Head-to-Head Winner

A candidate is a **Head-to-Head winner** if he or she beats all other candidates by majority rule when they meet head-to-head (one-on-one).

To decide if a Head-to-Head winner exists:

Every candidate is matched on a one-on-one basis with every other candidate.

Drawback: there may not exist a Head-to-Head winner.

Example – Head-to-Head Winner

Example: Suppose that three candidates, *A*, *B*, and *C* are ranked as follows:

Number of Votes	4	3	2
First Choice	*A*	*B*	*C*
Second Choice	*B*	*C*	*B*
Third Choice	*C*	*A*	*A*

***A* vs. *B*:** *B* wins 5 to 4
***A* vs. *C*:** *C* wins 5 to 4
***B* vs. *C*:** *B* wins 7 to 2

B is the Head-to-Head winner.

Note: if *C* were to drop out, the result is unchanged;
the Plurality winner is not the Head-to-Head winner.

Example

A group of 13 students have to decide among three types of pizza: Sausage (S), Pepperoni (P), and Cheese (C). Their preference rankings are shown below. Pepperoni pizza wins using the Borda count but Cheese is the head-to-head winner.

Number of votes	5	4	2	2
First choice	C	P	S	P
Second choice	P	C	C	S
Third choice	S	S	P	C

Borda count does not satisfy the Head-to-Head Criterion

Head-to-Head Criterion

If a candidate is the head-to-head winner, the voting method selects that candidate as the winner.

- The Borda Count, Plurality, and Plurality-with-Elimination methods do not satisfy the Head-to-Head Criterion.

Monotonicity

When a candidate wins an election and, in a reelection, the only changes are changes that favor that candidate, then that same candidate should win the reelection.

Number of votes	~~5~~ 6	~~4~~ 3
First choice	*A*	*B*
Second choice	*B*	*A*

Majority rule is monotone and is the only method for two-candidate elections that is monotone, treats voters equally, and treats both candidates equally.

Plurality-with-Elimination is Not Monotone

Monotonicity: When a candidate wins an election and, in a reelection, the only changes are changes that favor that candidate, then that same candidate should win the reelection.

Number of Votes	7	6	5	3
First choice	A	~~B~~	C	~~D~~
Second choice	~~B~~	A	~~B~~	C
Third choice	C	C	A	~~B~~
Fourth choice	~~D~~	~~D~~	~~D~~	A

D is eliminated.

B is eliminated.

A is the Winner.

Number of votes	7	6	5	3
First choice	A	B	C	D
Second choice	B	A	B	C
Third choice	C	C	A	B
Fourth choice	D	D	D	A

A is the winner, so now suppose the voters in the last column raise A to first place.

Number of Votes	7	6	5	3
First choice	A	B	~~C~~	A
Second choice	B	A	B	~~D~~
Third choice	~~C~~	~~C~~	A	~~C~~
Fourth choice	~~D~~	~~D~~	~~D~~	B

Eliminate D.

Eliminate C.

B wins!

Monotonicity Criterion

A voting method satisfies the Monotonicity Criterion if the method is monotone.

- The Plurality-with-Elimination method does not satisfy the Monotonicity Criterion.
- Plurality and the Borda Count do satisfy this criterion.

Irrelevant Alternatives Criterion

When a voting system satisfies the Irrelevant Alternatives Criterion, the winner under this system always remains the winner when a non-winner is dropped from the ballot.

Number of Votes	4	3	2
First Choice	*A*	*B*	*C*
Second Choice	*B*	*C*	*B*
Third Choice	*C*	*A*	*A*

If C drops out, B becomes the winner with the Plurality method.

Plurality Voting does not satisfy the Irrelevant Alternatives Criterion.

Fairness Criteria for Voting Methods

- Majority Criterion: If a candidate is the majority winner, the voting method selects that candidate as the winner.
- Head-to-Head Criterion: If a candidate is the head-to-head winner, the voting method selects that candidate as the winner.
- Monotonicity Criterion: If a candidate is the winner using the voting method, then the same candidate wins in a reelection where the only changes are changes that favor the candidate.
- Irrelevant Alternatives Criterion: If a candidate is the winner using the voting method, then the same candidate would win if a non-winner were to drop out.

Is there a voting method that satisfies all of these criteria?

Arrow’s Impossibility Theorem

With three or more candidates, there cannot exist a voting system that always produces a winner and satisfies all four of the fairness criteria.

This theorem is named for Kenneth Arrow who proved a version of this theorem in 1951.