Contents

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Second Quantization of Recurrences

By cyclically extending recurrences, we find a hierarchy of hierarchies of identities. The technique employed was developed for finding matching polynomials of cyclically labelled paths. The method is based on trace formulas for matrices acting on the space of symmetric tensors. Borrowing terminology from quantum field theory, the action of operators on this space is called "second quantization". [1]

0.1 Introduction

First we recall some matching polynomials related to paths. Observing that the polynomials satisfy recurrence relations, we review correspondences between matrices and recurrences, providing the connection with our principal tools.

The main object of study is the recurrence which is the periodic extension [constant coefficients] of a given recurrence [non-constant coefficients]. "Second quantization" appears in the context of the Symmetric Trace Theorem for a matrix acting on the space of symmetric tensors. If the matrix has a fine structure, being itself the product of matrices depending on underlying variables, we discover a family of identities in those variables.

Here is an outline of our approach:

1. Start with a general r-term linear recurrence. Rewrite it in terms of r-by-r matrices.

2. Run the r-term recurrence t steps. This yields a product of t matrices. Call this the composite matrix. The columns of the composite matrix consist of the fundamental solutions to the recurrence. In particular, the last column produces the first fundamental solution.

3. Now use the composite matrix of #2 to generate a recurrence (via the Cayley-Hamilton theorem).

[1]

Based on a talk given at
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4. Writing the first fundamental solution to the recurrence of #3 in various ways yields identities.

5. The Symmetric Trace Theorem yields identities linking the coefficients of the original r-term recurrence with the solution to the recurrence generated by the composite matrix.

We emphasize the essential feature being the fact that the action on symmetric tensors is a multiplicative homomorphism.

The case of 2×2 matrices is presented in detail. Products of *cyclic binomials* appear quite naturally and some resulting identities are given in detail. The very simplest cases already yield interesting identities for hypergeometric functions.

Let us review some background for the present article. Matching polynomials on cyclically labelled paths is the topic of [3], which presents complete details. Looking at path covers and trellises yields combinatorial models for general linear recurrences, [4]. For material on matching and matching polynomials in the combinatorial setting, see [2, 5, 6].

For representations of matrices and discussion of tensor powers, we refer to [8]. For invariant theory and the Symmetric Trace Theorem specifically, see [7]. MacMahon's Master Theorem interestingly appears in mathematical physics contexts as well, [9, 10].

Finally, for special functions in general, hypergeometric functions and identities in particular, we refer to [1].

0.2 Matching polynomials

First, a path with all labels equal to x.

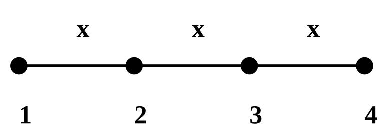

FIGURE 1: Matching polynomial is $1 + 3x + x^2$

For a path with n edges, the matching polynomial is known to be

$$\phi_n(x) = \sum_k \binom{n+1-k}{k} x^k$$

which we may call Reciprocal-Chebyshev 2^{nd} kind. Note that the total number of matchings is $\phi_n(1) = F_{n+2}$, the $(n+2)^{\text{nd}}$ Fibonacci number.

For the path with general labels such as

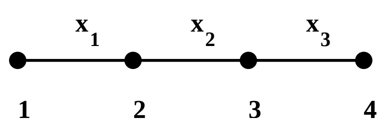

FIGURE 2: Matching polynomial is $1 + x_1 + x_2 + x_3 + x_1x_3$

the matching polynomial is the nc-function $\phi_n(x_1, \ldots, x_n)$, the sum of all *nonconsecutive* monomials in the variables $x_1, x_2, \ldots, x_n$. By nonconsecutive we mean that no products x_ix_{i+1} appear in any of the terms.

The nc-function ϕ_n satisfies this recurrence, called the nc-recurrence,

$$\phi_n = \phi_{n-1} + x_n\phi_{n-2}$$

with initial conditions $\phi_{-1} = 1$, $\phi_0 = 1$. The *first fundamental solution*, f_n, has initial conditions $\phi_{-1} = 0$, $\phi_0 = 1$, while the *second fundamental solution*, g_n, has initial conditions $\phi_{-1} = 1$, $\phi_0 = 0$. Thus, $\phi_n = f_n + g_n$.

Now consider the cycle with variable labels.

