ON THREE NOTIONS OF EFFECTIVE COMPUTATION OVER \( \mathbb{R} \)

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Abstract. We compare three notions of effectiveness on uncountable structures. The first notion is that of a \( \mathbb{R} \)-computable structure, based on a model of computation proposed by Blum, Shub, and Smale, which uses full-precision real arithmetic. The second notion is that of an \( F \)-parameterizable structure, defined by Morozov and based on Mal’tsev’s notion of a constructive structure. The third is \( \Sigma \)-definability over \( HF(\mathbb{R}) \), defined by Ershov as a generalization of the observation that the computably enumerable sets are exactly those \( \Sigma_1 \)-definable in \( HF(\mathbb{N}) \).

We show that every \( \mathbb{R} \)-computable structure has an \( F \)-parameterization, but that the expansion of the real field by the exponential function is \( F \)-parameterizable but not \( \mathbb{R} \)-computable. We also show that the structures with \( \mathbb{R} \)-computable copies are exactly the structures with copies \( \Sigma \)-definable over \( HF(\mathbb{R}) \). One consequence of this equivalence is a method of approximating certain \( \mathbb{R} \)-computable structures by Turing computable structures.

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1. Introduction

In the mid-twentieth century, Russian and Western mathematicians gave definitions clarifying which algebraic structures could be considered computable. The Western approach focused on the effectiveness of the atomic diagram of the structure:

**Definition 1.1** ([9, 18]). Let $\mathcal{A}$ be a structure in a finite language. We say that $\mathcal{A}$ is computable if and only if the atomic diagram of $\mathcal{A}$ is Turing computable.

Of course, many equivalent definitions could be given for Turing computation, and the definition is often broadened to allow for a computable language, rather than a merely finite one. Also, this notion is not isomorphism invariant. Many structures are not themselves computable, but they have isomorphic copies that are. The key Russian definition was equivalent, but had a different focus:

**Definition 1.2** ([12]). Let $\mathcal{A}$ be a structure in a finite language. Let $\nu : D \to \mathcal{A}$ be surjective, for some $D \subseteq \omega$. Then the enumerated structure $(\mathcal{A}, \nu)$ is said to be constructive if and only if the following hold:

1. The set $\nu^{-1}$ is Turing computable.
2. For each predicate $P$ in the language (including the graphs of any functions in the language), the set $\nu^{-1}(P)$ is Turing computable.

A key restriction in each definition is that the structure $\mathcal{A}$ must be countable in order to satisfy either. In the last few years, attention from various directions has turned to the question of effectiveness for uncountable structures. In the most technical sense, the additive group of real numbers fails to be effective (computable). However, in any sense but the most technical, addition of real numbers is usually thought to be quite effective.

There have been many attempts to formalize this intuition. One model, proposed by the present author in joint work with Porter [6], is to generalize Definition 1.1 by replacing Turing computability with a model of computation proposed by Blum, Shub, and Smale, in which full-precision arithmetic is axiomatically “computable” (a more careful definition will be given later). A structure satisfying this hypothesis is said to be $\mathbb{R}$-computable. Another, proposed by Morozov [17], generalizes Definition 1.2 by replacing $\nu$ with a map from $\mathcal{A}$ into Baire space, where “Turing computable” is replaced by “analytic” (again, a formal definition will follow). Such a structure is said to be $F$-parameterizable. A third, proposed by Ershov [8], generalizes a classical theorem that computably enumerable sets are exactly those $\Sigma_1$-definable in $HF(\mathbb{N})$ by replacing $\mathbb{N}$ with the real field. This kind of structure is said to be $\Sigma$-definable in $HF(\mathbb{R})$. The purpose of the present paper is to compare these three models for effectiveness on uncountable structures.

Other formalizations of effectiveness for uncountable structures include: the notion of “local computability,” studied by R. Miller and others [13, 14]; infinite time machines, studied by Hankins, R. Miller, Seabold, and Warner [10]; Büchi automata [11]; and Borel structures, studied by H. Friedman and C. Steinhorn [19].

In Section 2 we will give the definitions of the three classes of structures. In Section 3, we will show that the class of $F$-parameterizable structures properly contains that of $\mathbb{R}$-computable functions. In Section 4, we will show that the classes of $\mathbb{R}$-computable and $\Sigma$-definable in $HF(\mathbb{R})$ structures are equivalent.
2. Definitions

2.1. \( \mathbb{R} \)-Computability. The definition of \( \mathbb{R} \)-computable structures comes from [6], and the BSS model of computation is explained in detail there. The idea of this model is that full precision real number arithmetic is axiomatically effective. We give here an outline of the relevant definitions.

