of such E is given by

$$\sum_{s=1}^{k+1}\left(\sum_{Z\subseteq[k]}\prod_{j\in Z}\frac{x_j(x_j+1)}{2}\cdot 2^{|Z|}\right) = \prod_{j=1}^{k}(x_j^2+x_j+1).$$

Lemma 5.1.

$$\mathrm{sce}(n,k) = \frac{n}{4k}\sum_{\underline{x}}\left(\sum_{i=1}^{k}\left\{\prod_{\substack{j=1\\ j\neq i}}^{k}(x_j^2+x_j+1)(x_i^2+x_i)^2\right\}\right)$$
$$+\frac{n}{k}\sum_{\underline{x}}\prod_{j=1}^{k}(x_j^2+x_j+1). \tag{7}$$

Let

$$D_1(x) = \sum_{r\geqslant 1}(r^2+r+1)x^r = \frac{x(3-2x+x^2)}{(1-x)^3} \quad \text{and}$$

$$D_2(x) = \sum_{r\geqslant 1}(r^2+r)^2x^r = \frac{4x(1+4x+x^2)}{(1-x)^5}.$$

Lemma 5.2.

$$\mathrm{sce}(n,k)=\begin{cases} 2n(2n-1)+1, & k=0,\\ [x^n]\{\frac{n}{4}D_1(x)^{k-1}D_2(x)+\frac{n}{k}D_1(x)^k\}, & k\geqslant 1.\end{cases}$$

This gives us

$$\mathrm{sce}(n)=2n(2n-1)+1 \\ +[x^n]\frac{x(4-13x+x^2-44x^3+54x^4-25x^5+x^6-2x^7)}{(1-x)^3(1-6x+5x^2-2x^3)^2}.$$

Partial fractions and De Moivre's theorem give:

Theorem 5.3.

$$\mathrm{sce}(n)\approx(0.6612n+1)(5.0958)^n+(2.1430)(0.6264)^n n\sin(0.7646n)+n^2 \\ -(0.6612n-2)(0.6264)^n\cos(0.7646n)-2n-2.$$

The first few terms in the sequence $\{\mathrm{sce}(n)\colon\ n\geqslant 2\}$ are $\{60,397,2464,14809,\ldots\}$.

6. Single-component forests (scf)

A *single-component forest*, F, of M_n is a forest which contains exactly one non-trivial connected component, all other components being isolated vertices.

An incomplete join $[u,v)$ of F must have at most two non-trivial components, each component must contain an edge incident to either u or v.

Thus, using Theorem A(iii), we have:

Lemma 6.1. *Let F be a (n,k)-graph. Then F is a single-component forest of M_n if and only if*

(i) *F is join-pair incomplete;*

(ii) *F has exactly $k-1$ complete joins; and*

(iii) *each incomplete join $[u,v)$ of F has at most two non-trivial components, each component containing an edge incident to either u or v.*

For $k=0$, we have $\mathrm{scf}(n,0)=2n(2n-1)$.

Again, if $[u,v)$ has size r, then the number of possible incomplete joins is $r(r+1)/2$.

For $k\geqslant 1$ let K be the (n,k)-graph of Fig. 3. Following Section 2 with $s=k-1$, the number of single-component forests which can be formed from K is given by

$$\left(2^{k-1}\prod_{j=1}^{k}\frac{x_j(x_j+1)}{2}\right)\cdot\sum_{i=1}^{k}\frac{x_i(x_i+1)}{2}=\frac{1}{4}\sum_{i=1}^{k}\left\{\prod_{\substack{j=1\\ j\neq i}}^{k}(x_j^2+x_j)(x_i^2+x_i)^2\right\}. \quad (8)$$

So

$$\mathrm{scf}(n,k)=\frac{n}{4k}\sum_{\underline{x}}\left(\sum_{i=1}^{k}\left\{\prod_{\substack{j=1\\ j\neq i}}^{k}(x_j^2+x_j)(x_i^2+x_i)^2\right\}\right).$$

Let

$$E(x)=\sum_{r\geqslant 1}(r^2+r)x^r=\frac{2x}{(1-x)^3} \quad \text{and} \quad D_2(x) \text{ be as in Section 5.}$$

Lemma 6.2.

$$\mathrm{scf}(n,k)=\begin{cases} 2n(2n-1), & k=0,\\ [x^n]\frac{n}{4}E(x)^{k-1}D_2(x), & k\geqslant 1.\end{cases}$$

Thus,

$$\mathrm{scf}(n)=2n(2n-1)+[x^n]\frac{x(1+9x-30x^2-20x^3+27x^4-9x^5-2x^6)}{(1-x)^3(1-5x+3x^2-x^3)^2}.$$

Theorem 6.3.

$$\mathrm{scf}(n)\approx 0.8757n(4.3652)^n-1.5432n(0.4786)^n\cos(0.8458n+0.9674)+n^2-2n.$$

The first few terms in the sequence $\{\mathrm{scf}(n)\colon n\geqslant 2\}$ are $\{34,222,1280,6955,\ldots\}$. The sequence $\{\mathrm{scf}(n,n)\}=\{n2^{n-1}\}$ appears as M3444 in [9].