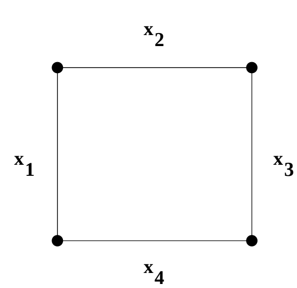

FIGURE 3: Matching polynomial is
$1 + x_1 + x_2 + x_3 + x_4 + x_1x_3 + x_2x_4$

The matching polynomial is now the ncc-function τ_n, the sum of all *nonconsecutive, cyclic* monomials in the variables $x_1, x_2, \ldots, x_n$. In this case the wrap-around product $x_n x_1$ is forbidden.

For the n-cycle with all edges labelled x, the matching polynomial is known to be

$$\tau_n(x) = \sum_k \binom{n-k}{k} \frac{n}{n-k}\, x^k$$

which we may call Reciprocal-Chebyshev 1^{st} kind, the total number of matchings in this case given by $L_n = F_{n+1} + F_{n-1}$, the n^{th} Lucas number.

Now consider the multivariable path repeated cyclically. For example,

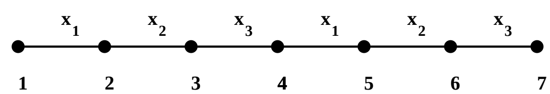

FIGURE 4: Matching polynomial is

$$\begin{aligned} &1 + 2x_1 + 2x_2 + 2x_3 + x_1^2 + 2x_1x_2 + 3x_1x_3 \\ &\quad + x_2^2 + 2x_2x_3 + x_3^2 + x_1^2x_3 + 2x_1x_2x_3 + x_1x_3^2 \end{aligned}$$

The study of the multivariable cyclic path is the subject of [3]. Here we extract the techniques used there for evaluating the matching polynomials for cyclic multivariable paths and look at some of the algebraic consequences.

0.3 Recurrences and matrices

0.3.1 From recurrence to matrices

This approach works for general r-term linear recurrences

$$\psi_n = a_{n1}\psi_{n-1} + \cdots + a_{nr}\psi_{n-r} = \sum_{j=1}^{r} a_{nj}\psi_{n-j}\ .$$

We will illustrate for $r = 3$ to see how the matrix approach works.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & b_1 & a_1 \end{pmatrix} \begin{pmatrix} \psi_{-2} \\ \psi_{-1} \\ \psi_0 \end{pmatrix} = \begin{pmatrix} \psi_{-1} \\ \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_0 \\ a_1\psi_0 + b_1\psi_{-1} + c_1\psi_{-2} \end{pmatrix}$$

In other words,

$$\psi_1 = a_1\psi_0 + b_1\psi_{-1} + c_1\psi_{-2}$$

exactly as we want for the recurrence to hold. Iterating gives

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_n & b_n & a_n \end{pmatrix} \begin{pmatrix} \psi_{n-3} \\ \psi_{n-2} \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} \psi_{n-2} \\ \psi_{n-1} \\ \psi_n \end{pmatrix}$$

with

$$\psi_n = a_n\psi_{n-1} + b_n\psi_{n-2} + c_n\psi_{n-3} \ .$$

Generally, set

$$\xi_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_i & b_i & a_i \end{pmatrix} \ .$$

Running the recurrence n steps we get

$$X = \xi_n\xi_{n-1}\cdots\xi_1 = \begin{pmatrix} \eta_{n-2} & \zeta_{n-2} & \theta_{n-2} \\ \eta_{n-1} & \zeta_{n-1} & \theta_{n-1} \\ \eta_n & \zeta_n & \theta_n \end{pmatrix}$$

say. The last row of X consists of the n^{th} term of the recurrence with initial values corresponding to the standard basis vectors. The last column is comprised of the *first fundamental solution* to the recurrence, with the remaining columns correspondingly named progressing to the left.

Remark 1. Reference [4] presents a combinatorial approach to general linear recurrences using path polynomials.

0.3.1.1 nc-Recurrence

Consider

$$\phi_n = \phi_{n-1} + x_n\phi_{n-2} \ .$$

The nc-function ϕ_n satisfies this recurrence with initial conditions $\phi_{-1} = 1$, $\phi_0 = 1$.

Denoting by f_n and g_n the fundamental solutions to this recurrence, we have

$$X = X_n = \begin{pmatrix} g_{n-1} & f_{n-1} \\ g_n & f_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x_n & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{n-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ x_1 & 1 \end{pmatrix} \ .$$

Note that the ncc-function $\tau_n = g_{n-1} + f_n$ is the trace of X_n.

0.3.2 From matrix to recurrence

On the other hand take any 3×3 matrix X. Then the Cayley-Hamilton theorem gives a relation

$$X^3 = aX^2 + bX + cI$$

with coefficients coming from the characteristic polynomial of X, essentially the elementary symmetric functions in the eigenvalues of X, alternatively, traces of exterior powers of X.