The definition of a BSS machine comes from [4], and the concept is more fully described in [3]. Let \( \mathbb{R}^\infty \) be the set of finite sequences of elements from \( \mathbb{R} \), and \( \mathbb{R}_\infty \) the bi-infinite direct sum

\[
\bigoplus_{i \in \mathbb{Z}} \mathbb{R}.
\]

**Definition 2.1.** A machine \( M \) over \( \mathbb{R} \) is a finite connected directed graph, containing five types of nodes: input, computation, branch, shift, and output, with the following properties:

1. The unique input node has no incoming edges and only one outgoing edge.
2. Each computation and shift node has exactly one output edge and possibly several input branches.
3. Each output node has no output edges and possibly several input edges.
4. Each branch node \( \eta \) has exactly two output edges (labeled \( 0_\eta \) and \( 1_\eta \)) and possibly several input edges.
5. Associated with the input node is a linear map \( g_I : \mathbb{R}^\infty \to \mathbb{R}_\infty \).
6. Associated with each computation node \( \eta \) is a rational function \( g_\eta : \mathbb{R}_\infty \to \mathbb{R}_\infty \).
7. Associated with each branch node \( \eta \) is a polynomial function \( h_\eta : \mathbb{R}_\infty \to \mathbb{R} \).
8. Associated with each shift node is a map \( \sigma_\eta : \{ \sigma_1, \sigma_r \} \), where \( \sigma_1(x)_i = x_{i+1} \) and \( \sigma_r(x)_i = x_{i-1} \).
9. Associated with each output node \( \eta \) is a linear map \( O_\eta : \mathbb{R}_\infty \to \mathbb{R}^\infty \).

A machine may be understood to compute a function in the following way:

**Definition 2.2.** Let \( M \) be a machine over \( \mathbb{R} \).

1. A path through \( M \) is a sequence of nodes \( (\eta_i)_{i=0}^n \) where \( \eta_0 \) is the input node, \( \eta_n \) is an output node, and for each \( i \), we have an edge from \( \eta_i \) to \( \eta_{i+1} \).
2. A computation on \( M \) is a sequence of pairs \( (\eta_i, x_i)_{i=0}^n \) with a number \( x_{n+1} \), where \( (\eta_i)_{i=0}^n \) is a path through \( M \), where \( x_0 \in \mathbb{R}^\infty \), and where, for each \( i \), the following hold:
   (a) If \( \eta_i \) is an input node, \( x_{i+1} = g_I(x_i) \).
   (b) If \( \eta_i \) is a computation node, \( x_{i+1} = g_\eta(x_i) \).
   (c) If \( \eta_i \) is a branch node, \( x_{i+1} = x_i \) and \( \eta_{i+1} \) determined by \( h_\eta \) so that if \( h_\eta(x_i) \geq 0 \), then \( \eta_{i+1} \) is connected to \( \eta_i \) by \( 1_\eta \) and if \( h_\eta(x_i) < 0 \), then \( \eta_{i+1} \) is connected to \( \eta_i \) by \( 0_\eta \). (Note that in all other cases, \( \eta_{i+1} \) is uniquely determined by the definition of path.)
   (d) If \( \eta_i \) is a shift node, \( x_{i+1} = \sigma_\eta(x_i) \)
   (e) If \( \eta_i \) is an output node, \( x_{i+1} = O_\eta(x_i) \).

The proof of the following lemma is an obvious from the definitions.

**Lemma 2.3.** Given a machine \( M \) and an element \( z \in \mathbb{R}^\infty \), there is at most one computation on \( M \) with \( x_0 = z \).
Definition 2.4. The function $\varphi_M : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is defined in the following way: For each $z \in \mathbb{R}^\infty$, let $\varphi_M(z)$ be $x_{n+1}$, where $((\eta_i, x_i))_{i=0}^n, x_{n+1})$ is the unique computation, if any, where $x_0 = z$. If there is no such computation, then $\varphi_M$ is undefined on $z$.

Since a machine is a finite object, involving finitely many real numbers as parameters, it may be coded by a member of $\mathbb{R}^\infty$.

Definition 2.5. If $\sigma$ is a code for $M$, we define $\varphi_{\sigma} = \varphi_M$.

