A Combinatorial Interpretation of Lommel Polynomials and Their Derivatives

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Abstract. In this paper we present interpretations of Lommel polynomials and their derivatives. A combinatorial interpretation uses matchings in graphs. This gives an interpretation for the derivatives as well. Then Lommel polynomials are considered from the point of view of operator calculus. A step-3 nilpotent Lie algebra and finite-difference operators arise in the analysis.

I. Introduction

Interpretations of orthogonal polynomials in terms of combinatorial models has received a lot of attention over the last decade. S. Dulucq and L. Favreau [3] recently presented a combinatorial model for Bessel polynomials. A general combinatorial theory for orthogonal polynomials has been developed in the work of X.G. Viennot [12], de Médicis and Viennot [2], and in the theory of species, formalized by F. Bergeron [1], A. Joyal [6][7], G. Labelle [8] and P. Leroux [9] . An analytical study of the zeros of Lommel polynomials may be found in [5]. The basics of the operator calculus approach are in [4].

Lommel polynomials arise in the study of Bessel functions as the linearization coefficients expressing $J_{\nu+n}$ in terms of J_ν and $J_{\nu-1}$, cf. Watson [13]. They may be given explicitly in the form

$$R_n(\xi,\nu)=\sum_{k=0}^{[n/2]}\binom{n-k}{k}(-1)^k\frac{\Gamma(\nu+n-k)}{\Gamma(\nu+k)}\,(2/\xi)^{n-2k}$$

Changing variables to $\phi_n(x,\varepsilon)=R_n(-2/\varepsilon,-x/\varepsilon)$, with ε understood as a parameter, we have the recurrence

$$x\phi_n=\phi_{n+1}+\varepsilon n\phi_n+\phi_{n-1} \tag{1.1}$$

with initial conditions $\phi_{-1}=0$, $\phi_0=1$. Thus, these are a family of orthogonal polynomials. They have the form

$$\phi_n(x,\varepsilon)=\sum_k\binom{n-k}{k}(-1)^k\,(x-k\varepsilon)\cdots(x-(n-k-1)\varepsilon) \tag{1.2}$$

In this paper we present some interpretations of Lommel polynomials. We proceed from a general approach and then specialize to the case of Lommel polynomials. For example, note that with $\varepsilon = 0$ in equation (1.1), we have the recurrence for Chebyshev polynomials. In fact, with $\varepsilon = 0$, the polynomials ϕ_n become the Chebyshev polynomials of the second kind.

II. Basic construction: matchings

Let G be a simple graph on n vertices with vertex labels 1 to n and weight w_i on vertex i, $1 \leq i \leq n$. A *matching*, M, of G is a set of disjoint edges, a set of edges pairwise having no vertex in common. If a vertex i is incident to an edge of M, we write $i \in M$, otherwise $i \notin M$. We define the weight of M, $W_G(M)$, to be

$$W_G(M) = \prod_{i \notin M} w_i$$

If G has an even number of vertices and M is a perfect matching of G, in other words, if all vertices of G lie in M, then $W_G(M)$ is defined to equal 1. Note that every graph has the empty matching, containing no edges.

For the rest of this work, we will take G to be P_n, the path on n vertices, running from left to right. We denote $W_G(M)$, then, by $W_n(M)$, the weight of the matching M of P_n.

Define

$$\mathcal{P}_n = \sum_M (-1)^{|M|}\, W_n(M)$$

with $|M|$ the number of edges in M, summing over all matchings M of P_n.

2.1 Proposition. *$\mathcal{P}_n$ satisfies the recurrence*

$$\mathcal{P}_{n+1} = w_{n+1}\mathcal{P}_n - \mathcal{P}_{n-1}$$

Proof: Let M be a matching of $\mathcal{P}_{n+1}$. Either M does not contain the edge $(n, n+1)$ or it does. Every case where the edge $(n, n+1)$ is not in M corresponds to some matching on n vertices with the additional factor of w_{n+1} since $n+1$ is in none of them. On the other hand, in all matchings including the edge $(n, n+1)$, removing it leaves a matching on $n-1$ vertices with the removal of the edge contributing a minus sign. ■

We define $\mathcal{P}_{-1} = 0$, $\mathcal{P}_0 = 1$. From the Proposition, this gives $\mathcal{P}_1 = w_1$, which agrees with the scheme, since the only matching of P_1, a single vertex, is the empty matching, with weight w_1. Similarly, we have from the Proposition that $\mathcal{P}_2 = w_1w_2 - 1$. The graph P_2 has two matchings — the empty matching with weight w_1w_2 and the matching which contains the single edge $(1, 2)$ of weight 1.