7. Strong single-component edge-subgraphs (ssce)

A *strong single-component edge-subgraph* is a strong edge-subgraph which contains exactly one non-trivial connected component. Clearly, $\mathrm{ssce}(n,0)=2n+1$.

If E, an (n,k)-graph, is a strong single-component edge-subgraph of M_n then every incomplete join of E is the complete join less one edge; hence an incomplete join of size r has r possibilities.

Following (7) from Section 5 we have:

Lemma 7.1.

$$\mathrm{ssce}(n,k)=\frac{n}{k}\sum_{\underline{x}}\left(\sum_{i=1}^{k}\left\{\prod_{\substack{i=1\\ j\neq i}}^{k}(2x_j+1)x_i^2\right\}\right)+\frac{n}{k}\sum_{\underline{x}}\prod_{j=1}^{k}(2x_j+1).$$

Let

$$F_1(x) = \sum_{r\geqslant 1}(2r+1)x^r = \frac{x(3-x)}{(1-x)^2} \quad \text{and} \quad F_2(x) = \sum_{r\geqslant 1} r^2x^r = \frac{x(1+x)}{(1-x)^3}.$$

Lemma 7.2.

$$\mathrm{ssce}(n,k) = \begin{cases} 2n+1, & k=0, \\ [x^n]\left\{nF_1(x)^{k-1}F_2(x) + \frac{n}{k}F_1(x)^k\right\}, & k\geqslant 1. \end{cases}$$

Thus,

$$\mathrm{ssce}(n) = 2n+1+[x^n]\frac{x(4-15x+2x^2+5x^3)}{(1-x)^2(1-5x+2x^2)^2}.$$

Theorem 7.3.

$$\mathrm{ssce}(n) = \left(\left\{\frac{17-\sqrt{17}}{34}\right\}n+1\right)\left(\frac{5+\sqrt{17}}{2}\right)^n + \left(\left\{\frac{17+\sqrt{17}}{34}\right\}n+1\right)\left(\frac{5-\sqrt{17}}{2}\right)^n + n - 1.$$

The first few terms in the sequence $\{\mathrm{ssce}(n)\colon n\geqslant 2\}$ are $\{38, 205, 1092, 5719, \ldots\}$.

8. Spanning trees (st)

If a (n,k)-graph is a spanning tree of M_n then, as in Section 7 above, an incomplete join must be the complete join less one edge.

Lemma 8.1. *Let F be a (n,k)-graph. Then F is a spanning tree of M_n if and only if*
(i) *F is join-pair incomplete*;
(ii) *F has exactly $k-1$ complete joins*; *and*
(iii) *each incomplete join of F is the complete join less one edge.*

For $k=0$ we have $\mathrm{st}(n,0)=2n$. The number of possible incomplete joins for a join of size r is r. So, for $k\geqslant 1$, cf., (8) from Section 6, the number of spanning trees which can be formed from K of Fig. 3 is

$$\left(2^{k-1}\prod_{j=1}^{k} x_j\right)\cdot\sum_{i=1}^{k} x_i = n\,2^{k-1}\prod_{j=1}^{k} x_j.$$

Then

$$\mathrm{st}(n,k)=\frac{n^2 2^{k-1}}{k}\sum_{\underline{x}}\prod_{j=1}^{k}x_j.$$

Letting

$$G(x)=\sum_{r\geqslant 1}rx^r=\frac{x}{(1-x)^2},$$

we have:

Lemma 8.2.

$$\mathrm{st}(n,k)=\begin{cases}2n, & k=0,\\ [x^n]\dfrac{n^2 2^{k-1}}{k}G(x)^k, & k\geqslant 1.\end{cases}$$

So

$$\mathrm{st}(n)=2n+[x^n]\frac{x(1+2x-10x^2+2x^3+x^4)}{(1-x)^2(1-4x+x^2)^2}.$$

Theorem 8.3.

$$\mathrm{st}(n)=\frac{n}{2}[(2+\sqrt{3})^n+(2-\sqrt{3})^n]+n.$$

The first few terms in the sequence $\{\mathrm{st}(n)\colon n\geqslant 2\}$ are $\{16,81,392,1815,\ldots\}$.

According to p. 42 of Biggs [1] there are two known methods to compute the number of spanning trees of M_n. The first is the Matrix Tree Theorem, see [1, pp. 39–40]; and the second is a recursive method mentioned at the end of Section 2; [2,8]. This, then, provides a third method.

9. Restricted edge-subgraphs (re)

A *restricted edge-subgraph* of M_n is an edge-subgraph with maximum degree 2, i.e., every component is an isolated vertex, a path, or a cycle.

It is straightforward to show that an edge-subgraph, E, of M_n is restricted if and only if it is the complement in M_n of a strong edge-subgraph, (Section 3). Clearly, the number of complements of strong edge-subgraphs is equal to the number of strong edge-subgraphs, hence $\mathrm{re}(n)=\mathrm{se}(n)$. Thus:

Theorem 9.1.

$$\mathrm{re}(n)\approx(6.4188)^n+(-0.8056)^n+(0.3867)^n-(-1)^n.$$

10. Restricted forests (rf)

In a *restricted forest* every component is an isolated vertex or a path. Unfortunately, we are unable to use this counting technique to compute the exact value of $\mathrm{rf}(n)$. However, we can use it to obtain the following theorem, the details are omitted.