Thus, matrix elements, for given 3-vectors $\mathbf{v}$ and $\mathbf{w}$,

$$\psi_n = \langle \mathbf{v}, X^n\mathbf{w}\rangle$$

will satisfy the constant coefficient recurrence

$$\psi_n = a\psi_{n-1} + b\psi_{n-2} + c\psi_{n-3}\ .$$

There are three aspects to the first fundamental solution to this recurrence:

1. The first fundamental solution is given by h_n, the homogeneous symmetric functions in the eigenvalues of X.

2. One has
$$\frac{1}{\det(I-cX)} = \sum_{n=0}^{\infty} c^n h_n$$
where the denominator is effectively the reciprocal polynomial to the characteristic polynomial. This is a generating function for the sequence h_n.

3. The Symmetric Trace Theorem, discussed in detail below, gives a way to calculate the h_n directly in terms of the matrix X as the traces on the symmetric powers of X, i.e., the matrices induced by acting on symmetric tensor powers of the underlying vector space.

0.3.3 Matrices with fine structure

Start with a general linear recurrence and run it to index t. For example, in our 3×3 case, fix t, and let

$$X = \xi_t\xi_{t-1}\cdots\xi_1$$

Now, X satisfies a 3-term recurrence (in general, r-term recurrence)

$$X^3 = aX^2 + bX + c\ .$$

We wish to study the first fundamental solution to this recurrence. In particular, how to express it in terms of the underlying variables a_i, b_i, c_i. For the composite matrix, X, we use N for the index of the recurrence, i.e., we have

$$\psi_N = a\psi_{N-1} + b\psi_{N-2} + c\psi_{N-3}\ .$$

If we have a multiplicative homomorphism, σ, say, on matrices, then we have

$$\sigma(X) = \sigma(\xi_t)\sigma(\xi_{t-1})\cdots\sigma(\xi_1)$$

and we can take the trace of both sides, determinants, etc. We get identities using the entries of $\sigma(X)$ on the left side and the underlying variables in the matrices ξ_i on the right.

0.3.4 2-term recurrences

For the remainder of this work, we restrict to 2-term recurrences, where all details can be worked out smoothly.

First consider the recurrence

$$\psi_n = a_n\psi_{n-1} + x_n\psi_{n-2}$$

Then we have

$$\xi_i = \begin{pmatrix} 0 & 1 \\ x_i & a_i \end{pmatrix}$$

noting that the case all $a_i = 1$ recovers the nc-functions. Running this for t steps, we have

$$X = X_t = \begin{pmatrix} g_{t-1} & f_{t-1} \\ g_t & f_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x_t & a_t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{t-1} & a_{t-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ x_1 & a_1 \end{pmatrix}\ .$$

Cayley-Hamilton says that $X^2 = \tau X - \Delta I$, with $\tau = \operatorname{tr} X$, $\Delta = \det X$.

Any sequence of matrix elements $\psi_N = \langle \mathbf{u}, X^N \mathbf{v}\rangle$, $\mathbf{u}, \mathbf{v} \in \mathbf{R^2}$, satisfies the τ-Δ *recurrence*

$$\psi_N = \tau\,\psi_{N-1} - \Delta\,\psi_{N-2}$$

with the trace τ given by the complete ncc-function in the variables x_i in the case all $a_i = 1$, and, in all cases $\Delta = (-1)^t x_1 x_2 \cdots x_t$.

0.3.5 First fundamental solution of the τ-Δ recurrence

The τ-Δ recurrence corresponds to the matrix

$$\Gamma = \begin{pmatrix} 0 & 1 \\ -\Delta & \tau \end{pmatrix} .$$

We denote the first fundamental solution to the recurrence by Φ_N and it is given explicitly in terms of Chebyshev polynomials of the second kind

$$\Phi_N = \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N-k}{k} \tau^{N-2k} (-\Delta)^k = \Delta^{N/2}\, U_N\left(\frac{\tau}{2\sqrt{\Delta}}\right) .$$

Remark 2. *In the paper [3], this solution was denoted G_N.*

This formula may be checked by induction and derived either from the generating function

$$\frac{1}{\det(I - cX)} = \frac{1}{\det(I - c\Gamma)} = \sum_{N=0}^{\infty} c^N\, \Phi_N$$

or by application of the Symmetric Trace Theorem presented in the next section.