We can now say that a set is $\mathbb{R}$-computable if and only if its characteristic function is $\varphi_M$ for some $M$. Now by identifying a structure with its atomic diagram in the usual way, we define a set to be $\mathbb{R}$-computable if and only if its atomic diagram is $\mathbb{R}$-computable.

2.2. $F$-Parameterizability. In [17], Morozov introduced a concept that he called $F$-parameterizability in order to understand the elementary substructure relation on both automorphism groups and the structure of hereditarily finite sets over a given structure. In a talk at Stanford University, though, he identified this notion as one “which generalizes the notion of computable” [16].

Definition 2.6 ([17]). Let $\mathcal{M}$ be a structure in a finite relational language $(P_n^n)_{n \leq k}$. We say that $\mathcal{M}$ is $F$-parameterizable if and only if there is an injection $\xi : \mathcal{M} \to \omega^\omega$ with the following properties:

1. The image of $\xi$ is analytic in the Baire space, and
2. For each $n$, the set \[ \{ (\xi(a_i))_{i \leq k_n} : \mathcal{M} \models P_n(\bar{a}) \} \] is analytic.

The function $\xi$ is called an $F$-parameterization of $\mathcal{M}$. Morozov also introduced the following stronger condition, essentially requiring that $\mathcal{M}$ be able to define its own $F$-selfparameterization.

Definition 2.7 ([17]). Let $\mathcal{M}$ be an $F$-parameterizable structure. We say that $\mathcal{M}$ is weakly selfparameterizable if and only if there are functions $\Xi, p : \mathcal{M} \times \omega \to \omega$, both definable without parameters in $HF(\mathcal{M})$, with the following properties:

1. For all $x \in \mathcal{M}$ and all $m \in \omega$, we have $\Xi(x,m) = \xi(x)[m]$, and
2. For all $f \in \omega^\omega$ there is some $x \in \mathcal{M}$ such that for all $n \in \omega$ we have $p(x,n) = f(n)$.

In making sense of effectiveness on uncountable structures, a major motivation is to describe a sense in which real number arithmetic — an operation that, while not Turing computable, does not seem horribly ineffective — can be considered to be effective. Morozov proved the following result, which is a major source of motivation for the present paper.

Proposition 2.8 ([17]). The real field is weakly $F$-selfparameterizable.

Outline of proof. Define a function $\xi : \mathbb{R} \to \omega^\omega$ maps $x$ to its decimal expansion. This function is definable without parameters in $HF(\mathbb{R})$, in the sense required by Definition 2.7. □

2.3. $\Sigma$-definability. In the present paper, when $\mathbb{R}$ denotes a structure, it is the ordered field of real numbers. The following definition is standard, and appears in equivalent forms in [2], [8], and [7].

Definition 2.8. The function $\varphi_M : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is defined in the following way: For each $z \in \mathbb{R}^\infty$, let $\varphi_M(z)$ be $x_{n+1}$, where $((\eta_i, x_i))_{i=0}^n, x_{n+1})$ is the unique computation, if any, where $x_0 = z$. If there is no such computation, then $\varphi_M$ is undefined on $z$.

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1. For all $x \in \mathcal{M}$ and all $m \in \omega$, we have $\Xi(x,m) = \xi(x)[m]$, and
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2.3. $\Sigma$-definability. In the present paper, when $\mathbb{R}$ denotes a structure, it is the ordered field of real numbers. The following definition is standard, and appears in equivalent forms in [2], [8], and [7].
Definition 2.9. Given a structure $\mathcal{M}$ with universe $M$, we define a new structure $HF(\mathcal{M})$ as follows.

(1) The universe of $HF(\mathcal{M})$ is the union of the chain $HF_n(M)$ defined as follows:
\begin{enumerate}
\item $HF_0(M) = \emptyset$
\item $HF_{n+1}(M) = \mathcal{P}^{\leq \omega}(M \cup HF_n(M))$, where $\mathcal{P}^{\leq \omega}(S)$ is the set of all finite subsets of $S$
\end{enumerate}

(2) The language for $HF(\mathcal{M})$ consists of a binary predicate $\in$ interpreted as membership, plus a predicate symbol $\sigma^*$ for each predicate symbol $\sigma$ of the language of $\mathcal{M}$, interpreted in such a way that
$$HF(\mathcal{M}) \models \sigma^*(\{x_0\}, \{x_1\}, \ldots) \iff \mathcal{M} \models \sigma(x_0, x_1, \ldots),$$
and similar additions for each function and constant symbol.

