CYCLIC PERMUTATIONS IN DOUBLY-TRANSITIVE GROUPS

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INTRODUCTION

Let \( \Omega \) be a finite set of size \( n \). A **cyclic permutation** on \( \Omega \) is a permutation whose cycle decomposition is one cycle of length \( n \). This paper classifies all finite doubly-transitive permutation groups which contain a cyclic permutation. The classification appears in Table 1.

We use \((G, \Omega)\) for a finite doubly-transitive permutation group \( G \) acting on a finite set \( \Omega \). For other notation and definitions see the self-contained article Cameron [1].

CLASSIFICATION

\((G, \Omega)\) has a unique minimal normal subgroup \( N = \text{soc}(G) \), which is either elementary abelian or simple.

In the first case suppose \((G, \Omega)\) has an elementary abelian regular normal subgroup \( N \) of size \( p^d \), where \( d \geq 1 \). Let \( g \in G \) be a cyclic permutation, it has order \( p^d \). Now \( G \leq AGL(d, p) \leq GL(d + 1, p) \). By considering the JCF of \( g \) we have \( p^{d-1} + 1 \leq d + 1 \), so \( d = 1 \) or \( p = d = 2 \). So \( G \) contains no cyclic permutations unless \( d = 1 \) or \( p = d = 2 \). See Table 1, \( d = 1 \) corresponds to row \( a \) and \( p = d = 2 \) to row \( b \).

In the second case, when \( N \) is simple, \( N \) is known because of the classification of the finite simple groups. Cameron [1] tabulates all simple groups, \( N \), which occur as socles of finite doubly-transitive groups.

We have \( N \leq G \leq \text{Aut}(N) \). For each row of the table in [1] we will check such \( G \) for cyclic permutations:

- **N = A\(_n\)**: Clearly \( A_n \) contains a cyclic permutation if and only if \( n \) is odd. When \( n \geq 5 \) and \( n \) is odd, then \( \text{Aut}(A_n) \cong S_n \). Hence \( G \cong A_n \) or \( S_n \), see rows \( c \) and \( d \) of Table 1.
\( N = \text{PSL}(d, q) \): Here Zsigmondy’s theorem may be used. If \( G = \text{PSL}(2, 8) \) there is nothing to prove. Consider \( GL(1, q^d) \triangleleft \Gamma L(1, q^d) \leq \Gamma L(d, q) \). Except for the case that \( d = 2 \) and \( q \) is a Mersenne prime, let \( p \) be a primitive prime divisor of \( q^d - 1 \) and let \( P \) be a Sylow \( p \)-subgroup of \( GL(1, q^d) \). We may check that \( \Gamma L(1, q^d) = N_{\Gamma L(d, q)}(P) \) and \( GL(1, q^d) = C_{\Gamma L(1, q^d)}(P) \).

Now \( p \) does not divide \( q - 1 \), so any cyclic permutation must be the image in \( PTL(d, q) \) of a cyclic subgroup of \( \Gamma L(d, q) \) containing \( P \) or a conjugate, and so must be a conjugate of the image of \( GL(1, q^d) \). Hence such a cyclic permutation must lie in \( PGL(d, q) \). Finally, if \( d = 2 \) and \( q \) is a Mersenne prime, a similar argument can be made with a subgroup \( P \) of order 4.

Hence, for every \( d \geq 2 \) and prime power \( q \), a group \( G \) for which \( \text{PSL}(d, q) \leq G \leq \text{PTL}(d, q) \) contains a cyclic permutation if and only if \( PGL(d, q) \leq G \). See row e of Table 1. Thus, we have decided which subgroups of \( P\Gamma L(d, q) \) have cyclic permutations, see p.179 of Feit [3].

\( N = \text{PSU}(3, q) \): Here we use Liebeck, Praeger, and Saxl [4] which lists all maximal factorizations of all finite simple groups and their automorphism groups. Let \( g \in G \) be a cyclic permutation. In this case \( N \) is already doubly-transitive and so we need only consider \( G = N\langle g \rangle \). If \( M \) is any maximal subgroup of \( G \) containing \( g \), then \( G = MG_\alpha \) is a maximal factorization and appears in these lists.

From the lists on p.13 of [4] only \( G = \text{PSU}(3, q) \) for \( q = 3, 5, \) and 8 has a maximal factorization. In the first two cases the group \( A \) does not contain an element of order \( q^d + 1 \), so we may exclude them. In the final case, since \( G = N\langle g \rangle \), so \( G/N \) is cyclic, and then this case is out by their remark. Hence, \( \text{PSU}(3, q) \) contains no cyclic permutations.

\( N = 3\text{B}_2(q) \) and \( 2\text{G}_2(q) \): The lists also take care of these two groups.

\( N = \text{PSp}(2d, 2) \): Here both permutation representations have even degree, hence a cyclic permutation is an odd permutation, but \( N \) is complete.

For the remaining cases we refer to the “Atlas of Finite Groups” by Conway, Curtis, Norton, Parker, and Wilson [2]. The only groups which contain cyclic permutations are those with prime degree, see the last three rows of Table 1. (See also p.179 of Feit [3].)

This completes the examination of the Table in [1]. For every finite doubly-transitive group \( G \) we have determined whether or not it contains a cyclic permutation, those which do are listed in Table 1.
TABLE 1

<table>
<thead>
<tr>
<th>G</th>
<th>n</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $AGL(1, p)$, $p$ any prime</td>
<td>$p$</td>
<td>$C_p$</td>
</tr>
<tr>
<td>b) $S_4$</td>
<td>4</td>
<td>$C_2 \times C_2$</td>
</tr>
<tr>
<td>c) $S_n$, $n \geq 5$</td>
<td>$n$</td>
<td>$A_n$</td>
</tr>
<tr>
<td>d) $A_n$, $n$ odd and $\geq 5$</td>
<td>$n$</td>
<td>$A_n$</td>
</tr>
<tr>
<td>e) Any $G$ with $PGL(d, q) \leq G \leq PTL(d, q)$ $(d, q) \neq (2, 2)$, $(2, 3)$, or $(2, 4)$</td>
<td>$(q^d - 1)/(q - 1)$</td>
<td>$PSL(d, q)$</td>
</tr>
<tr>
<td>f) $PSL(2, 11)$</td>
<td>11</td>
<td>$PSL(2, 11)$</td>
</tr>
<tr>
<td>g) $M_{11}$</td>
<td>11</td>
<td>$M_{11}$</td>
</tr>
<tr>
<td>h) $M_{23}$</td>
<td>23</td>
<td>$M_{23}$</td>
</tr>
</tbody>
</table>

REMARKS

(i) The groups $S_2$ and $S_3$ occur in row a as $AGL(1, 2)$ and $AGL(1, 3)$ respectively.

(ii) Groups in rows a and b have an elementary abelian socle, groups in rows c – h a non-abelian simple socle.

(iii) No two groups from Table 1 are isomorphic except $S_5$ from row c and $PGL(2, 5)$ from row e, these two groups have inequivalent representations being of degrees 5 and 6 respectively.

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REFERENCES