0.4 The Symmetric Representation, MacMahon's Master Theorem, and Evaluation of Φ_N

Consider polynomials in the variables $u_1, \ldots, u_d$. We will work with the vector space whose basis elements are the homogeneous polynomials of degree N in these variables, *i.e.*, with

$$\{u_1^{n_1} \cdots u_d^{n_d} \,|\, n_1 + \cdots + n_d = N, \text{each } n_\ell \geq 0\},$$

This vector space has dimension $\binom{N+d-1}{N}$.

The symmetric representation of a $d \times d$ matrix $A = (a_{\ell\ell'})$ is the action on polynomials induced by:

$$u_1^{n_1} \cdots u_d^{n_d} \to v_1^{n_1} \cdots v_d^{n_d},$$

where

$$v_\ell = \sum_{\ell'} a_{\ell\ell'} u_{\ell'}$$

or, more compactly, $v = Au$. That is, define the matrix element $\left\langle \begin{matrix} m_1, \ldots, m_d \\ n_1, \ldots, n_d \end{matrix} \right\rangle_A$ to be the coefficient of $u_1^{n_1} \cdots u_d^{n_d}$ in $v_1^{m_1} \cdots v_d^{m_d}$. Then, for a fixed $(m_1, \ldots, m_d)$, we have

$$v_1^{m_1} \cdots v_d^{m_d} = \sum_{(n_1,\ldots,n_d)} \left\langle \begin{matrix} m_1, \ldots, m_d \\ n_1, \ldots, n_d \end{matrix} \right\rangle_A u_1^{n_1} \cdots u_d^{n_d}. \tag{1}$$

Observe that the total degree $N = |n| = \sum n_\ell = |m| = \sum m_\ell$, i.e., homogeneity of degree N is preserved. We use multi-indices: $m = (m_1, \ldots, m_d)$ and $n = (n_1, \ldots, n_d)$. Then, for a fixed m, (1) becomes

$$v^m = \sum_n \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_A u^n.$$

Successive application of B then A shows that this is a homomorphism of the multiplicative semi-group of square $d \times d$ matrices into the multiplicative semi-group of square $\binom{N+d-1}{N} \times \binom{N+d-1}{N}$ matrices, namely

Proposition 0.4.1. *Matrix elements satisfy the homomorphism property*

$$\left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{AB} = \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_A \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle_B.$$

Proof. Let $v = (AB)u$ and $w = Bu$. Then,

$$\begin{aligned} v^m &= \sum_n \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{AB} u^n \\ &= \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_A w^k \\ &= \sum_n \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_A \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle_B u^n. \end{aligned}$$

□

Definition 0.4.2. Fix the degree $N = \sum n_\ell = \sum m_\ell$. Define $\mathrm{tr}_{\mathrm{Sym}}^N(A)$, the *symmetric trace* of A in degree N, as the sum of the diagonal elements $\left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_A$, *i.e.*,

$$\mathrm{tr}_{\mathrm{Sym}}^N(A) = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_A.$$

Equality such as $\mathrm{tr}_{\mathrm{Sym}}(A) = \mathrm{tr}_{\mathrm{Sym}}(B)$ means that the symmetric traces are equal in every degree $N \geq 0$.

Remark 3. The action defined here on polynomials is equivalent to the action on symmetric tensor powers, see [8], pp. 472–5. The matrices induced at each level N of a matrix A acting on the space of symmetric tensors is the second quantization of A.

It is straightforward to see directly (cf. the diagonal case shown in the Corollary below) that if A is upper-triangular, with eigenvalues $\lambda_1, \ldots, \lambda_d$, then $\mathrm{tr}^N_{\mathrm{Sym}}(A) = h_N(\lambda_1, \ldots, \lambda_d)$, the N^{th} homogeneous symmetric function. The homomorphism property, Proposition 0.4.1, shows that, as usual, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and that similar matrices have the same trace. Again by the homomorphism property, if two $d \times d$ matrices are similar, $A = MBM^{-1}$, then that relation extends to their respective symmetric representations in every degree. Recall that any matrix is similar to an upper-triangular one with the same eigenvalues. Thus, (see [7], pp. 51–2)

Theorem 0.4.3. Symmetric Trace Theorem
We have

$$\frac{1}{\det(I - cA)} = \sum_{N=0}^{\infty} c^N \mathrm{tr}^N_{\mathrm{Sym}}(A).$$

Proof. With λ_ℓ denoting the eigenvalues of A,

$$\begin{aligned} \frac{1}{\det(I - cA)} &= \prod_\ell \frac{1}{1 - c\lambda_\ell} \\ &= \sum_{N=0}^{\infty} c^N h_N(\lambda_1, \ldots, \lambda_d) \\ &= \sum_{N=0}^{\infty} c^N \mathrm{tr}^N_{\mathrm{Sym}}(A). \end{aligned}$$

□

As a Corollary we have MacMahon's Master Theorem, which we express in the above terminology.