Now recall that a three-term recurrence of the form

$$x\phi_n = \phi_{n+1} + b_n\phi_n + c_n\phi_{n-1}$$

with $c_n \geq 0$ yields a sequence of orthogonal polynomials. In Proposition 2.1, we can introduce a variable x either additively or multiplicatively into the weights. I.e., consider

$$(x - u_{n+1})\mathcal{P}_n = \mathcal{P}_{n+1} + \mathcal{P}_{n-1}$$

which means that x as the eigenvalues of the corresponding tridiagonal matrices are the zeros of the polynomials $\mathcal{P}_n$. Or we can write $w_{n+1} = xu_{n+1}$ with the recurrence taking the form

$$xu_{n+1}\mathcal{P}_n = \mathcal{P}_{n+1} + \mathcal{P}_{n-1}$$

for example, with constant $u_{n+1} = 2$, we have the recurrence for the Chebyshev polynomials.

2.1 Lommel polynomials and their derivatives

The recurrence (1.1) corresponds to the weights $w_i = x - (i-1)\varepsilon$. For fixed ε, we treat ϕ_n as a function of one variable, x, and denote it by R_n. Here are some explicit expressions:

$$R_1 = x \qquad R_3 = x^3 - 3\varepsilon x^2 + 2\varepsilon^2 x - 2x + 2\varepsilon$$
$$R_2 = x^2 - \varepsilon x - 1 \qquad R_4 = x^4 - 6\varepsilon x^3 + 11\varepsilon^2 x^2 - 6\varepsilon^3 x - 3x^2 + 9\varepsilon x - 6\varepsilon^2 + 1$$

Now, let G be a simple graph with n vertices labelled 1 to n having weight $x-(i-1)\varepsilon$ on vertex i. Define a *k-extended matching* of G as a set $\{ v_1, \ldots, v_k, M \}$ of k vertices of G together with a matching M such that $v_j \notin M$ for $1 \leq j \leq k$. I.e., no vertex v_j is incident to any edge in M.

Denoting the k-extended matching $\{ v_1, \ldots, v_k, M \}$ by E_M, M is called the matching of E_M. For a vertex $v \in G$, we write $v \in E_M$ if either $v = v_j$ for some j, $1 \leq j \leq k$, or $v \in M$. Let $|E_M| = |M|$ denote the number of edges in the matching of E_M.

The weight of E_M, $W_G(E_M)$ is given by:

$$W_G(E_M) = \prod_{i \notin E_M} (x - (i-1)\varepsilon)$$

As above, if every vertex of G lies in E_M, then $W_G(E_M) = 1$. We take $G = P_n$, the path, and denote $W_G(E_M)$ by $W_n(E_M)$.

2.1.1 Derivatives of Lommel polynomials

For a combinatorial interpretation of the kth derivative $R_n^{(k)}$, start from the contribution of the matching M, $W_n(M) = \prod_{i \notin M}(x-(i-1)\varepsilon)$. Consider $W_n(M)$ as a function of x. Since the derivative of each factor in $W_n(M)$ is 1, for the kth derivative we have, the summation taken over all k-subsets of vertices $\{v_{i_1}, \ldots, v_{i_k}\}$ of G such that no vertex v_{i_j} is in M,

$$\begin{aligned} W_n^{(k)}(M) &= \sum_{\{i_1,\ldots,i_k\}} \frac{k!\, W_n(M)}{\prod_j \big(x-(i_j-1)\varepsilon\big)} \\ &= k! \sum_{\substack{\{i_1,\ldots,i_k\} \\ v_{i_j} \notin M}} W_n\big(\{i_1,\ldots,i_k,M\}\big) \\ &= k! \sum_{E_M} W_n(E_M) \end{aligned}$$

this last summation taken over all k-extended matchings E_M with matching M. Hence

2.1.1.1 Proposition. *The kth derivative of the Lommel polynomial R_n is given by*

$$R_n^{(k)} = k! \sum_E (-1)^{|E|}\, W_n(E)$$

where the summation is over all k-extended matchings of P_n, with $|E|$ denoting the number of edges in the matching of E.