Theorem 10.1. *For large n,*

$$(4.8820)^n < \mathrm{rf}(n) < (6.4188)^n.$$

The first few terms in the sequence $\{\mathrm{rf}(n)\colon n \geqslant 2\}$ are $\{34, 241, 1582, \ldots\}$, which suggests that the value of $\mathrm{rf}(n)$ is about $(6.3)^n$. In some of the following sections we will need knowledge of cycles and unions of cycles in M_n.

Clearly, each union of cycle(s) in M_n has either $k=0$ or $k \geqslant 1$ diagonals, so

(i) $k=0$: we have one cycle of M_n, i.e., C_{2n};

(ii) $k \geqslant 1$: to transform the empty (n,k)-graph K of Fig. 3 into a union of cycle(s) and isolated vertices without increasing the number of diagonals, we must complete every second join beginning at X_1 or at X_2. Either way will result in a single cycle if k is odd; or a union of $(k/2)$ cycles, each containing 2 diagonals, if k is even.

These constructions determine all cycles and union of cycles of M_n.

11. Union of cycles (uc)

An edge-subgraph of M_n is a *union of cycles* if each component is a cycle or an isolated vertex.

Clearly $\mathrm{uc}(n,0)=2$. For $k \geqslant 1$, by the comments above, each K of Fig. 3 determines 2 such edge-subgraphs, so $\mathrm{uc}(n,k) = (n/k)\sum_{\underline{x}} 2 = \binom{n}{k}2$.

Theorem 11.1.

$$\mathrm{uc}(n) = 2^{n+1}.$$

12. Cycles (*c*)

A *cycle* of M_n is an edge-subgraph with exactly one cycle, all other components being isolated vertices. We have: $c(n,0)=1$, $c(n,2)=2$, and $c(n,k)=0$ for k even and $\geqslant 4$; and $c(n,k)=2$ for k odd. Hence,

$$c(n) = 1 + 2\binom{n}{2} + \sum_{k \text{ odd}} 2\binom{n}{k}.$$

Theorem 12.1.

$$c(n) = 2^n + n^2 - n + 1.$$

This formula appears in Entringer and Slater [4]. The first few terms in the sequence $\{c(n)\colon n \geqslant 2\}$ are $\{7, 15, 29, 53, \ldots\}$.

13. Hamiltonian cycles (hc)

Clearly $\mathrm{hc}(n,0) = 1$. If K of Fig. 3 determines the 2-composition $1, n-1$ of n then K gives rise to two Hamiltonian cycles for $n = 2$ and one if $n \geqslant 3$. Hence, $\mathrm{hc}(2,2) = 2$ and $\mathrm{hc}(n,2) = 1 \cdot n = n$ if $n \geqslant 3$; this deals with even k. For odd k, the graph K must determine the k-composition $\underbrace{1 \cdots 1}_{k}$ of n, which gives two more Hamiltonian cycles, hence $k = n$.

Theorem 13.1.

$$\mathrm{hc}(n) = \begin{cases} n+1, & n \text{ even}, \\ n+3, & n \text{ odd}. \end{cases}$$

14. Paths (p)

Let F be a (n,k)-graph which is a path of M_n, with first vertex v_f and last vertex v_ℓ, $v_f \neq v_\ell$; as usual all other components of F are isolated vertices. We will abuse notation and denote by F the non-trivial connected component of F.

Now F is acyclic and connected, and so, from Theorem A(iii), it is join-pair incomplete and has exactly $k-1$ complete joins. Hence F has exactly one join-pair, X_1X_1', in which both joins are incomplete. All other join-pairs have exactly one join complete and the other incomplete.

Let $k \geqslant 4$. Consider the join-pairs X_2X_2' and X_kX_k' to the right and left of X_1X_1', as indicated in Fig. 4; exactly one join in each is complete. There are two cases to consider:

(a) Suppose that both X_2 and X_k are complete, see Fig. 4(a). Without loss of generality, $v_f \in X_k' \cup X_1$, (i.e., v_f is connected to 1); for, suppose not, then at least one of X_k' or X_1 is complete, a contradiction. Similarly $v_\ell \in X_1' \cup X_2'$.

Hence any join between X_2 and X_k, and between X_2' and X_k', is either complete or empty because it cannot contain v_f or v_ℓ. Thus, any join-pair X_jX_j' for $3 \leqslant j \leqslant k-1$ has one join complete and the other empty, this forces k to be even.