Corollary 0.4.4. MacMahon's Master Theorem.
The diagonal matrix element $\left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_A$ *is the coefficient of* $u^m = u_1^{m_1} \cdots u_d^{m_d}$ *in the expansion of* $\det(I - UA)^{-1}$ *where* $U = diag(u_1, \ldots, u_d)$ *is the diagonal matrix with entries* $u_1, \ldots, u_d$ *on the diagonal.*

Proof. From Theorem 0.4.3, with $c = 1$, we want to calculate the symmetric trace of UA. By the homomorphism property,

$$\begin{aligned} \mathrm{tr}^N_{\mathrm{Sym}}(UA) &= \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_{UA} \\ &= \sum_m \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_U \left\langle \begin{matrix} k \\ m \end{matrix} \right\rangle_A . \end{aligned}$$

Now, with $v = Uw$ and $v_\ell = u_\ell w_\ell$, then

$$v^m = (u_1 w_1)^{m_1} \cdots (u_d w_d)^{m_d} = u^m w^m = \sum_k \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_U w^k,$$

i.e.,

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle_U = u_1^{m_1} \cdots u_d^{m_d} \delta_{k_1 m_1} \cdots \delta_{k_d m_d}$$

so that

$$\mathrm{tr}^N_{\mathrm{Sym}}(UA) = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_A u^m.$$

□

Remark 4. For interesting background and applications including MacMahon's Master Theorem, see [9] and [10].

Now we restrict ourselves to $d = 2$, and return to the τ-Δ recurrence.

Take our composite matrix

$$\begin{aligned} X &= \begin{pmatrix} 0 & 1 \\ x_t & a_t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{t-1} & a_{t-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ x_1 & a_1 \end{pmatrix} \\ &= \xi_t \xi_{t-1} \cdots \xi_1, \end{aligned}$$

where $\xi_i = \begin{pmatrix} 0 & 1 \\ x_i & a_i \end{pmatrix}$ for $1 \le i \le t$. Let

$$\mathrm{tr}(X) = \tau \quad \text{and} \quad \det(X) = \Delta,$$

and let Φ_N be the first fundamental solution to the τ-Δ recurrence:

$$\psi_N = \tau\, \psi_{N-1} - \Delta\, \psi_{N-2}. \tag{2}$$

Then

$$\begin{aligned} \sum_{N=0}^{\infty} c^N \Phi_N &= \frac{1}{1 - \tau c + \Delta c^2} \\ &= \frac{1}{\det(I - cX)} \\ &= \sum_{N=0}^{\infty} c^N \mathrm{tr}^N_{\mathrm{Sym}}(X). \end{aligned}$$

So

$$\Phi_N = \mathrm{tr}^N_{\mathrm{Sym}}(X) = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_X = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_{\xi_t \xi_{t-1} \cdots \xi_1}.$$

We need to calculate the symmetric trace of X and so identify Φ_N. By the homomorphism property, we need only find the matrix elements for each matrix ξ_i, multiply together and take the trace.

For $\xi_i = \begin{pmatrix} 0 & 1 \\ x_i & a_i \end{pmatrix}$ the mapping induced on polynomials is

$$v_1 = u_2 \,, \quad v_2 = x_i \, u_1 + a_i \, u_2. \tag{3}$$

For any integer $N \geq 0$, the expansion of $v_1^m v_2^{N-m}$ in powers of u_1 and u_2 is of the form

$$v_1^m v_2^{N-m} = \sum_n \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{\xi_i} u_1^n u_2^{N-n}, \tag{4}$$

with the notation for the matrix elements abbreviated accordingly. From (3) and (4), the binomial theorem yields

$$\left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{\xi_i} = \binom{N-m}{n} x_i^n a_i^{N-m-n}.$$

For example, when $t = 3$, the product $X = \xi_3 \xi_2 \xi_1$ gives the matrix elements, for homogeneity of degree N,

$$\begin{aligned} \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_X &= \sum_{(k_2,k_3)} \left\langle \begin{matrix} m \\ k_3 \end{matrix} \right\rangle_{\xi_3} \left\langle \begin{matrix} k_3 \\ k_2 \end{matrix} \right\rangle_{\xi_2} \left\langle \begin{matrix} k_2 \\ n \end{matrix} \right\rangle_{\xi_1} \\ &= \sum_{(k_2,k_3)} \binom{N-m}{k_3} \binom{N-k_3}{k_2} \binom{N-k_2}{n} x_1^n x_2^{k_2} x_3^{k_3} a_1^{N-k_2-n} a_2^{N-k_3-k_2} a_3^{N-k_3-m} \end{aligned}$$