Barwise [2] noted the following connection to computation, although he recorded that it was already well-known.

Theorem 2.10. Let $S$ be a relation on $\mathbb{N} = (\mathbb{N}, +, \cdot)$. Then

(1) $S$ is (classically) computably enumerable if and only if $S$ is $\Sigma_1$ on $HF(\mathbb{N})$.
(2) $S$ is (classically) computable if and only if $S$ is $\Delta_1$ on $HF(\mathbb{N})$.

This fact gives rise to Ershov’s definition [8] of a notion generalizing computability to structures other than $\mathbb{N}$. We will first give Barwise’s definition of the class of $\Sigma$-formulas.

Definition 2.11. The class of $\Sigma$-formulas are defined by induction.

(1) Each $\Delta_0$ formula is a $\Sigma$-formula.
(2) If $\Phi$ and $\Psi$ are $\Sigma$-formulas, then so are $(\Phi \land \Psi)$ and $(\Phi \lor \Psi)$.
(3) For each variable $x$ and each term $t$, if $\Phi$ is a $\Sigma$-formula, then the following are also $\Sigma$-formulas:
\begin{enumerate}
\item $\exists x \in t \Phi$
\item $\forall x \in t \Phi$, and
\item $\exists x \Phi$.
\end{enumerate}

A predicate $S$ is called a $\Delta$-predicate if both $S$ and its complement are defined by $\Sigma$-formulas.

Definition 2.12. Let $\mathcal{M}$ and $\mathcal{N} = (\mathbb{N}, P_0, P_1, \ldots)$ be structures. We say that $\mathcal{N}$ is $\Sigma$-definable in $HF(\mathcal{M})$ if and only if there are $\Sigma$-formulas $\Psi_0, \Psi_1, \Phi_0, \Phi_1, \Phi_1^*, \Phi_1^*, \ldots$ such that

(1) $\Psi_0^{HF(\mathcal{M})} \subseteq HF(\mathcal{M})$ is nonempty,
(2) $\Psi_1$ defines a congruence relation on $(\Psi_0^{HF(\mathcal{M})}, \Phi_0^{HF(\mathcal{M})}, \Phi_1^{HF(\mathcal{M})}, \ldots)$,
(3) $(\Psi_1^{HF(\mathcal{M})})^{HF(\mathcal{M})}$ is the relative complement in $(\Psi_0^{HF(\mathcal{M})})^2$ of $\Psi_1^{HF(\mathcal{M})}$,
(4) For each $i$, the set $(\Phi_1^*)^{HF(\mathcal{M})}$ is the relative complement in $\Psi_0^{HF(\mathcal{M})}$ of $\Phi_1^{HF(\mathcal{M})}$, and
(5) $\mathcal{N} \simeq (\Psi_0^{HF(\mathcal{M})}, \Phi_0^{HF(\mathcal{M})}, \Phi_1^{HF(\mathcal{M})}, \ldots) / \Psi_1^{HF(\mathcal{M})}$.
3. \( \mathbb{R} \)-COMPUTATION AND \( F \)-PARAMETERIZABILITY

We will first show that every \( \mathbb{R} \)-computable structure is \( F \)-parameterizable. We will also show that the converse of this theorem does not hold.

**Theorem 3.1.** Let \( \mathcal{M} \) be a \( \mathbb{R} \)-computable structure. Then \( \mathcal{M} \) is \( F \)-parameterizable.

**Proof.** Let \( \xi \) be an \( F \)-parameterization of the real field. Then consider the natural product map \( \xi_\infty : \mathbb{R}_\infty \rightarrow \omega^\omega \). By Path Decomposition (Theorem 1 in Chapter 2 of [3]), the universe of \( \mathcal{M} \) must be a countable union of semialgebraic sets of \( \mathbb{R}_\infty \), and so \( \text{im}(\xi_\infty | _{\mathcal{M}}) \) must be analytic. Similarly, for each predicate \( P \) in the language of \( \mathcal{M} \), the set \( P(\mathcal{M}) \) is a countable union of semialgebraic sets, so that \( \xi_\infty(P(\mathcal{M})) \) is also analytic. Thus, \( \xi_\infty | _{\mathcal{M}} \) is an \( F \)-parameterization of \( \mathcal{M} \). \( \square \)

The converse of this theorem is not true. Indeed, a mathematically familiar structure meets the stronger condition of being weakly \( F \)-selfparameterizable but not \( \mathbb{R} \)-computable.