Since v_f is connected to 1 there are $(x_k + x_1 - 1)$ vertices where v_f can be placed (the edges between v_f and 1 are then included as part of F), and v_ℓ can be placed with a choice of $(x_1 + x_2 - 1)$ places. Each placement can then be extended uniquely

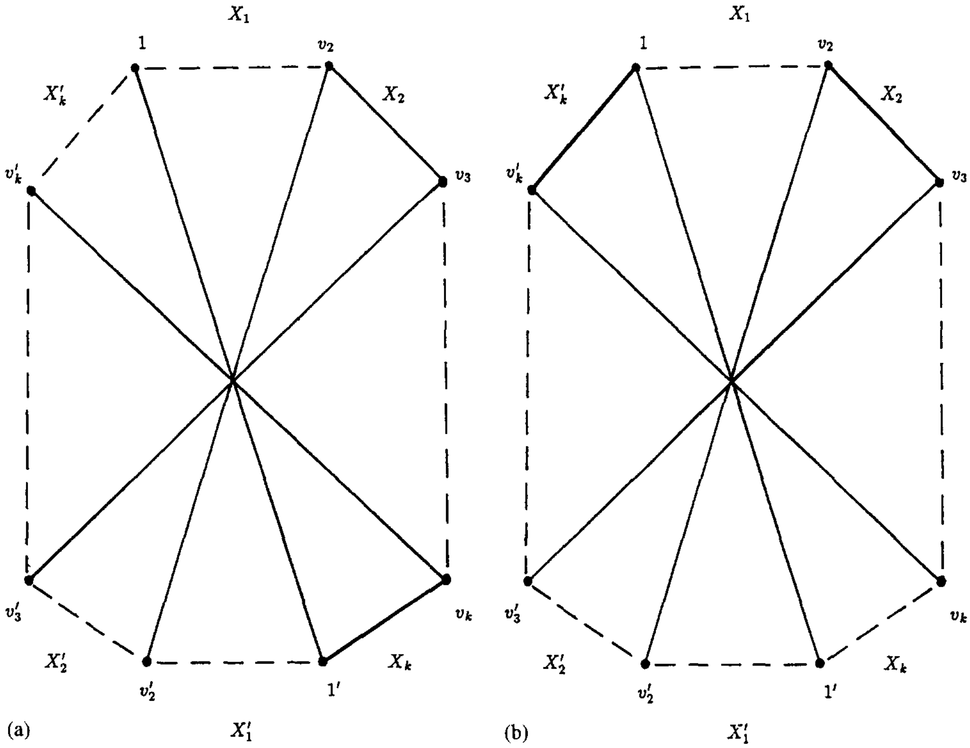

Fig. 4.

to form a path. Similarly if both X'_2 and X'_k are complete. Also, the unique join-pair with both joins incomplete can be any of the k join-pairs $X_jX'_j$ for $1 \leqslant j \leqslant k$.

Hence, the total number of paths for this case is

$$k \cdot \frac{n}{k} \sum_{\underline{x}} 2(x_1 + x_2 - 1)(x_1 + x_k - 1) = 10n\binom{n}{k+1} + 2n\binom{n-1}{k-1}.$$

(b) Suppose that both X_2 and X'_k are complete, see Fig. 4(b); then k is odd and $\geqslant 3$. Without loss of generality, $v_f \in X'_1 \cup X'_2$ and $v_\ell \in X_k \cup X'_1$.

If $v_f \in X'_2$ (with x_2 places) then the number of places for v_ℓ is $(x_1 + x_k - 1)$, yielding $x_2(x_1 + x_k - 1)$ paths.

If $v_f \in X'_1$, the number of places for v_ℓ is $\sum_{t=1}^{x_1-1}(x_1 + x_k - 1 - t)$, where t is the number of edges between v_f and v'_2.

From the previous two paragraphs and the comments in (a) above, the number of paths for this case is

$$\begin{aligned} &k \cdot \frac{n}{k} \sum_{\underline{x}} 2\left(\frac{x_1^2}{2} + x_1x_2 + x_1x_k + x_2x_k - \frac{3}{2}x_1 - x_2 - x_k + 1\right) \\ &\quad = 8n\binom{n}{k+1} + 2n\binom{n-1}{k-1}. \end{aligned}$$

The formula for $p(n,2)$ may be incorporated into the formula for $p(n,k)$ derived in (a) above.

Lemma 14.1.

$$p(n,k)=\begin{cases} 2n(2n-1), & k=0,\\ n(3n^2-3n+1), & k=1,\\ 10n\binom{n}{k+1}+2n\binom{n-1}{k-1}, & k \text{ even and } \geqslant 2,\\ 8n\binom{n}{k+1}+2n\binom{n-1}{k-1}, & k \text{ odd and } \geqslant 3. \end{cases}$$

Theorem 14.2.

$$p(n)=10n2^n-n^3-5n^2-11n.$$

The first few terms in the sequence $\{p(n)\colon n\geqslant 2\}$ are $\{30,135,452,1295,\ldots\}$. The sequence $\{p(n,1)\}=\{n(3n^2-3n+1)\}$ is M4933 in [9].

15. Hamiltonian paths (hp)

In this section let $\underline{x}=x_1\cdots x_k=x_1,\ldots,x_k$.

Again, let F be a (n,k)-graph which is a Hamiltonian path for M_n, then using the same reasoning as in Section 14, there are two cases to consider:

(a) Both X_2 and X_k are complete. Since F is a Hamiltonian path then K of Fig. 4(a) determines the composition

$$\underline{y}=x_1,x_2,\underbrace{1,\ldots,1}_{k-3},n-x_1-x_2-k+3,$$

or otherwise F will contain isolated vertices. Furthermore, k is even and $\geqslant 4$, and at least one of $x_1=1$ or $x_k=1$, and at least one of $x_1=1$ or $x_2=1$, or, again, there will be isolated vertices in F.