Thus, the symmetric trace $\mathrm{tr}^N_{\mathrm{Sym}}(X) = \sum_m \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle_X$ is

$$\sum_{(k_1,k_2,k_3)} \binom{N-k_2}{k_1} \binom{N-k_3}{k_2} \binom{N-k_1}{k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3} \, a_1^{N-k_1-k_2} a_2^{N-k_2-k_3} a_3^{N-k_3-k_1},$$

a *cyclic binomial*. In general, for a product of arbitrary length, the symmetric trace is given by the corresponding cyclic binomial. Combining these observations yields the main identity:

Theorem 0.4.5. *Let* $X = \xi_t \xi_{t-1} \cdots \xi_1$*, with* $\xi_i = \begin{pmatrix} 0 & 1 \\ x_i & a_i \end{pmatrix}$ *for* $1 \le i \le t$*, and let* $\tau = \mathrm{tr}(X)$ *and* $\Delta = \det(X)$*. Let* Φ_N *denote the first fundamental solution to the* τ*-*Δ *recurrence.*

Then we have the cyclic binomial identity

$$\Phi_N = \sum_{k_1,\ldots,k_t} \binom{N-k_2}{k_1}\binom{N-k_3}{k_2}\cdots\binom{N-k_t}{k_{t-1}}\binom{N-k_1}{k_t} \times x_1^{k_1}\cdots x_t^{k_t} a_1^{N-k_1-k_2} a_2^{N-k_2-k_3}\cdots a_t^{N-k_t-k_1}$$

$$= \Delta^{N/2} U_N\left(\frac{\tau}{2\sqrt{\Delta}}\right)$$

$$= \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N-k}{k} \tau^{N-2k}(-\Delta)^k$$

$$= \sum_{m,k} \binom{m}{k}\binom{N-m}{m-k} f_t^{N-2m+k} g_{t-1}^k (f_{t-1}g_t)^{m-k}$$

where U_N *denotes the Chebyshev polynomial of the second kind.*

Remark 5. Recall f_t and g_t are the fundamental solutions to the underlying 2-term recurrence.

Proof. The first equality follows from the above observations, and the second and third from well-known properties of the Chebyshev polynomials of the second kind. The last formulation is found via the Symmetric Trace Theorem using the expression for X in terms of the fundamental solutions of the underlying recurrence. □

Note that $\Phi_{-1} = 0$ and $\Phi_0 = 1$, so $\Phi_1 = \tau$ using the τ-Δ recurrence. This also follows directly from the condition $k_{s-1} + k_s \le 1$ for non-zero terms in the cyclic binomial summation.

Example 0.4.6. Denote the symmetric representation in degree N of the matrix A by $A^{\mathrm{Sym(N)}}$. Consider the case where all a_i are equal to 1.

Let $N = 2$ and $t = 3$.

We have

$$\begin{aligned}\Phi_2 &= \sum_{(k_1,k_2,k_3)} \binom{2-k_2}{k_1}\binom{2-k_3}{k_2}\binom{2-k_1}{k_3} x_1^{k_1}x_2^{k_2}x_3^{k_3} \\ &= 1+2x_1+2x_2+2x_3+x_1^2+2x_1x_2+2x_1x_3+x_2^2+2x_2x_3+x_3^2+x_1x_2x_3.\end{aligned}$$

Here $N = 2$ and $d = 2$ so $\binom{N+d-1}{N} = 3$, and $\xi_i = \begin{pmatrix} 0 & 1 \\ x_i & 1 \end{pmatrix}$ for $1 \le i \le 3$, so

$$X = \xi_3\xi_2\xi_1 = \begin{pmatrix} x_1 & x_2+1 \\ x_1x_3+x_1 & x_2+x_3+1 \end{pmatrix}.$$