Note that with $k=0$ we recover the original case of R_n, considering a matching as a 0-extended matching.

Example. Consider the second derivative of R_4. We have

$$\begin{aligned} R_4 &= x^4 - 6\varepsilon x^3 + 11\varepsilon^2 x^2 - 6\varepsilon^3 x - 3x^2 + 9\varepsilon x - 6\varepsilon^2 + 1 \\ \tfrac{1}{2}\, R_4'' &= 6x^2 - 18\varepsilon x + 11\varepsilon^2 - 3 \end{aligned}$$

The 2-extended matchings with corresponding weights are given in the following table:

$2-$ ext. match.	weight
$\{1,2,\emptyset\}$	$(x-2\varepsilon)(x-3\varepsilon)$
$\{1,3,\emptyset\}$	$(x-\varepsilon)(x-3\varepsilon)$
$\{1,4,\emptyset\}$	$(x-\varepsilon)(x-2\varepsilon)$
$\{2,3,\emptyset\}$	$x\,(x-3\varepsilon)$
$\{2,4,\emptyset\}$	$x\,(x-2\varepsilon)$
$\{3,4,\emptyset\}$	$x\,(x-\varepsilon)$
$\{3,4,(1,2)\}$	1
$\{1,4,(2,3)\}$	1
$\{1,2,(3,4)\}$	1

III. Lommel polynomials and finite-difference calculus

Here we show how to write Lommel polynomials in terms of a recurrence with operator coefficients. Recall that the solution to the recurrence

$$f_{n+1} = af_n + bf_{n-1}$$

with initial conditions $f_{-1} = 0$, $f_0 = 1$, is given by

$$f_n = \sum_k \binom{n-k}{k} a^{n-2k} b^k$$

(In other terms, we can express the solution to

$$f_{n+1} = af_n - bf_{n-1}$$

with the same initial conditions, in terms of Chebyshev polynomials of the second kind: $f_n = b^{n/2} U_n(a/(2\sqrt{b}))$.) This formula holds for a and b operators, e.g., matrices, with $f_0 = I$, the identity, as long as a and b commute. If they do not commute, we apply them on different sides:

3.1 Proposition. *For operators a and b, the solution to the recurrence*

$$f_{n+1} = f_n a + bf_{n-1}$$

with initial conditions $f_{-1} = 0$, $f_0 = I$, is given by

$$f_n = \sum_k \binom{n-k}{k} b^k a^{n-2k}$$

And similarly with a acting on the left and b on the right.

For Lommel polynomials, introduce the shift operator T_ε acting on functions f by

$$T_\varepsilon f(x) = f(x - \varepsilon)$$

We denote the operator of multiplication by x by X. Using the relation

$$(XT_\varepsilon)^n 1 = x(x - \varepsilon)(x - 2\varepsilon) \cdots (x - (n-1)\varepsilon)$$

where 1 denotes the constant function 1, we have from equation (1.2),

$$R_n(x) = \sum_k \binom{n-k}{k} (-T_\varepsilon)^k (XT_\varepsilon)^{n-2k}\, 1 \tag{3.1}$$

Comparing with Proposition 3.1, we see

3.2 Proposition. *Define operators F_n by the recurrence*

$$F_{n+1} = F_n X T_\varepsilon - T_\varepsilon F_{n-1}$$

with $F_{-1} = 0$, $F_0 = I$. Then the Lommel polynomials are given by

$$R_n(x) = F_n\, 1$$

For example,

$$F_1 = XT_\varepsilon\,, \quad F_2 = (XT_\varepsilon)^2 - T_\varepsilon\,, \quad F_3 = (XT_\varepsilon)^3 - 2T_\varepsilon XT_\varepsilon$$

etc.

