Probability

The study of probability is concerned with random phenomena. Even though we cannot be certain whether a given result will occur, we often can obtain a good measure of its *likelihood*, or **probability**.

Example: Tossing a Coin

If a single fair coin is tossed, find the probability that it will land heads up.

Solution

The set of possible outcomes is $S = \{h, t\}$, and the event whose probability we seek consists of a single outcome: $E = \{h\}$.

A "fair" coin means that heads and tails are equally likely to occur. Therefore:

$$P(\text{heads}) = P(E) = 1/2.$$

Probability Model

- The **sample space *S*** of a random phenomenon is the set of all possible outcomes.
- An **event** is any outcome or any set of outcomes of a random phenomenon. That is, an event is a subset of the sample space.
- A **probability model** is a mathematical description of a random phenomenon consisting of two parts: a sample space *S* and a way of assigning probabilities to events.

Theoretical Probability Formula

If all outcomes in a sample space S are equally likely, and E is an event within that sample space, then the **theoretical probability** of the event E is given by

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)}.$$

Example: Card Hands

There are 2,598,960 possible hands in poker. If there are 36 possible ways to have a straight flush, find the probability of being dealt a straight flush.

Solution

$$P(\text{straight flush}) = \frac{36}{2{,}598{,}960} \approx .0000139$$

Example: Coin Tossing

Experiment: Toss a coin three times in succession. Record whether heads (h) or tails (t) is obtained with each toss. What are the possible outcomes?

The sample space is set of all possible outcomes:
S = {hhh, hht, hth, htt, thh, tht, tth, ttt}

Sequence of Coin Tosses

- Events are subsets of the sample space such as: {hht}, {hht, hth, thh} (the event of tossing 2 heads), $\emptyset$ = { } (the empty event), etc.
- Assuming a fair coin, all eight outcomes in *S* are equally likely. Thus:

$$P(\{\text{htt}\}) = 1/8,$$

$$P(2 \text{ heads}) = P(\{\text{hht, hth, thh}\}) = 3/8,$$

$$P(\{\,\}) = 0/8 = 0$$

Empirical Probability Formula

If E is an event that may happen when an experiment is performed, then the **empirical probability** of event E is given by

$$P(E) \approx \frac{\text{number of times event } E \text{ occurred}}{\text{number of times the experiment was performed}}.$$

Probability in Genetics

Gregor Mendel, an Austrian monk used the idea of randomness to establish the study of genetics. To study the flower color of certain pea plants he found that: *Pure red crossed with pure white produces red.*

Mendel theorized that red is "dominant" (symbolized by R), while white is recessive (symbolized by r). The pure red parent carried only genes for red (R), and the pure white parent carried only genes for white (r).

Probability in Genetics

Every offspring receives one gene from each parent which leads to the tables below. Every second generation flower is red because R is dominant.

1st to 2nd Generation

offspring		Second Parent	
		r	r
First Parent	R	Rr	Rr
	R	Rr	Rr

2nd to 3rd Generation

offspring		Second Parent	
		R	r
First Parent	R	RR	Rr
	r	rR	rr

Example: Probability of Flower Color

Determine the probability that a third generation will be

a) red b) white

		Second Parent	
		R	r
First Parent	R	RR	Rr
	r	rR	rr

a) Since red dominates white, any combination with R will be red. Three out of four have an R, so $P(\text{red}) = 3/4$.

b) Only one combination rr has no gene for red, so $P(\text{white}) = 1/4$.

Properties of Probability

Let E be an event from the sample space S. That is, E is a subset of S. Then the following properties hold.

1. $0 \le P(E) \le 1$ (The probability of an event is between 0 and 1, inclusive.)

2. $P(\varnothing) = 0$ (The probability of an impossible event is 0.)

3. $P(S) = 1$ (The probability of a certain event is 1.)

Addition Rule of Probability (for *A* or *B*)

Two events *A* and *B* are **mutually exclusive events** if they have no outcomes in common. (Mutually exclusive events cannot occur simultaneously.)

If *A* and *B* are **mutually exclusive**, then

$$P(A \text{ or } B) = P(A) + P(B).$$

Example: Probability Involving "Or"

A single die is rolled. $S = \{1, 2, 3, 4, 5, 6\}$. What is the probability of a 2 or odd?

Solution

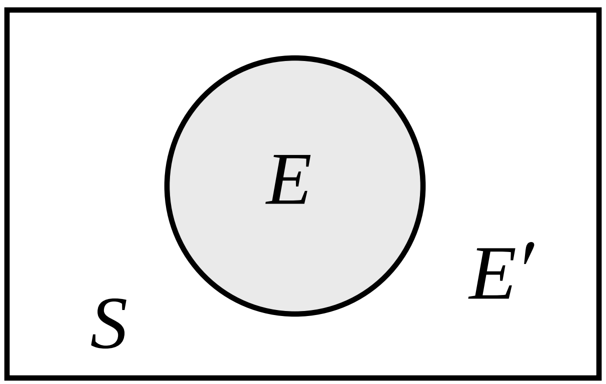
These are mutually exclusive events.

$$P(2 \text{ or odd}) = P(2) + P(\text{odd})$$

$$= \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}.$$

Probability of a Complement

The probability that an event E will *not* occur is equal to one minus the probability that it *will* occur.



$$P(E) + P(E') = P(S) = 1$$

$$P(\text{not } E) = P(E')$$

$$= 1 - P(E)$$

Example: Probability of a Complement

A fair coin is tossed three times. What is the probability of getting at least one head?

Solution

If E is the event of getting no heads, then E' is the event of getting at least one head.

$$P(E') = 1 - P(E) = 1 - \frac{1}{8} = \frac{7}{8}$$

Odds

Odds compare the likelihood of favorable outcomes to the likelihood of unfavorable outcomes.

If all outcomes in a sample space are equally likely, *a* of them are favorable to the event *E*, and the remaining *b* outcomes are unfavorable to *E*, then the **odds in favor** of *E* are *a* to *b*, and the **odds against** *E* are *b* to *a*.

Example: Odds

200 tickets were sold for a drawing to win a new television. If Matt purchased 10 of the tickets, what are the odds in favor of Matt's winning the television?

Solution

Matt has 10 chances to win and 190 chances to lose. The odds in favor of winning are 10 to 190, or 1 to 19.

Example: Converting Probability to Odds

Suppose the probability of rain today is .43. Give this information in terms of odds.

Solution

We can say that $P(\text{rain}) = .43 = \dfrac{43}{100}.$

So P(no rain) = 1 – .43 = .57 = 57/100. The odds in favor of rain are 43 to 57 and the odds against rain are 57 to 43.

Example: Converting Odds to Probability

Your odds of completing College Algebra class are 16 to 9. What is the probability that you will complete the class?

Solution

For every 16 favorable outcomes there are 9 unfavorable. This gives 25 possible outcomes. So

$$P(\text{completion}) = \frac{16}{25} = .64.$$