Now ${\xi_i}^{\mathrm{Sym}(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & x_i & 1 \\ x_i^2 & 2x_i & 1 \end{pmatrix}$ for $1 \le i \le 3$, so

$$\begin{aligned} X^{\mathrm{Sym}(2)} &= {\xi_3}^{\mathrm{Sym}(2)}{\xi_2}^{\mathrm{Sym}(2)}{\xi_1}^{\mathrm{Sym}(2)} \\ &= \begin{pmatrix} x_1^2 & 2x_1x_2+2x_1 & x_2^2+2x_2+1 \\ x_1^2x_3+x_1^2 & \begin{array}{c} x_1x_2x_3+2x_1x_2 \\ +2x_1x_3+2x_1 \end{array} & \begin{array}{c} x_2^2+x_2x_3+2x_2 \\ +x_3+1 \end{array} \\ x_1^2x_3^2+2x_1^2x_3+x_1^2 & \begin{array}{c} 2x_1x_2x_3+2x_1x_3^2+2x_1x_2 \\ +4x_1x_3+2x_1 \end{array} & \begin{array}{c} x_3^2+2x_2x_3+x_2^2 \\ +2x_2+2x_3+1 \end{array} \end{pmatrix}. \end{aligned}$$

We check that $\Phi_2 = \mathrm{tr}(X^{\mathrm{Sym}(2)})$, as required.

0.4.1 Matching polynomials

We quote some results of [3]:

$\Phi_N + (\phi_t - \tau_t)\Phi_{N-1}$ is the matching polynomial for the N-fold repeated path of length Nt.
$2\,\Delta^{N/2}\,T_N\left(\dfrac{\tau}{2\sqrt{\Delta}}\right)$ is for the corresponding cycle, with T_N the Chebyshev polynomial of the first kind.

In [3], matching polynomials for cyclically repeated paths, cycles, and trees are found.

0.4.2 Evaluations involving Fibonacci numbers

Consider the path of length n. Write

$$\begin{array}{rcl}\phi_n(x) & = & \phi_n(x,\dots,x)\ .\\ \tau_n(x) & = & \tau_n(x,\dots,x)\ .\end{array}$$

Recall the Fibonacci sequence $\{F_m \,|\, m \geq 1\}$

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	$\cdots$
1	1	2	3	5	8	13	21	$\cdots$

Some observations/properties:

1. There are $\phi_n(1) = F_{n+2}$ terms in $\phi_n(x_1,\dots,x_n)$.

2. There are $\tau_n(1) = F_{n-1} + F_{n+1}$ terms in $\tau_n(x_1,\dots,x_n)$.

3. We have explicitly

$$\begin{array}{rcl}\phi_{n-1}(-x) = \sum_k \binom{n-k}{k}(-x)^k & = & x^{n/2}\, U_n\left(\dfrac{1}{2\sqrt{x}}\right)\\ \tau_n(-x) = \sum_k \binom{n-k}{k}\dfrac{n}{n-k}\,(-x)^k & = & 2x^{n/2}\, T_n\left(\dfrac{1}{2\sqrt{x}}\right)\end{array}$$

4. Let $x_i = -x$ for all i, all $a_i = 1$. Then

$$\begin{array}{l}\tau = 2x^{n/2}T_n(1/(2\sqrt{x}))\ ,\\ \Delta = x^n\ .\end{array}$$

We get for Φ_N,

$$\begin{array}{rcl}\sum \binom{N-k_2}{k_1}\cdots\binom{N-k_1}{k_n}(-1)^{|k|}x^{|k|} & = & x^{nN/2}U_N\left(T_n\left(\dfrac{1}{2\sqrt{x}}\right)\right)\\ & = & x^{nN/2}\dfrac{U_{Nn+n-1}(1/(2\sqrt{x}))}{U_{n-1}(1/(2\sqrt{x}))}\end{array}$$

The substitution $x = \frac{1}{4}\sec^2(\alpha/n)$ gives the formula:

$$\sum \binom{N-k_2}{k_1} \cdots \binom{N-k_1}{k_n} \frac{(-1)^{|k|}}{4^{|k|}} (\sec^2 \frac{\alpha}{n})^{|k|}$$
$$= (\tfrac{1}{2}\sec\alpha)^{nN}\, U_N\left(T_n(\cos\frac{\alpha}{n})\right)$$
$$= (\tfrac{1}{2}\sec\alpha)^{nN}\, U_N(\cos\alpha)$$
$$= (\tfrac{1}{2}\sec\alpha)^{nN}\, \frac{\sin(N+1)\alpha}{\sin\alpha}$$

using the principal property $T_n(\cos\theta) = \cos n\theta$ and similarly for U_N.