Another approach to expressions of the form

$$\sum_k \binom{n-k}{k} a^{n-2k} b^k$$

involves the nilpotent Lie algebra generated by the operators $D^2 = (d/dx)^2$ and X. The commutator $[D^2, X] = D^2X - XD^2 = 2D$, while $[D, X] = 1$. Thus, $\{\, D^2, D, X, 1 \,\}$ form the basis for a nilpotent Lie algebra of step 3, i.e., all commutators of length greater than 3 vanish. Now,

3.3 Proposition. *Let $f_n(x) = \sum_k \binom{n-k}{k} b^k x^{n-2k}$. Then*

$$f_n(x) = {}_0F_1\left(\begin{matrix} - \\ -n \end{matrix} \,\Big|\, -bD^2 \right) x^n$$

Proof: Expanding the ${}_0F_1$ function gives

$$\sum_k \frac{(-bD^2)^k}{(-n)_k\, k!}\, x^n = \sum_k \frac{(n-k)!}{n!\, k!}\, b^k D^{2k}\, x^n$$

from which the result is clear. ■

To see the connection with Lommel polynomials, first we review the basic operator calculus needed. Consider the formal series in one variable

$$V(z) = \sum_{n=0}^{\infty} a_n z^n$$

this is the *symbol* of the generalized differential operator $V(D)$, which acts on polynomials in the variable x. This satisfies

$$[V(D), X] = V'(D) \tag{3.2}$$

$V'(z)$ denoting the derivative of the series $V(z)$. We assume that $a_0 = 0$, $a_1 = 1$. Thus, V' has a formal multiplicative inverse $1/V'(z)$, which we denote by $W(z)$. From equation (3.2), we see that, defining $\xi = XW(D)$, we have

$$[V(D), \xi] = 1$$

From which the usual rules of polynomial calculus follow, such as

$$V\xi^n 1 = n\xi^{n-1}1$$

The shift operator T_ε has symbol $e^{-\varepsilon z}$. Thus, for the operator $\xi = XT_\varepsilon$, we have the corresponding operator $V(D)$ with symbol

$$V(z) = \frac{1}{\varepsilon}\left(e^{\varepsilon z} - 1\right) \tag{3.3}$$

which is the (forward) finite-difference operator with step size ε. Next, define the *finite-difference Laplacian* with symbol

$$\Delta_\varepsilon(z) = \frac{1}{\varepsilon^2}\left(e^{\varepsilon z} + e^{-\varepsilon z} - 2\right)$$

Now we have

3.4 Proposition. *The Lommel polynomials $R_n(x)$ satisfy*

$$R_n(x) = {}_0F_1\left(\begin{matrix} - \\ -n \end{matrix} \,\middle|\, \Delta_\varepsilon\right) \xi^n 1$$

with $\xi = XT_\varepsilon$.

Proof: Write equation (3.1) in the form

$$R_n(x) = \sum_k \binom{n-k}{k} (-T_\varepsilon)^k \xi^{n-2k} 1$$

Then, as in Proposition 3.3,

$$R_n(x) = {}_0F_1\left(\begin{matrix} - \\ -n \end{matrix} \,\middle|\, T_\varepsilon V(D)^2\right) \xi^n 1$$

with $V(D)$ the difference operator in equation (3.3). Now calculating with symbols, we see that

$$T_\varepsilon(z)V(z)^2 = \frac{1}{\varepsilon}\left(e^{\varepsilon z} - 1\right)\left(1 - e^{-\varepsilon z}\right) = \Delta_\varepsilon(z)$$

and hence the result. ■

IV. Concluding remarks

It would be interesting to consider the combinatorial approach for other families of orthogonal polynomials as indicated in §II. In [4], the function ${}_0F_1$ arises naturally in the sl(2) calculus yielding eigenfunctions of the radial Laplacian in Euclidean space. Here, we see that the Lommel polynomials correspond to the finite-difference Laplacian. It appears that the Lommel polynomials play a natural role in harmonic analysis on a lattice and merit further study in this context.

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