**Theorem 3.2.** The structure \( (\mathbb{R}, +, \cdot, 0, 1, e^x) \) is weakly \( F \)-selfparameterizable but not \( \mathbb{R} \)-computable.

**Proof.** Suppose \( (\mathbb{R}, +, \cdot, 0, 1, f) \) is a \( \mathbb{R} \)-computable structure. By Path Decomposition (Theorem 1 in Chapter 2 of [3]), the atomic diagram must be a countable disjoint union of semi-algebraic sets, so that, in particular, the graph of \( f \) must be a countable disjoint union of semi-algebraic sets. The graph of \( e^x \) does not have this property, so \( (\mathbb{R}, +, \cdot, 0, 1, e^x) \) is not \( \mathbb{R} \)-computable.

Let \( \xi \) be the standard weak \( F \)-selfparameterization of the reals. It suffices to show that \( \xi(e^x) \) is analytic. We first show that \( \xi(\ln(x)) \) is analytic, and the result for the exponential function will follow. Since

\[
\ln x = \int_1^x \frac{1}{x} \, dx
\]

the actual value of \( \ln x \) must be between the upper and lower Riemann sum approximations to this integral. If the interval \([1, x]\) is partitioned into \( n \) equal intervals, the upper Riemann sum is

\[
U_n = \sum_{k=0}^{n-1} \left( \frac{x - 1}{n} \cdot \frac{1}{1 + k(x-1)/n} \right)
\]

and the lower sum is

\[
L_n = \sum_{k=1}^{n} \left( \frac{x - 1}{n} \cdot \frac{1}{1 + k(x-1)/n} \right)
\]

so that we have the error estimate

\[
|\ln x - U_n| \leq U_n - L_n.
\]

Now the difference of the sums is given by

\[
U_n - L_n = \frac{x^2 - 2x + 1}{nx}.
\]

Consequently, we have the definition

\[
\ln x = y \iff \bigcap_{n\in\omega} \{U_n - y\} \leq \frac{x^2 - 2x + 1}{nx},
\]

so that \( \xi(\ln(x)) \) is Borel. This completes the proof. \( \square \)
It seems natural to ask for some topological characterization of the \( \mathbb{R} \)-computable structures among the \( F \)-parameterizable structures. We might hope, for instance, that by replacing “analytic” with “\( \Sigma^0_1 \)” or some such class in Definition 2.6, we might find a class that coincides exactly with the \( \mathbb{R} \)-computable structures. A more careful analysis of the foregoing proof and of Morozov’s proof of Proposition 2.8, however, shows that such a characterization is impossible, at least with the standard parameterization of the real field.

**Lemma 3.3.** Let \( \xi \) be the weak \( F \)-selfparameterization of the real field given in the proof of Proposition 2.8. Then the following sets are \( \Pi^0_1 \):

1. \( \xi(=) \)
2. \( \xi(+) \)
3. \( \xi(\cdot) \)

**Proof.** For equality, it suffices to check, for each \( n \), whether the decimal approximation of the two numbers up to \( 10^{-n} \) either match exactly or match except for a terminal sequence of 9’s starting at the \( 10^{-k} \) place, where the decimal approximation of the two numbers up to \( 10^{-k+1} \) differ by \( 10^{-k+1} \). Since this is a \( \Delta^0_1 \) condition for each \( n \in \omega \), the equality relation is \( \Pi^0_1 \).

The cases of addition and multiplication are nearly identical to one another. Let \( x_n \) be the decimal approximation of \( x \) up to \( 10^{-n} \). To determine whether \( x + y = z \), it suffices to check whether, for each \( n \), we have

\[
|x_n + y_n - z_n| \leq 10^{-n+1}.
\]

Again, the condition is \( \Delta^0_1 \) for each \( n \in \omega \), so that the addition relation is \( \Pi^0_1 \). \( \square \)

**Lemma 3.4.** The set \( \xi(=) \) is \( \Pi^0_1 \) hard.

**Proof.** Let \( Q \) be a \( \Pi^0_1 \) set, defined by \( \forall x \bar{Q}(x, S) \), where \( S \) ranges over sets of natural numbers. For each set \( S \), define the set \( T_S \) to be the set of all \( x \) such that \( \neg \bar{Q}(x, S) \). Now \( Q(S) \) holds if and only if \( T(S) \) is empty.

