**Extra Credit Assignment–The Arc Length of an Ellipse**
Calculus II–Hughes

Using integrals to calculate the arc length of an ellipse results in an integration problem that cannot be solved with the elementary functions used in first-year Calculus. However, it is possible to use infinite series to represent these integrals and so approximate the arc length of an ellipse.

Suppose $a > b > 0$. Consider an ellipse where $a$ is the length of the semi-major axis and $b$ is the length of the semi-minor axis. Let $c = \sqrt{a^2 - b^2}$. Then the eccentricity of the ellipse is $e = c/a$. In the following, $e$ will represent the eccentricity of an ellipse, a constant between 0 and 1, and not the base of the natural exponential. We examine equations for an ellipse of three different types.

One way to represent an ellipse in Cartesian coordinates is through a pair of parametric equations

$$x = a \sin t, \quad y = b \cos t, \quad 0 \le t \le 2\pi. \tag{1}$$

Due to the symmetry of the ellipse, the entire perimeter of the ellipse can be found by multiplying the length of the arc from $t = 0$ to $t = \pi/2$ by four.

> *Exercise 1*:
> a) Set up an integral for the total arc length (perimeter) of the ellipse given by (1).
> b) Use the relationship $b^2 = a^2 - e^2a^2$ to rewrite the integral in part (a) in terms of $a$ and $e$.
> c) Note that the arc length of the ellipse is $k \displaystyle\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t}\, dt$. What is $k$?

By eliminating the parameter in parametric equations (1) the equation for the upper half of the ellipse can be written as a function $y = f(x)$.

> *Exercise 2*:
> a) Show that this function is of the form $f(x) = K\sqrt{a^2 - x^2}$. Use the relationship $b^2 = a^2 - e^2a^2$ to write $K$ in terms of the eccentricity $e$.
> b) Use the function in part (a) to set up an integral for the total arc length (upper and lower half) of the ellipse.
> c) Make the trigonometric substitution $x = a \sin t$ in this integral. You should get the same integral as you found in Exercise 1(b).

Another equation for an ellipse with semi-major axis $a$ and eccentricity $e$ can be given in terms of polar coordinates:

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}, \quad 0 \le \theta \le 2\pi. \tag{2}$$

*Exercise 3*: Use polar coordinates to set up an integral for the total arc length (perimeter) of the ellipse.

Our techniques of integration do not allow us to find exact values for the integrals found above, but the integrals can be approximated using numerical techniques. Consider the formula for the arc length of an ellipse given by a multiple of the integral $\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t}\, dt$ where $e$ is the eccentricity of the ellipse, $0 < e < 1$. In order to compute the integral, we might first represent the integrand as a power series. The power series for $\sqrt{1 - e^2 \sin^2 t}$ in powers of $t$ would be very complicated. Instead, consider the power series in $x$ where $x = \sin t$. This can be calculated using the binomial series.

*Exercise 4*:

a) Use the binomial series to write the Maclaurin series for $(1 + x)^{1/2}$.

b) Use the series in (a) to write the Maclaurin series for $(1 - e^2 x^2)^{1/2}$.

c) Use the series in (b), replace $x$ with $\sin t$, and verify that the result is the following series:

$$\sqrt{1 - e^2 \sin^2 t} = 1 - \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{e^{2n}}{2n-1} \sin^{2n} t. \tag{3}$$

Because $|\sin t| \leq 1$ and $e < 1$, the series in (3) is absolutely convergent and equal to the given function for all $t$. Further, it can be shown that, as in the case of a power series, the definite integral $\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t}\, dt$ can be computed by integrating the series term-by-term. This requires the integration of powers of $\sin t$.

*Exercise 5*:

a) Use integration by parts to derive the reduction formula

$$\int \sin^{2n} t\, dt = -\frac{1}{2n} \sin^{2n-1} t\, \cos t + \frac{2n-1}{2n} \int \sin^{2n-2} t\, dt. \tag{4}$$

b) Apply equation (4) to the definite integral from $t = 0$ to $t = \pi/2$ to get a reduction formula for the definite integral.

c) By repeated application of the reduction formula in part (b), show that

$$\int_0^{\pi/2} \sin^{2n} t\, dt = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2} \tag{5}$$

c) Use equation (5) to integrate the series in (3) term-by-term to find a power series in $e$ for the integral $\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t}\, dt$. (Don't forget to integrate the constant term, too.)

The value of the integral can be approximated with partial sums of the power series:

$$\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t}\, dt \approx \frac{\pi}{2}\left[1 - \sum_{n=1}^{N} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right)^2 \cdot \frac{e^{2n}}{2n-1}\right]. \tag{6}$$

This series is not alternating so the Alternating Series Estimation Theorem cannot be used. Instead we estimate the error in this approximation by estimating the remaining terms. The terms are sufficiently complicated to make a precise estimate of the remaining terms difficult. We can simplify this considerably by using the following inequality:

$$\left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right)^2 \cdot \frac{1}{2n-1} \leq \frac{1}{4n^2}. \tag{7}$$

(This inequality can be verified by mathematical induction on $n$.) We get the following error estimate:

$$\begin{aligned} \text{error} &= \frac{\pi}{2} \sum_{n=N+1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right)^2 \cdot \frac{e^{2n}}{2n-1} \\ &\leq \frac{\pi}{2} \sum_{n=N+1}^{\infty} \frac{e^{2n}}{4n^2} \leq \frac{\pi}{8(N+1)^2} \sum_{n=N+1}^{\infty} e^{2n} \\ &= \frac{\pi}{8(N+1)^2} \cdot \frac{e^{2N+2}}{1-e^2}, \end{aligned} \tag{8}$$

where the last equality comes from the geometric series formula.

*Exercise 6*: Use formula (6) to approximate the integral for the following values of $e$ using the given number of terms $N$. Then use formula (8) to estimate the error in this approximation.

a) $e = 3/5$, $N = 5$

b) $e = 3/5$, $N = 10$

c) $e = 7/25$, $N = 5$

d) $e = 7/25$, $N = 10$

**References.**

Stewart, J., *Essential Calculus: Early Transcendentals,* 2nd Ed., Brooks/Cole, 2013.