(i) If $x_1=1$ then

$$\underline{y}=1,x_2,\underbrace{1,\ldots,1}_{k-3},n-x_2-k+2,$$

where $1\leqslant x_2\leqslant n-k+1$.

Hence, cf. Section 14, the total number of Hamiltonian paths for this case is $k\cdot(n/k)\sum_{\underline{y}}2=2n(n-k+1)$.

(ii) If $x_1 > 1$ then $x_k = x_2 = 1$, and

$$\underline{y} = n-k+1, \underbrace{1,\ldots,1}_{k-1},$$

and the total number of Hamiltonian paths is $2n$.

(b) Both X_2 and X_k' are complete. Then k is odd and $\geqslant 3$, and the relevant composition determined by Fig. 4(b) is again

$$\underline{y} = 1, x_2, \underbrace{1,\ldots,1}_{k-3}, n - x_2 - k + 2,$$

which yields $2n(n-k+1)$ Hamiltonian paths.

Again, the formula for $\mathrm{hp}(n,2)$ may be incorporated into the formula for $\mathrm{hp}(n,k)$ derived in (a) above.

Lemma 15.1.

$$\mathrm{hp}(n,k) = \begin{cases} 2n, & k = 0, 1, \text{ and } k = n, \\ 2n(n-k+2), & k \text{ even and } 2 \leqslant k \leqslant n-1, \\ 2n(n-k+1), & k \text{ odd and } 3 \leqslant k \leqslant n-1. \end{cases}$$

Theorem 15.2.

$$\mathrm{hp}(n) = \begin{cases} n^3 + 2n, & n \text{ even}, \\ n^3 + 3n, & n \text{ odd}. \end{cases}$$

The first few terms in the sequence $\{\mathrm{hp}(n)\colon n \geqslant 2\}$ are $\{12, 36, 72, 140, \ldots\}$.

16. No-leaf edge-subgraphs (nle)

Call a vertex in a graph a *leaf* if it has degree 1. A *no-leaf edge-subgraph* of M_n is an edge-subgraph with no leaves, i.e., each of its vertices has degree 0, 2, or 3. Clearly, $\mathrm{nle}(n,0) = 2$. If E, an (n,k)-graph, is a no-leaf edge-subgraph then its joins must be empty or complete; moreover, its empty joins must be non-consecutive, cyclic. So the number of such E which come from an arbitrary K of Fig. 3 is $\mathscr{L}(2k)$.

Hence,

$$\mathrm{nle}(n,k) = \frac{n}{k} \sum_{\underline{x}} \mathscr{L}(2k) = \binom{n}{k} \mathscr{L}(2k).$$

Theorem 16.1.

$$\mathrm{nle}(n) = \left(\frac{5+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{2}\right)^n.$$

The first few numbers in the sequence $\{\mathrm{nle}(n)\colon n \geqslant 2\}$ are $\{15, 50, 175, 625, \ldots\}$.

17. Matchings (*m*)

A *matching*, E, of M_n is a collection of disjoint edges and isolated vertices. So $m(n,0) = \mathscr{L}(2n)$.

Let $\delta(r)$ be the number possibilities for a join of E of size r. Then $\delta(1) = \delta(2) = 1$, and $\delta(r) = \delta(r-1) + \delta(r-2)$ for $r \geqslant 3$.

By exact analogy with Section 3, the number of matchings which come from K of Fig. 3 is $\prod_{j=1}^{k} \delta(x_j)^2$. This gives us

$$m(n) = \mathscr{L}(2n) + [x^n]\frac{x(1-2x+4x^2-2x^3+x^4)}{(1+x)(1-3x+x^2)(1-3x-x^2+x^3)}.$$

Theorem 17.1.

$$m(n) \approx (3.2143)^n + (-0.6751)^n + (0.4608)^n - (-1)^n.$$

The first few numbers in the sequence $\{m(n)\colon n \geqslant 2\}$ are $\{10, 34, 106, 344, \ldots\}$, see Table 1 in Hosoya and Harary [7].

Let H_n be an arbitrary cubic bipartite graph on $2n$ vertices, then, by a similar argument to that used at the end of Section 3, the graph H_n contains at most 4^n matchings. Hence: For odd n, does there exist a family of cubic bipartite graphs, G_n, on $2n$ vertices, such that, for some n_0 and all odd $n > n_0$, the number of matchings of G_n is greater than $(3.2143)^n$?

18. One-factors (of) two-factors (tf)

A *one-factor*, E, of M_n is a collection of n disjoint edges, i.e., E is regular of degree one. For example, see Fig. 1(b), let B_n be the one-factor containing outside edge 12 and every second outside edge, and C_n the one-factor containing the outside edges not in B_n.

An *edge-pair* is a pair of opposite outside edges, i.e., the pair $e_j = \{j(j+1), j'(j+1)'\}$ for some j with $1 \leqslant j \leqslant n-1$, or the pair $e_n = \{n1', n'1\}$. Let us identify $[n]$ with the set of edge-pairs, $\mathscr{E}_n = \{e_1, \ldots, e_n\}$, in the obvious way. Recall the definition of an ncc set of $[n]$, so of $\mathscr{E}_n$, and the number of ncc t-sets of $[n]$, $\psi(n,t)$, from the Preliminaries.