Given a set \( S \subseteq \omega \), we can view \( S \) as a real number in \([0, 1]\) in a natural way, as

\[
x_S = \sum_{i=0}^{\infty} \chi_S(i)10^{-i}.
\]

Now to decide \( Q(S) \) from \( \xi(=) \), it suffices to check whether \( \xi(=) \) holds of the pair \((\xi(x_{T_S}), \xi(x_\emptyset))\). \( \square \)

Now in the proof of Theorem 3.2, we saw that \( \xi(e^x) \) is \( \Pi^0_1 \) over \( \xi(=) \oplus \xi(+) \oplus \xi(\cdot) \), but this set is \( \Pi^0_1 \) complete. Consequently, in the standard parameterization of the real field, the Borel hierarchy does not distinguish between the complexity of the field structure and that of the exponential function. Of course, it may be that there is a simpler parameterization of the real field in which this proof does not suffice, so we cannot as yet categorically rule out any such characterization.

4. \( \mathbb{R} \)-COMPUTATION AND \( \Sigma \)-DEFINABILITY

It should be noted that the work of this section is related to earlier work of Ashaev, Belyaev, and Myasnikov [1], who proved a similar but weaker result for the much more general definition of computation over an arbitrary structure given by Blum, Shub, and Smale.
4.1. \( \mathbb{R} \)-Computable Structures are \( \Sigma \)-definable in \( HF(\mathbb{R}) \).

**Proposition 4.1.** Every \( \mathbb{R} \)-computable structure is \( \Sigma \)-definable in \( HF(\mathbb{R}) \).

**Proof.** Let \( \mathcal{M} = (M, P_0, P_1, \ldots) \) be a \( \mathbb{R} \)-computable structure. Now \( M, \) and each \( P_i^{\mathcal{M}} \), are \( \mathbb{R} \)-computable sets, and so, in particular, both they and their complements are \( \mathbb{R} \)-semidecidable.

**Lemma 4.2.** The class \( \mathcal{RC} \) of \( \mathbb{R} \)-computable functions is the smallest class containing all polynomial functions, the characteristic function of \( < \), and the shift function, and closed under composition, juxtaposition, primitive recursion, and minimalization.

**Proof.** The finite-dimensional case of this result, including most of the important issues, is given in [4]. The polynomial functions, the characteristic function of \( < \), and the shift function are clearly \( \mathbb{R} \)-computable. That the class of \( \mathbb{R} \)-computable functions is closed under composition, juxtaposition, primitive recursion, and minimalization is shown in [4]. On the other hand, let \( f \) be a \( \mathbb{R} \)-computable function computed by a machine \( \Omega_f \). Then \( \Omega_f \) describes how to build \( f \) using only composition, juxtaposition, primitive recursion, and minimalization, from polynomials, the characteristic function of \( < \), and the shift function. \( \square \)

Now in \( HF(\mathbb{R}) \), we will represent elements \( (x_i)_{i \in \mathbb{Z} \geq 0} \in \mathbb{R}_{\infty} \) as sets of the form \( \{\{i\}, \{x_i\}\} : i \in I \) where \( I \subseteq \mathbb{Z} \) is the (finite) set of indices for nonzero elements.

**Lemma 4.3.** Every \( \mathbb{R} \)-computable function \( f \) is a \( \Sigma \)-function in \( HF(\mathbb{R}) \). Moreover, the complement of the graph of \( f \) is also defined by a \( \Sigma \)-formula.

**Proof.** The polynomial functions and the characteristic function of \( < \), by the definition of the structure \( HF(\mathbb{R}) \), are definable without quantifiers in \( HF(\mathbb{R}) \), as are their complements. The shift operators are defined as

\[
\sigma_l(\{\{i\}, \{x_i\}\} : i \in I) = (\{\{i\}, \{y_i\}\} : i \in I) \iff \bigwedge_{i \in I} y_i = x_{i+1}
\]

and

\[
\sigma_r(\{\{i\}, \{x_i\}\} : i \in I) = (\{\{i\}, \{y_i\}\} : i \in I) \iff \bigwedge_{i \in I} y_i = x_{i-1},
\]

both of which can be expressed by quantifier-free formulas in \( HF(\mathbb{R}) \), as can their complements. The juxtaposition of two \( \Sigma \) functions is a \( \Sigma \)-function (by clause 2 of the definition of \( \Sigma \)-formulas), and if the complements of their graphs are \( \Sigma \)-definable, then the same is true of the juxtaposition. The case of the complement is the same (by clause 3c).

