Announcements

- The second hour-exam will be held on Friday, July 12.
- The use of cell phones and other wireless devices is not permitted on the exam. You will need to bring a separate calculator for the exam.
- Sharing of calculators is not permitted during the exam.

Future value of an annuity

A = accumulated (future) value

d = uniform deposit per compounding period

r = nominal annual interest rate

t = number of years

$i = r/n$ = interest rate per compounding period

nt = number of compounding periods

$$A = d\left[\frac{(1+i)^{nt} - 1}{i}\right]$$

Annuity Example

An individual saves $100 per month, deposited directly into her credit union account on payday, the last day of the month. The account earns 6% per year, compounded monthly. How much will she have at the end of 5 years, assuming that the credit union continuous to pay the same interest rate?

$$A = d\left[\frac{(1+i)^{nt} - 1}{i}\right]$$

d = $100

$i = 0.06/12 = 0.005$

$nt = 12 \times 5 = 60$

Answer:

$$A = \$100\left[\frac{(1+0.005)^{60} - 1}{0.005}\right] = \$6977.00$$

Example

Suppose that a couple begins saving a regular amount of *d* per month after their child is born and want to have **$100,000** available for college when the child turns **18**. *How much do they have to save each month*, if their account earns **6.5%** interest per year, compounded monthly?

Solution: The monthly rate is 0.065/12 ≈ 0.0054166667.

A = \$100,000, $i \approx 0.0054166667$, and $nt = 12 \times 18 = 216$.

$$\$100{,}000 = d\left[\frac{(1+i)^{nt}-1}{i}\right] = d\left[\frac{(1.0054166667)^{216}-1}{0.0054166667}\right]$$

$$\$100{,}000 = d(408.3389)$$

$$d = \frac{\$100{,}000}{408.3389} = \$244.89$$

Conventional Installment Loan

Interest is compounded.

P = principal (amount borrowed)

r = nominal annual rate of interest

t = number of years

n = number of installments per year

$i = r/n$ = interest rate per compounding period

nt = number of installments

What payment should be made at the end of each compounding period in order to pay off the loan and any accumulated interest in t years?

Conventional Loan Payments

How much principal and interest would be due if all were paid at the end of *t* years?

$$A = P(1 + i)^{nt}$$

Using the savings plan, how much of a deposit, *d*, should one save *n* times per year to accumulate *A* at the end of *t* years?

$$A = d\left[\frac{(1 + i)^{nt} - 1}{i}\right]$$

Amortization Formula

P = principal (amount borrowed)
$i = r/n$ = interest rate per compounding period
nt = number of installments
d = payment made at end of each period

$$P(1+i)^{nt} = d\left[\frac{(1+i)^{nt}-1}{i}\right]$$

$$d = P\left(\frac{i}{1-(1+i)^{-nt}}\right)$$

Conventional Loan Example

Suppose that \$9,000 is borrowed for 5 years at 6% APR. What is the monthly payment for a conventional loan?

$$P(1+i)^{nt} = d\left[\frac{(1+i)^{nt}-1}{i}\right]$$

$$9{,}000(1.005)^{60} \approx 12{,}139.65$$

$$d\left[\frac{(1.005)^{60}-1}{0.005}\right] \approx d \times 69.77003$$

$$d \approx \frac{12{,}139.65}{69.77003} \approx 174.00$$

Student Loan Example

Suppose a student borrows \$10,000 at an interest rate of 6.9% compounded monthly. The loan accrues interest for two years before repayment begins. If the borrower has ten years to repay the accrued balance, what are the monthly payments?

After two years, the accrued balance is

$$A = 10{,}000\left(1 + \frac{0.069}{12}\right)^{24} = 10{,}000(1.00575)^{24} \approx 11{,}475.22$$

$$P(1 + i)^{nt} = d\left[\frac{(1 + i)^{nt} - 1}{i}\right]$$

$$\$11{,}475.22(1.00575)^{120} = d\left[\frac{1.00575^{120} - 1}{0.00575}\right]$$

$$\$22{,}833.16 = d \times 172.1356$$

$$d = \$22{,}833.16 / 172.1356 \approx \$132.65$$

Example: Home Mortgage

The price of a home is \$195,000. The bank requires a 10% down payment and two points at the time of closing. The cost of the home is financed with a 30-year fixed rate mortgage at 7.5%.

The required down payment is 10% of \$195,000:

$$0.10 \times \$195{,}000 = \$19{,}500.$$

The amount of the mortgage is the difference between the price of the home and the down payment:

$$\$195{,}000 - \$19{,}500 = \$175{,}500$$

Example continued: points

Each “point” is a one-time charge that equals 1% of the loan amount.

The cost of two points on a mortgage of \$175,500 is

$$0.02 \times \$175{,}500 = \$3510.$$

The down payment (\$19,500) is paid to the seller and the cost of two points (\$3510) is paid to the lending institution.

Example continued: monthly payment

We need to find the monthly mortgage payment for \$175,500 at 7.5% for 30 years. We use the loan payment formula for installment loans.

$$\$175{,}500\left(1.00625^{360}\right) = d \times \left[\frac{1.00625^{360} - 1}{0.00625}\right]$$

$$d \approx \$1227.12$$

The monthly mortgage payment for principal and interest is approximately \$1227.12.

In 30 years this totals \$441,763.20 in payments.

Of this, \$441,763 – 175,500 = \$266,263 is interest.

30-Year Mortgage

Suppose that a family buys a $120,000 house with $0 down payment and a 30-year fixed-rate mortgage with 6% APR and monthly payments. The monthly payment is $719.46.

What proportion of each payment is interest?

- In the first month, the interest is 0.5% of $120,000 = $600 or about 83% of the first payment.
- In the last month, the interest is a little less than 0.5% of the last payment or $3.58.

Amortization schedule

(monthly payment = \$719.46)

Month	Interest	Principal	Balance
			120,000.00
1	600.00	119.46	119,880.54
2	599.40	120.06	119,760.48
…			…
180	427.75	291.71	85,258.80
…			…
251	303.80	415.66	60,344.39
252	301.72	417.74	59,926.66
…			…
359	7.14	712.32	716.51
360	3.58	716.51	0.00

← starting balance

$0.005 \times 120{,}000 = 600.00$

$719.46 - 600.00 = 119.46$

← after 15 years

← paid off half

← Last payment is a little more due to rounding.

Paying off the loan

- The loan is paid off slowly at first since interest is a large portion of the initial payments.
- After half of the term of a mortgage has passed, the borrower has paid off less than half of the loan (in our example, after 15 years, the principal is only reduced by $34,741.20 or about 30% of $120,000).
- It might take 2/3 of the term of the mortgage to pay off 50% of the loan (in our example, it takes about 21 of 30 years to pay off $60,000).

30-Year vs. 15-Year Mortgage

Suppose that a family buys a $120,000 house with $0 down payment and a 30-year fixed-rate mortgage with 6% APR and monthly payments. What is the monthly payment?

$$d \approx \$719.46$$

What would the monthly payment be at the same 6% APR for a 15-year fixed-rate mortgage?

$$d \approx \$1012.63$$

In both cases, $600 of the first payment is interest.