An *edge-pair one-factor* is a one-factor in which all outside edges occur in edge-pairs. A *t-edge-pair one-factor* is an edge-pair one-factor which contains t edge-pairs.

Let F be a t-edge-pair one-factor of M_n, then its t edge-pairs form a ncc t-set of $\mathscr{E}_n$. Conversely, the edge-pairs corresponding to a ncc t-set of $\mathscr{E}_n$ can be uniquely extended to a t-edge-pair one-factor by adding on the diagonals ii' for each vertex i not already covered. Hence, the number of t-edge-pair one-factors of M_n is $\psi(n,t) = n/(n-t)\binom{n-t}{t}$, where $0 \leqslant t \leqslant \lfloor n/2 \rfloor$.

Here we have not used our composition technique, however, consider the following: Call a k-composition of n, $\underline{x} = x_1 x_2 \cdots x_k$, *odd* if each x_j is odd. By adding 1 to each such x_j, we can set up a bijection between odd k-compositions of n and k-compositions of $(n+k)/2$; hence, the number of odd k-compositions of n is $\binom{(n+k)/2-1}{k-1}$. Now, an (n,k)-graph, E, can be extended to a unique one-factor if and only if each of its joins is odd. Let $\varepsilon(r)$ denote the number of possibilities for a join of E of size r, then $\varepsilon(r) = 0$ if r is even and $\varepsilon(r) = 1$ if r is odd. Also, note that an edge-pair one-factor with k diagonals has $(n-k)/2$ edge-pairs. Hence,

$$\mathrm{of}(n,k) = \frac{n}{k} \sum_{\underline{x}} \prod_{j=1}^{k} \varepsilon(x_j)^2 = \frac{n}{k} \binom{\frac{n+k}{2} - 1}{k-1} = \frac{2n}{n+k} \binom{\frac{n+k}{2}}{\frac{n-k}{2}} = \psi\left(n, \frac{n-k}{2}\right)$$

as required. It is straightforward to prove that most one-factors of M_n are edge-pair one-factors:

Lemma 18.1. *All one-factors of M_n are edge-pair one-factors, except for B_n and C_n when n is odd.*

Hence, for odd n, $\mathrm{of}(n) = \sum_{t=0}^{\lfloor n/2 \rfloor} \psi(n,t) + 2$, where the $+2$ corresponds to the non-edge-pair one-factors B_n and C_n; we do not need the $+2$ for even n.

The numbers $\mathrm{of}(n)$ are closely related to the Lucas numbers, see the comments in the Preliminaries.

Now, a two-factor of M_n is the complement in M_n of a one-factor, and vice versa, so:

Theorem 18.2.

$$\mathrm{of}(n) = \mathrm{tf}(n) = \begin{cases} \mathscr{L}(n), & n \text{ even}, \\ \mathscr{L}(n) + 2, & n \text{ odd}. \end{cases}$$

This formula appears in [7]. The first few terms in the sequence $\{\mathrm{of}(n) \colon n \geqslant 2\}$ are $\{3, 6, 7, 13, \ldots\}$.

19. One-factorizations (oF)

In this section we do not use our composition technique. Our results give us a combinatorial interpretation of a sequence which occurs on p. 603 of Guy [5], it appears to be the first combinatorial interpretation of this sequence.

A *one-factorization* of M_n is a triple of one-factors which partition the edges of M_n. For example, see Fig. 1(b), let A_n be the one-factor of M_n consisting of its n diagonals, and B_n and C_n be as in Section 18, then $\{A_n, B_n, C_n\}$ is a one-factorization of M_n.

An *edge-pair cycle* is a cycle in which all outside edges occur in edge-pairs, each edge-pair cycle is an even cycle. An *edge-pair two-factor* is a two-factor in which each component is an edge-pair cycle.

As noted earlier, if we remove a one-factor from M_n we obtain a two-factor, moreover:

Lemma 19.1. *Let F be an edge-pair one-factor of M_n. Then, for $t \geqslant 1$, F is a t-edge-pair one-factor if and only if $M_n - F$ is an edge-pair two-factor with t components.*

Proof. Assume that F is a t-edge-pair one factor where $t \geqslant 1$.

Without loss of generality, let two consecutive edge-pairs of F be $\{c(c+1), c'(c+1)'\}$ and $\{d(d+1), d'(d+1)'\}$, see Fig. 5, with obvious notational changes if $d=n$. Then $c+1, c+2, \ldots, d-1, d, d', (d-1)', \ldots, (c+2)', (c+1)', c+1$, shown with an arrowed broken line in Fig. 5, is an even edge-pair cycle of length $2(d-c)$ which lies in $M_n - F$. Similarly, there is an even edge-pair cycle 'between' any pair of consecutive edge-pairs of F, and there are t such pairs of edge-pairs, and so t such cycles, which cover all vertices of M_n. Hence, $M_n - F$ is a edge-pair two-factor with t components.

