Scheduled Labs

- **Morris Library 174** (open lab)
  Jun 10-Aug 1| Closed Thursday, July 4
  Sun 6-9 | Mon 1:30-4 | Tues 2-5 | Wed 1-4 | Thurs 2-5
- **Faner Room 1032**
  Mon 11-12 | Wed 11-12
  (and 2:30-5 on Thurs, July 11 and July 25)
- **Faner Room 1024**
  Thurs 1:20-2:20 (except July 4)

Find the Generator Matrix Given Parity Check Equations

Construct the generator matrix of a (3, 5) block code with the following parity check equations:

$$c_1 = m_2 + m_3 \qquad c_2 = m_1 + m_3$$

$$\begin{array}{cc} & \begin{array}{ccccc} \\ \end{array} \\ \begin{array}{c} m_1 \\ m_2 \\ m_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \\ & \begin{array}{ccccc} m_1 & m_2 & m_3 & c_1 & c_2 \end{array} \end{array}$$

Producing a Linear Code with a Generator Matrix

We can use a generator matrix to create a collection of codewords that we call a "linear code". To produce a linear code, we start with all binary strings of a certain length. We will denote all strings of length k as W_k.

$$W_1 = \{0, 1\}$$

$$W_2 = \{00, 01, 10, 11\}$$

$$W_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

The set W_k **will have 2^k elements**.

(3,5) Linear Code generated by previous matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$c_1 = m_2 + m_3$$
$$c_2 = m_1 + m_3$$

$$W_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

$$C(3,5) = \{00000, 00111, 01010, 01101,$$
$$10001, 10110, 11011, 11100\}$$

Decoding Matrix Codes

- The last section described how to encode codewords using a generator matrix. In this section we will discuss how to decode a received codeword.
- The key notion will involve the **check matrix** of the code.
- All of the matrix codes we will work with have a specific structure, and this structure is the key to determining the check matrix of the code.

Structure of the Matrix for a (k,n) Matrix Code

Each generator matrix for a (k,n) code has two parts:

I, a $k \times k$ matrix on the left consisting of 1's in a diagonal, which repeats the message at the beginning;

J, a $k \times (n - k)$ matrix on the right, which calculates the check digits.

$$\boldsymbol{I} \qquad\qquad\qquad\qquad \boldsymbol{J}$$

$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Building the Check Matrix for a (k,n) Matrix Code

First write down the J portion of the matrix and then, below the J portion, complete the correct size I matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_{n-k}$$

The new matrix has n rows, which is the length of the code-words generated by the matrix.

Using the Check Matrix of a (k,n) Matrix Code

Produce a codeword using the (3,6) generator from the previous lecture, and then input that codeword into the check matrix:

$$\begin{array}{c} m_1 \\ m_2 \\ m_3 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \\ \begin{matrix} m_1 & m_2 & m_3 & c_1 & c_2 & c_3 \end{matrix} \end{array}$$

$$\begin{array}{c} m_1 \\ m_2 \\ m_3 \\ c_1 \\ c_2 \\ c_3 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} s_1 & s_2 & s_3 \end{matrix} \end{array}$$

The output of the check matrix is called the "**syndrome**" $s_1 \; s_2 \; s_3$.

Valid codewords produce all 0's in the syndrome

$$\begin{array}{cc}
\begin{matrix}1\\0\\1\end{matrix} & \begin{bmatrix}1&0&0&1&0&1\\0&1&0&1&1&0\\0&0&1&0&1&1\end{bmatrix} \\
 & \begin{matrix}1&0&1&1&1&0\end{matrix}
\end{array}
\qquad
\begin{array}{cc}
\begin{matrix}1\\0\\1\\1\\1\\0\end{matrix} & \begin{bmatrix}1&0&1\\1&1&0\\0&1&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix} \\
 & \begin{matrix}0&0&0\end{matrix}
\end{array}$$

Start with the message 101 and use the generator matrix to produce the codeword 101110. When we input the codeword into the check matrix, we get the syndrome 000. **Every valid codeword produces all 0's.**

Correcting transmission errors

Suppose an error during transmission turns the 3rd bit from a 1 to a 0. The received word is 100110. The syndrome of the the received word is 011, which is the same as the 3rd row of the check matrix. This locates the error in the 3rd bit of the message, so we can correct to the codeword, 101110, and message, 101.

$$\begin{array}{cc}
\begin{matrix}1\\0\\1\end{matrix} & \begin{bmatrix}1&0&0&1&0&1\\0&1&0&1&1&0\\0&0&1&0&1&1\end{bmatrix} \\
 & \begin{matrix}1&0&1&1&1&0\end{matrix}
\end{array}$$

$$\begin{array}{cc}
\begin{matrix}1\\0\\0\\1\\1\\0\end{matrix} & \begin{bmatrix}1&0&1\\1&1&0\\ \boxed{0\;\;1\;\;1}\\1&0&0\\0&1&0\\0&0&1\end{bmatrix} \\
 & \boxed{0\;\;1\;\;1}
\end{array}$$

Decoding using the Check Matrix

$$\begin{array}{c c} \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & \begin{array}{ccc} 1 & 1 & 1 \end{array} \end{array}$$

This syndrome does not match any row. The received word has too many errors to be corrected.

$$\begin{array}{c c} \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & \begin{array}{ccc} 0 & 0 & 0 \end{array} \end{array}$$

This syndrome is all 0's. We assume there are no errors and use the codeword as received.

All Matrix Codes are Linear Codes

A **binary linear code** consists of words composed of 0's and 1's obtained from possible binary messages by using parity-check sums to append check digits to the messages. The resulting strings are called **codewords**. A set of codewords is a linear code if the **binary sum of any two codewords is another codeword.** All matrix codes satisfy this property.

Suppose C is the code {0101, 1010, 1111, 0000}.
Is C a linear code?

Yes! If you add any two sequences, the result is in the set.

Suppose C is the code {0011, 1010, 1101, 0000}.
Is C a linear code?

No! 0011 + 1010 = 1001 and 1001 is not in the set.

Weight of a codeword

- The weight of a codeword is the number of 1's in the codeword.
- Weight of 100110001 is 4
- Weight of 000011000 is 2

Weight of a Code

Weight of a code – The minimum number of 1's that occur among all ***nonzero codewords*** of a code. In other words, the weight of a **code** is the smallest weight of any **nonzero codeword**.

Hamming (3,6) Codewords:

{000000, 001011, 010110, 011101,
100101, 101110, 110011, 111000}

Thus the weight of Hamming (3,6) Code is 3.

Weight is Minimum Distance

The weight of a linear code is the same as the minimum distance between codewords.

Let m be the weight of a linear code.

- The code will detect any $m - 1$ or fewer errors;
- If m is odd, the code will correct any $(m - 1)/2$ or fewer errors;
- If m is even, the code will correct any $(m - 2)/2$ or fewer errors;