Suppose that \( F : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{l} \rightarrow \mathbb{R} \) is a \( \Sigma \)-function (defined, say, by \( \Phi(t, \bar{x}, y) \)) and that the complement of its graph is \( \Sigma \)-definable (by \( \overline{\Phi}(t, \bar{x}, y) \)). Then \( L(\bar{x}) := \mu t[F(t, \bar{x}) = 0] \) is defined by

\[
\Theta(\bar{x}, t) := \Phi(t, \bar{x}, 0) \wedge \forall s \in t[\neg \Phi(s, \bar{x}, 0)],
\]

and its complement is defined similarly.

To show that the class of \( \Sigma \)-definable functions is closed under primitive recursion, suppose that \( h \) and \( g \) are \( \Sigma \)-definable functions, and \( f \) is defined by the
following schema:
\[
\begin{align*}
f(0, x_2, \ldots, x_k) &= g(x_2, \ldots, x_k) \\
f(y + 1, x_2, \ldots, x_k) &= h(y, f(y, x_2, \ldots, x_k)).
\end{align*}
\]
Now since \( g \) and \( h \) are \( \Delta \) relations, we can apply \( \Delta \) recursion (Corollary I.6.6 of [2]) to show that \( f \) is also a \( \Delta \) relation. □

Now since the characteristic function of \( M \), and of each \( P_i \) is \( \mathbb{R} \)-computable, they are also \( \Sigma \) functions of \( HF(\mathbb{R}) \), as are the characteristic functions of their complements. Let us say that, as \( U \) ranges over \( \{ M, M^c, P_0, P_0^c, P_1, P_1^c, \ldots \} \), the function \( \chi_U \) is defined by \( \Theta_U(x, \{ y \}) \); that is, suppose that \( \chi_U(x) = y \) if and only if \( \Theta_U(x, \{ y \}) \). Now \( \Theta_U(x, \{ y \}) \) is a \( \Sigma \)-formula defining \( U \). We let \( \Psi_1(x, y) \) be the relation \( x = y \), so that both \( \Psi_1 \) and its complement are defined by \( \Sigma \) formulas. This completes the proof. □

R. Miller and Mulcahey [14] raise the issue, in the context of a different notion of effectiveness for uncountable structures, of “simulating” an uncountable structure by a classically computable one. The following result shows a way in which this goal can be realized for certain \( \mathbb{R} \)-computable structures.

**Definition 4.4.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be structures in a common signature. We write that \( \mathcal{A} \leq_1 \mathcal{B} \) if \( \mathcal{A} \) is a substructure of \( \mathcal{B} \), and for all existential formulas \( \varphi(\bar{x}) \) and for all tuples \( \bar{a} \subseteq \mathcal{A} \), we have
\[
\mathcal{B} \models (\varphi(\bar{a}) \Rightarrow \mathcal{A} \models \varphi(\bar{a})).
\]

**Corollary 4.5.** For any \( \mathbb{R} \)-computable structure \( M \) whose defining machine involves only computable reals as parameters, there is a computable structure \( M^* \) such that \( M^* \leq_1 M \).

**Proof.** Morozov and Korovina [15] proved this result under the alternate hypothesis that \( M \) is \( \Sigma \)-definable without parameters over \( HF(\mathbb{R}) \). In fact, if the field of computable real numbers is used in place of the field of real algebraic numbers, their proof works without further modification for a structure \( \Sigma \)-definable with finitely many computable parameters. Since, by the theorem, \( M \) satisfies this hypothesis, the result follows. □

### 4.2. Structures \( \Sigma \)-Definable in \( HF(\mathbb{R}) \) are \( \mathbb{R} \)-Computable.

**Theorem 4.6.** Every structure \( \Sigma \)-definable over \( HF(\mathbb{R}) \) has an isomorphic copy which is \( \mathbb{R} \)-computable.

**Proof.** Let \( \mathcal{M} = (M, P_0, P_1, \ldots) \) be \( \Sigma \)-definable over \( HF(\mathbb{R}) \), via the scheme
\[
\Psi_0, \Psi_1, \Phi_0, \Phi_1, \ldots
\]
We may assume that all of these formulae are in prenex normal form.