Conversely, if X is an edge-pair two-factor with t components, then, clearly, $M_n - X$ is an edge-pair one-factor. Suppose $M_n - X$ has $s \neq t$ edge-pairs. Then, by above, $X = M_n - (M_n - X)$ has s components, a contradiction. Hence, $M_n - X$ is a t-edge-pair one-factor. □

An *edge-pair one-factorization* of M_n is a one-factorization in which all one-factors are edge-pair one-factors. It is straightforward to prove the following, (cf. Lemma 18.1):

Lemma 19.2. *All one-factorizations of M_n are edge-pair one-factorizations, except for $\{A_n, B_n, C_n\}$ when n is odd.*

Let the number of edge-pair one-factorizations of M_n be $\mathrm{eoF}(n)$.

Again, let F be a t-edge-pair one-factor where $t \geqslant 1$. Now, an even cycle has 2 one-factors, thus $M_n - F$, which contains t even cycles, has 2^t one-factors.

Let $\{F, F', F''\}$ be an edge-pair one-factorization of M_n which contains F. How many such one-factorizations are there? Such a one-factorization can be formed by choosing F first, then there are 2^t ways in which to choose F', and the remaining edges then form F''. However, this one-factorization could have been formed by choosing F'' second and leaving F'. Hence, the total number of edge-pair one-factorizations which contain F is $\frac{1}{2} \cdot 2^t = 2^{t-1}$.

Recall that the number of t-edge-pair one-factors of M_n is $\psi(n,t)$. Hence, for even n, we have $3\mathrm{eoF}(n) = \sum_{t=1}^{\lfloor n/2 \rfloor} \psi(n,t) 2^{t-1} + 1$, where the $+1$ corresponds to the 0-edge-pair one-factor A_n; we do not need the $+1$ for odd n. This gives: $\mathrm{eoF}(2) = \mathrm{eoF}(3) = 1$ and $\mathrm{eoF}(n) = \mathrm{eoF}(n-1) + 2\mathrm{eoF}(n-2)$ for $n \geqslant 4$. So $\mathrm{eoF}(n) = \frac{1}{3}\{2^{n-1} + (-1)^n\}$.

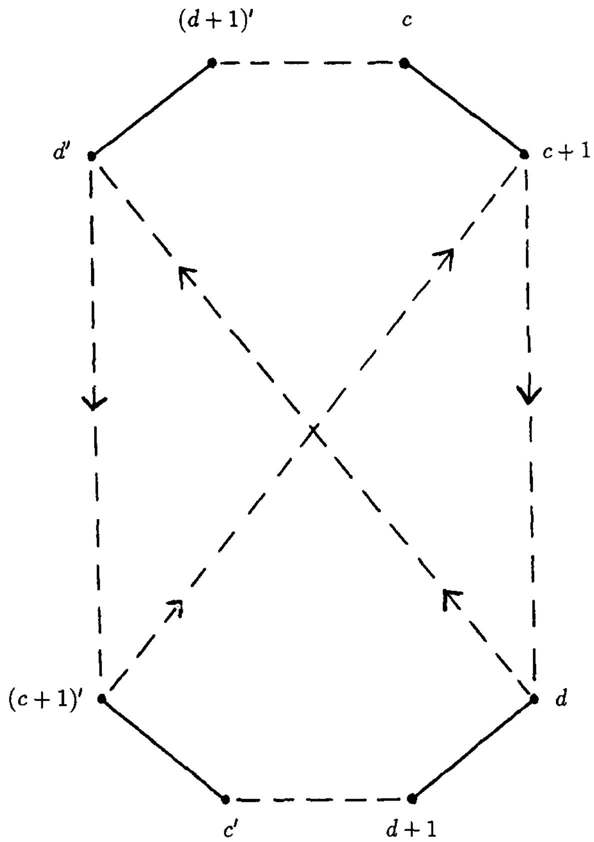

Fig. 5.

Now $\mathrm{oF}(n)=\mathrm{eoF}(n)$ if n is even, and $\mathrm{oF}(n)=\mathrm{eoF}(n)+1$ if n is odd via Lemma 19.2. Hence:

Theorem 19.3.

$$\mathrm{oF}(n)=\begin{cases}\frac{1}{3}(2^{n-1}+1), & n \textit{ even},\\ \frac{1}{3}(2^{n-1}+2), & n \textit{ odd}.\end{cases}$$

The first few terms in the sequence $\{\mathrm{oF}(n)\colon n\geqslant 2\}$ are $\{1,2,3,6,\ldots\}$. This sequence is M0788 in [9], and appears at the bottom of p. 603 in [5].

20. Perfect one-factorizations (poF)

A *perfect one-factorization* of M_n is a one-factorization in which the union of any 2 one-factors is a Hamiltonian cycle.

If $n=2$ or if n is odd and $\geqslant 5$ then, via Lemma 19.1, $\{A_n, B_n, C_n\}$ is the only perfect one-factorization of M_n. If $n=3$ there is one more perfect one-factorization, there are none for n even and $\geqslant 4$.

Theorem 20.1.

$$\mathrm{poF}(n)=\begin{cases} 1, & n=2 \text{ or } n \text{ is odd and } \geqslant 5,\\ 2, & n=3,\\ 0, & n \text{ even and } \geqslant 4. \end{cases}$$

21. Elementary edge-subgraphs (ee)

An *elementary edge-subgraph*, E, of M_n is an edge-subgraph in which every component is regular of degree 0, 1, or 2; i.e., every component is an isolated vertex, a single edge, or a cycle. Clearly $\mathrm{ee}(n,0)=\mathscr{L}(2n)+1$.

