Announcements

- The second hour-exam will be held on Friday, July 12.
- I have extended the due dates on the Practice Quizzes in MyLabsPlus.
- Homework from Chapter 8 needs to be finished right away.

Compound Interest (Exponential Growth)

A = accumulated amount (future value)
P = principal amount (present value)
r = nominal annual interest rate
t = number of years
n = number of compounding periods per year
i = r/n = interest rate per compounding period

$$A = P(1+i)^{nt} = P\left(1+\frac{r}{n}\right)^{nt}$$

Example

If you deposit $2000 at 4% compounded quarterly, what is the balance after 2 years?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$\$2000\left(1 + \frac{0.04}{4}\right)^{4\times 2} = \$2000 \times 1.01^{8} = \$2000 \times 1.082857$$

$$= \$2165.71$$

Note: the amount accrued using 4% simple interest is $A = P(1 + rt)$: $2000(1 + 2 × 0.04) = $2160.

Example: frequency of compounding

Suppose that you have a principal of P = \$1000 invested at 10% nominal interest per year. Determine the amount in the account after 10 years with the compounding period:

a) Annual compounding.

b) Quarterly compounding.

c) Monthly compounding.

$$A = P(1 + i)^{nt}$$

a) The annual rate of 10% gives i = 0.10. After 10 years, the account has $\$1000(1 + 0.10)^{10} = \$1000(1.10)^{10} = \$2593.74$.

b) The quarterly rate is i = 0.10/4 = 0.025. After 10 years (40 quarters), the account has $\$1000(1.025)^{40} = \$2685.06$.

c) The monthly rate is i = 0.10/12 = 0.00833333. After 10 years (120 months) the account has $\$1000(1.00833333)^{120} = \$2707.04$.

No matter how often we compound, the maximum is \$2718.28!

Continuous Compounding

Payment of interest at the upper limit of what can be obtained by compounding more and more frequently is called continuous compounding. For a principal *P* deposited in an account at a nominal annual rate *r*, compounded continuously, the balance after *t* years is

$$A = Pe^{rt}$$

where *e* is a special number, $e \approx 2.71828$.

Example: Suppose *P*=$1000 is invested at 10% nominal interest per year. Determine the amount in the account after 10 years compounding continuously. Here $r = 0.10$ and $t = 10$ so

$$A = 1000 \times e^{0.1 \times 10} = 1000 \times e^{1} \approx 2718.28$$

Example: Certificate of Deposit

Suppose that you will need $15,000 to pay for a year of college for your child 18 years in the future, and you can buy a certificate of deposit whose interest rate of 10% compounded quarterly is guaranteed for the period. How much do you need to deposit?

$$A = P(1 + i)^{nt}$$

In this case, A= \$15,000, i = 0.10/4 = 0.025, and nt = 72.

$$\$15000 = P(1+0.025)^{72} \rightarrow \$15000 = P \times 5.91723$$

$$P = \$15000 / 5.91723 = \$2534.94$$

Answer: you need to make an initial deposit \$2534.94.

Effective Rate

The **effective rate** is the actual percentage rate of increase for a length of time, taking into account compounding.

When interest is compounded n times at a rate of i per period,

$$\text{effective rate} = (1 + i)^n - 1.$$

When the time period is one year the effective rate is called:

The **annual percentage yield (APY)** or **effective annual rate (EAR)** is the actual percentage rate of increase over a year, taking into account compounding.

$$\text{APY} = \left(1 + \frac{r}{n}\right)^n - 1$$

Example

What is the APY for 5.3% compounded quarterly?

$$\mathbf{APY} = \left(1 + \frac{\boldsymbol{r}}{\boldsymbol{n}}\right)^{n} - 1$$

$$\text{APY} = \left(1 + \frac{0.053}{4}\right)^{4} - 1 = 0.05406 \approx 5.4\%$$

Interest Models

A = accumulated (total) amount

P = principal (beginning) amount

r = annual rate of interest (as decimal)

t = number of years of investment

Simple Interest: $A = P + Prt = P(1 + rt)$

Compounding ***n*** times per year:

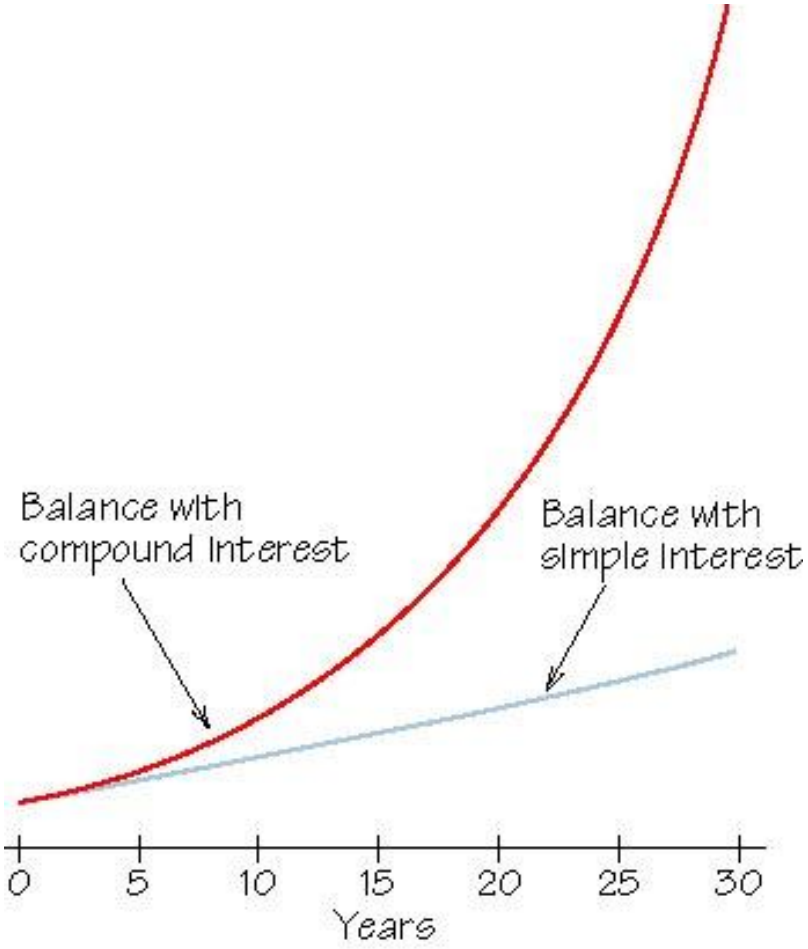
$i = r/n$ = interest rate per compounding period

nt = number of compounding periods

Compound Interest: $A = P(1 + i)^{nt}$

Continuous Compounding: $A = Pe^{rt}$

Linear vs. Exponential Growth



exponential growth: increases by constant ratio each year

linear growth: increases by fixed amount each year

Exponential Growth

The growth of an investment through compound interest is an example of **exponential growth**. Quantities that show exponential growth increase by the same proportion over the same period of time.

Problem: How long does it take for an investment to double if the annual rate is r?

Rule of 72: If the annual interest rate is r, then the time it takes to double is about 72/(100r) years.

Question: How long does it take for your investment to double at 8% interest, compounded quarterly?

Answer: $r = 0.08$ so it takes about 72/8 = 9 years.

Savings Plan (annuity)

Example: An individual saves \$100 per month, deposited on the last day of the month. The account earns 6% per year, compounded monthly. How much is accumulated in one year?

Savings Plan (continued)

$100 deposit at end of	**Dollars accumulated at end of year:**
1st month	$100(1 + 0.005)^{11} = 100 \times 1.005^{11}$
2nd month	$100(1 + 0.005)^{10} = 100 \times 1.005^{10}$
3rd month	$100(1 + 0.005)^{9} = 100 \times 1.005^{9}$
. . .	. . .
11th month	$100(1 + 0.005)^{1} = 100\times 1.005^{1}$
12th month	$100(1 + 0.005)^{0} = 100 \times 1$

$$\text{Total for year} = \$100(1 + 1.005 + 1.005^2 + \cdots + 1.005^{11})$$
$$\approx \$100 \times 12.335562 \approx \$1233.56$$

Future value of an annuity

A = accumulated (future) value

d = uniform deposit per compounding period

$i = r/n$ = interest rate per compounding period

nt = number of compounding periods

$$A = d\left[1 + (1+i) + (1+i)^2 + (1+i)^3 + \cdots + (1+i)^{nt-1}\right]$$

Use the formula for the sum of a geometric series to give:

$$A = d\left[\frac{(1+i)^{nt} - 1}{i}\right]$$

Annuity Example

An individual saves $100 per month, deposited directly into her credit union account on payday, the last day of the month. The account earns 6% per year, compounded monthly. How much will she have at the end of 5 years, assuming that the credit union continuous to pay the same interest rate?

$$A = d\left[\frac{(1 + i)^{nt} - 1}{i}\right]$$

d = $100
$i = 0.06/12 = 0.005$
$nt = 12 \times 5 = 60$

Answer:

$$A = \$100\left[\frac{(1 + 0.005)^{60} - 1}{0.005}\right] = \$6977.00$$

Example

Suppose that a couple begins saving a regular amount of *d* per month after their child is born and want to have **$100,000** available for college when the child turns **18**. *How much do they have to save each month*, if their account earns **6.5%** interest per year, compounded monthly?

Solution: The monthly rate is 0.065/12 ≈ 0.0054166667.

A = $100,000, $i \approx 0.0054166667$, and $nt = 12 \times 18 = 216$.

$$\$100{,}000 = d\left[\frac{(1+i)^{nt}-1}{i}\right] = d\left[\frac{(1.0054166667)^{216}-1}{0.0054166667}\right]$$

$$\$100{,}000 = d(408.3389)$$

$$d = \frac{\$100{,}000}{408.3389} = \$244.89$$