5. We know that the number of matchings in the path with $m-1$ edges is F_{m+1}. In [3] it is shown (Theorem 4.5) how to write the fundamental solution Φ_N as the ratio of two matching polynomals. It turns out that the numerator polynomial corresponds to the path with $(N+1)t-2$ edges, and the denominator polynomial to the path with $t-2$ edges. Specializing our expression for Φ_N with all $x_i = 1$ and all $a_i = 1$ we have an evaluation of the cyclic binomial sum as

$$\frac{F_{(N+1)t}}{F_t} = \sum_{(k_1,\ldots,k_t)} \binom{N-k_2}{k_1}\binom{N-k_3}{k_2} \cdots \binom{N-k_1}{k_t}.$$

0.5 Some identities illustrated

Looking at the identity for Φ_N for various values of t yields summation identities by matching coefficients of the resulting expressions.

Remark 6. See [1] for information on hypergeometric identities and related material.

0.5.1 Gauss' sum

Let $t = 2$. Write $\tau = 1 + x + y$, $\Delta = xy$. We have

$$\sum \binom{N-B}{A}\binom{N-A}{B} x^A y^B = \sum \binom{N-k}{k} (-1)^k (1+x+y)^{N-2k} (xy)^k$$

Expanding the right-hand side ($s = i + j$, $S = A + B$)

$$(1 + x + y)^{N-2k}(xy)^k = \sum \frac{(N-2k)!}{i!\, j!\, (N-2k-s)!}\, x^{i+k} y^{j+k}$$

Equating coefficients, $A = i + k$, $B = j + k$, $s = A + B - 2k$, yields

$$\binom{N-B}{A}\binom{N-A}{B} = \sum \frac{(N-k)!}{k!} \frac{(-1)^k}{(A-k)!\,(B-k)!} \frac{1}{(N-S)!}$$

Rewriting yields the finite Gauss summation (Chu-Vandermonde summation)

$${}_2F_1\left(\begin{matrix} -A, -B \\ -N \end{matrix} \,\middle|\, 1\right) = \frac{(N-A)!\,(N-B)!}{N!\,(N-(A+B))!}\ .$$

0.5.2 Pfaff-Saalschütz sum

Let $t = 3$. Write $\tau = 1 + x + y + z$, $\Delta = -xyz$. Now expand $\tau(x, y, z)^{N-2k}(xyz)^k$ with $s = a + b + c$, $S = A + B + C$

$$(1+x+y+z)^{N-2k}(xyz)^k = \sum \frac{(N-2k)!}{a!\, b!\, c!\, (N-2k-s)!}\, x^{a+k} y^{b+k} z^{c+k}$$

Comparing coefficients, $A = a + k$,$B = b + k$, $C = c + k$, $S = s + 3k$, yields

$$\binom{N-B}{A}\binom{N-C}{B}\binom{N-A}{C} = \frac{N!}{A!\,B!\,C!\,(N-S)!}\, {}_3F_2\left(\begin{matrix} -A,\ -B,\ -C \\ -N,\ N-S+1 \end{matrix} \,\middle|\, 1\right)$$

Rewriting gives the Pfaff-Saalschütz summation

$${}_3F_2\left(\begin{matrix} -A,\ -B,\ -C \\ -N,\ N-S+1 \end{matrix} \,\middle|\, 1\right) = \frac{(N-A)!\,(N-B)!\,(N-C)!\,(N-A-B-C)!}{N!\,(N-A-B)!\,(N-A-C)!\,(N-B-C)!}\ .$$

0.5.3 General values of t

For $t \geq 4$, the identities are rather more complex. For general t, denote by $x(\mathbf{i})$ monomials appearing in τ. Then we have, $\xi(\mathbf{i})$ denoting the exponent of $x(\mathbf{i})$, $s = \sum \alpha(\mathbf{i})$,

$$\Phi_N = \sum \frac{(-1)^{k(t+1)}\,(N-k)!}{k!\,(N-2k-s)!}\,(x_1 \cdots x_t)^k \prod \frac{x(\mathbf{i})^{\alpha(\mathbf{i})}}{\alpha(\mathbf{i})!}$$

0.6 Conclusion

We have developed a machine that produces a *hierarchy of hierarchies of identities*. Start with an r-term recurrence. Run it for t steps, $t > r$. Now run the recurrence generated by the resulting composite matrix, effectively t steps extended periodically. Then each index of the recurrence for the composite matrix yields a hierarchy of corresponding identities.

We have seen how the case $r = 2$ for low values of t already yield interesting identities for hypergeometric functions. The consequences for $r > 2$ and looking at larger values of t will provide families of identities most worthy of further study. In particular, we may ask if there are connections with mathematical objects such as multivariate Chebyshev polynomials.

The use of the multiplicative properties of the second quantization provides a way for deeper analysis in contexts where MacMahon's theorem is used as a counting or analytic technique. Such possibilities are not yet commonly recognized, thus leaving open great opportunities for exploration.

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