See K of Fig. 3, and let $k\geqslant 2$. In order to extend K to an elementary edge-subgraph we can first form cycles, then add on disjoint edges. Now, from Section 12, cycles of M_n can contain either two diagonals, or k diagonals for odd k.

For any k, our extended K, call it E, may contain cycles with two diagonals. These cycles are formed by choosing both joins in an arbitrary join-pair to be complete. These join-pairs must be a ncc (sub)set of $\mathscr{X}_k=\{X_1X_1', \ldots, X_kX_k'\}$, denote this set by N. Those joins of E which are incomplete are either empty or a matching. Hence, such a join of size r has $\delta(r)$ possibilities where $\delta(r)$ is defined as in Section 17. Conversely, any ncc set $N\subset\mathscr{X}_k$ gives rise to $|N|$ cycles by completing the join-pairs corresponding to the elements of N; this graph can then be extended to elementary edge-subgraphs by adding disjoint, outside edges. Let us identify $\mathscr{X}_k$ with $[k]$ in the obvious manner. Hence, the number of elementary edge-subgraphs which come from K is $\sum_{N\subset[k]}\prod_{j\notin N}\delta(x_j)^2$.

If k is odd, we may have a cycle with k diagonals. Here either join X_1 or X_2 of E is complete and then every second join is complete. Again, incomplete joins have $\delta(r)$ possibilities, hence K gives rise to $2\prod_{j=1}^{k}\delta(x_j)$ elementary edge-subgraphs.

Theorem 21.1.

$$\mathrm{ee}(n,k)=\begin{cases} \mathscr{L}(2n)+1, & k=0,\\ \frac{n}{k}\sum_{\underline{x}}\left\{\sum_{\substack{N\subset[k]\\ N \text{ ncc set}}}\prod_{j\notin N}\delta(x_j)^2\right\}+\frac{2n}{k}\sum_{\underline{x}}\prod_{j=1}^{k}\delta(x_j), & k \text{ odd},\\ \frac{n}{k}\sum_{\underline{x}}\left\{\sum_{\substack{N\subset[k]\\ N \text{ ncc set}}}\prod_{j\notin N}\delta(x_j)^2\right\}, & k \text{ even}. \end{cases}$$

It does not seem possible to compute a concise expression for $\mathrm{ee}(n)$.

The first few terms in the sequence $\{\mathrm{ee}(n)\colon n\geqslant 2\}$ are $\{17, 58, 181, 602, \ldots\}$.

22. Strong elementary edge-subgraphs (see)

A *strong elementary edge-subgraph* (sometimes called *sesquivalent*) of M_n is an elementary edge-subgraph which is strong. An even join of K of Fig. 3 cannot be made strong using disjoint edges; an odd join can be made strong in one way.

The formulas for $\mathrm{see}(n,k)$ are as in Section 21 except that $\mathrm{see}(n,0)=3$, and $\delta(r)$ must be replaced by $\varepsilon(r)$ where $\varepsilon(r)=0$ if r is even, and $\varepsilon(r)=1$ if r is odd, ($\varepsilon(r)$ as in Section 18).

Again, there does not seem to be a more precise expression for $\mathrm{see}(n)$.

The first few terms in the sequence $\{\mathrm{see}(n)\colon n\geqslant 2\}$ are $\{6,21,26,81,\ldots\}$. The sequence $\{\mathrm{see}(n,n-1)\}=\{(n/\sqrt{5})\{((1+\sqrt{5})/2)^{n-2}-((1-\sqrt{5})/2)^{n-2}\}\}$, appears as M2362 in [9], Generalized Lucas Numbers. Finally, the degree of a vertex in an edge-subgraph of M_n is 0, 1, 2, or 3. Hence, we may choose any subset of $\{0,1,2,3\}$ and count the subgraphs of M_n whose vertex degrees are equal to this subset. For example, in Section 17, we counted matchings which corresponds to the subset $\{0,1\}$; also, by taking complements in M_n, we counted the edge-subgraphs corresponding to the subset $\{3-0,3-1\}=\{2,3\}$. Barring trivial cases, we have considered all subsets except $\{1,2\}$, which is our final section.

23. Strong restricted edge-subgraphs (sre)

A *strong restricted edge-subgraph* of M_n is an edge-subgraph in which the degree of each vertex is 1 or 2, i.e., every component is a path with at least one edge, or a cycle.

We are unable to use this counting technique to compute the value of $\mathrm{sre}(n)$ exactly. However, we have the following symmetric relationship, since the class of strong restricted edge-subgraphs is closed under complements in M_n: $\mathrm{sre}(n,k)=\mathrm{sre}(n,n-k)$. We also have:

Theorem 23.1. *For large n,*

$$(3.2143)^n<\mathrm{sre}(n)<(6.4188)^n.$$

The first few terms in the sequence $\{\mathrm{sre}(n)\colon n\geqslant 2\}$ are $\{18,102,418,2006,\ldots\}$, which suggests that the value of $\mathrm{sre}(n)$ is about $(4.6)^n$.

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