**Lemma 4.7.** There is an enumeration function \( e : \mathbb{R}_\infty \rightarrow HF(\mathbb{R}) \) with the following properties:

1. \( e \) is a bijection;
2. \( e \) is a \( \Sigma \) function on \( HF(\mathbb{R}) \);
3. \( e \) is \( \mathbb{R} \)-computable; and
4. For any \( \Delta_0 \) formula \( \varphi(\bar{x}) \), the relation \( HF(\mathbb{R}) \models \varphi(e(n_1), \ldots, e(n_k)) \) is \( \mathbb{R} \)-computable.
Let $T$ be the class of finite trees, and let $T : \omega \to \omega$ be a classically computable Friedberg enumeration of $T$. That such an enumeration must exist is guaranteed by results in [5]. Let $T_k : \omega \to \omega$ be a classically computable Friedberg enumeration of the elements of $T$ with exactly $k$ terminal nodes. Let $\epsilon : \mathbb{Z} \to \omega$ by

$$
\epsilon(i) := \begin{cases} 
2i & \text{if } i \geq 0 \\
-2i - 1 & \text{if } i < 0 
\end{cases}.
$$

Let $x := (x_i : i \in \mathbb{Z}) \in \mathbb{R}_\infty$. Suppose $i_0$ is the first and $i_0 + k$ the last such that $x_{i_0}$ and $x_{i_0 + k}$ are nonzero. Now to specify $\epsilon(x)$, we take the classically computable tree with index $T_{k+1}(\epsilon(i_0))$, order its terminal nodes lexicographically from 0 to $k$, and label terminal node $j$ with the real number $x_j$. The interpretation of this tree with these labels as a member of $HF(\mathbb{R})$ is straightforward.

The function $\epsilon$ defined in this way is bijective and $\mathbb{R}$-computable. By Lemma 4.3, it is also a $\Sigma$-function in $HF(\mathbb{R})$. To decide if $\epsilon(n) \in \epsilon(m)$, it suffices to check whether there is an embedding of the appropriate finite trees, preserving the real labels on terminal nodes. This operation is $\mathbb{R}$-computable. Satisfaction of other $\Delta_0$ formulas is $\mathbb{R}$-computable, by induction on their form. 

Let $U$ range over the various $\Psi_i$, $\Psi_i^*$, $\Phi_i$, and $\Phi_i^*$. Now suppose that $U$ is of the form $\exists y_1, \ldots, y_n \phi(\bar{y}, \bar{x})$, where $\phi$ is $\Delta_0$. Then $U(\bar{x})$ holds exactly when

$$
\exists j_1, \ldots, j_n, \ell_1, \ldots, \ell_m \left[ \left( \bigwedge_{i \leq m} e(\ell_i) = x_i \right) \land \phi(e(j_1), \ldots, e(j_n), e(\ell_1), \ldots, e(\ell_m)) \right].
$$

The part within the brackets is $\mathbb{R}$-computable.

Now suppose both $U$ and its complement have definitions of the form $\exists \bar{x}[R(\bar{x})]$, where $R$ is $\mathbb{R}$-computable. Using Path Decomposition [3], we can write $R(\bar{x})$ as a countable disjoint union of semialgebraic sets $R_i(\bar{x})$, so that $U$ is defined by

$$
\exists \bar{x} \bigwedge_{i \in \omega} R_i(\bar{x}) \iff \bigcup_{i \in \omega} \exists \bar{x} R_i(\bar{x}).
$$

By Tarski’s elimination theory for the real ordered field, the condition on the right is a countable disjoint union of semialgebraic sets. Now both $\Psi_1$ and each of the $\Phi_i$ are $\mathbb{R}$-computable.

It remains to show that $\mathcal{M}$ has an isomorphic copy in which $\Psi_0$ is $\mathbb{R}$-computable. We may suppose $\Psi_0$ is in prenex normal form, so that $\Psi_0(\bar{x}) = \exists \bar{y}[\psi(\bar{x}, \bar{y})]$, where $\psi$ is quantifier-free. Let $M' \subseteq (HF(\mathbb{R})^{n_0} \times \mathbb{R}_\infty)$, where $n_0$ is the arity of $\Psi_0$, be such that $(\bar{x}, \bar{t}) \in M'$ if and only if $\psi(\bar{x}, \epsilon(\bar{t}))$. We will interpret all symbols of the language by their meaning on the first coordinate (so that, in particular, $(M', \Phi_0, \Phi_1, \ldots) \models (\bar{x}, \bar{t}) = (\bar{x}, \bar{s})$ for all $\bar{t}, \bar{s}$. Since $\psi$ is quantifier-free, the structure $(M', \Phi_0, \Phi_1, \ldots)$ is $\mathbb{R}$-computable.

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