On the spectra of computable bounded analytic functions

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Brian Zilli
joint work with T.H. McNicholl

Department of Mathematics
Iowa State University

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Zero sets of bounded analytic functions

Notation

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

Theorem (Identity Theorem, see Greene and Krantz 2006)

If $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $a_0, a_1, a_2, \ldots$ is a sequence of zeros of f with a limit point in $\mathbb{D}$, then $f \equiv 0$.

That is, if f is holomorphic, has infinitely many zeros, and is *not* identically zero, then its zero set *must* accumulate to $\partial\mathbb{D}$.

Zero sets of bounded analytic functions

Theorem (see Greene and Krantz 2006)

If f is a nonconstant bounded holomorphic function on $\mathbb{D}$ and $a_0, a_1, \ldots$ are the zeros of f (repeated according to their multiplicities), then

$$\sum_{j=0}^{\infty}(1 - |a_j|) < \infty.$$

Zero sets of bounded analytic functions

Conversely,

Theorem (see Greene and Krantz 2006)

If the points $a_0, a_1, \ldots$ in $\mathbb{D}$ satisfy

$$\sum_{j=0}^{\infty}(1 - |a_j|) < \infty,$$

then there is a bounded holomorphic function on $\mathbb{D}$ which has zero set consisting precisely of the a_j's, repeated according to their multiplicities.

The Blaschke product

- Specifically,

$$B(z) = \prod_{j=0}^{\infty} b_{a_j}(z) \quad \text{where} \quad b_{a_j}(z) = \begin{cases} \frac{|a_j|}{a_j} \frac{a_j - z}{1 - \bar{a} z} & \text{if } a_j \neq 0 \\ z & \text{if } a_j = 0. \end{cases}$$

  converges uniformly on compact subsets of $\mathbb{D}$ and has zeros that are precisely the a_j's.
- The individual terms b_{a_j} of the product are *Blaschke factors* and the product B is a *Blaschke product*.
- We thus call a sequence $a = (a_j)_{j=0}^{\infty}$ satisfying $\sum_{j=0}^{\infty}(1 - |a_j|) < \infty$ a *Blaschke sequence*. The sum is called the *Blaschke sum* and is denoted Σ_a.
- This canonical construction is analogous to the construction of a polynomial from linear factors corresponding to its zeros.
- We denote by B_a the Blaschke product corresponding to the Blaschke sequence a.

Spectra of bounded analytic functions on $\mathbb{D}$

Definition

bounded

The *spectrum* of a non-identically zero analytic function on $\mathbb{D}$ is the set of accumulation points of its zero sequence.

Notation

For a function B, we denote its spectrum by $\operatorname{spec} B$. For a Blaschke sequence $a = (a_j)_{j=0}^{\infty}$, we also denote its set of accumulation points by $\operatorname{spec} a$.

$\operatorname{spec} B_a = \operatorname{spec} a$

Remark

Such a spectrum is always a closed subset of $\partial\mathbb{D}$.

In fact,

Theorem (folklore)

For any nonempty, closed subset A of $\partial\mathbb{D}$, there exists a Blaschke product B with $\operatorname{spec} B = A$.

Computability of complex-valued functions

Definition (McNicholl 2013)

A function $f : \mathbb{D} \to \mathbb{C}$ is *computable* if there is an algorithm P with the following properties:

- Approximation. On input $D_1 \subseteq \mathbb{D}$ (an open rational disk) with $\overline{D_1} \subseteq \mathbb{D}$, either P does not halt or returns another open rational disk D_2.
- Correctness. If P halts on D_1 and returns D_2, then $f[D_1] \subseteq D_2$.
- Convergence. For all $z \in \mathbb{D}$, if $U \ni f(z)$ is open, then there exists $D_1 \ni z$ such that, when D_1 is input to P, P returns D_2 such that $f(z) \in D_2 \subseteq U$.

This definition is most likely equivalent to others you have seen.

Computability of Blaschke products

Given the one-to-one correspondence between Blaschke products and Blaschke sequences, it is natural to ask if there is a correspondence between the computability of a Blaschke sequence and the computability of its corresponding Blaschke product.

Theorem (Matheson and McNicholl 2008, McNicholl 2013)

A Blaschke product is computable if and only if it has a computable zero sequence with computable Blaschke sum.

Complexity notions for closed subsets of $\partial\mathbb{D}$

Definition

An *open rational arc* of $\partial\mathbb{D}$ is a set of the form

$$A(\alpha,\beta) := \{\exp\theta i : \alpha < \theta < \beta\}$$

where $\alpha < \beta$ are both elements of $\pi\mathbb{Q}$. We denote

$$\mathcal{A} := \{A : A \text{ is an open rational arc}\}$$

and fix an effective enumeration $(A_n)_{n=0}^{\infty}$ of $\mathcal{A}$.

Complexity notions for closed subsets of $\partial\mathbb{D}$

Definition (cf. Weihrauch 2000)

For a closed subset A of $\partial\mathbb{D}$, we say A is Σ^0_n, Π^0_n, *or* Δ^0_n *closed* if

$$\mathcal{S}_A = \{m \in \mathbb{N} : A_m \cap A \neq \emptyset\}$$

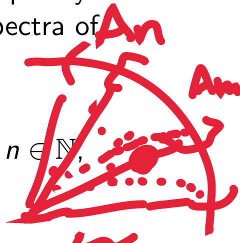
is a Σ^0_n, Π^0_n, or Δ^0_n (respectively) subset of $\mathbb{N}$.

Notation

When $B : \mathbb{D} \to \mathbb{C}$ is analytic ($B \not\equiv 0$) and $a = (a_n)_{n=0}^{\infty}$ is a Blaschke sequence, we denote $\mathcal{S}_{\mathrm{spec}\,B}$ and $\mathcal{S}_{\mathrm{spec}\,a}$ by $\mathcal{S}_B$ and $\mathcal{S}_a$, respectively.

Complexity of spectra

Given that the spectrum of a bounded analytic function is a closed subset of $\partial\mathbb{D}$ and we have established a notion of complexity for such sets, it is natural to analyze the complexity of spectra of computable bounded analytic functions.



Observation

For a computable Blaschke sequence $a = (a_j)_{j=0}^{\infty}$ and $n \in \mathbb{N}$,

$$\begin{aligned} n \in \mathcal{S}_a &\iff \operatorname{spec} a \cap A_n \neq \emptyset \\ &\iff (\exists m \in \mathbb{N})(\exists^\infty j \in \mathbb{N})\ \overline{A_m} \subset A_n \wedge \frac{a_j}{|a_j|} \in A_m, \end{aligned}$$

a Σ_3^0-sentence. That is, if B is a computable bounded analytic function on $\mathbb{D}$, then $\operatorname{spec} B$ is Σ_3^0–closed.

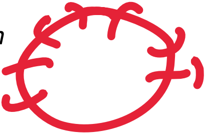
A Σ_3^0–complete spectrum

$= \{e \in \mathbb{N} : W_e \text{ is cofinite}\}$

Theorem (McNicholl & Z., 2023)

There exists a computable Blaschke product B for which $Cof \leq_m \mathcal{S}_B$. Consequently, $\mathcal{S}_B$ is Σ_3^0–complete.

C_e



Proof sketch.

Let $(C_e)_{e\in\mathbb{N}}$ be an effective sequence of disjoint rational open arcs in $\partial\mathbb{D}$ not containing 1.

Uniformly in e, construct a computable Blaschke product B_e for which $\operatorname{spec} B_e \subseteq C_e \cup \{1\}$,

$$\operatorname{spec} B_e \cap C_e \neq \emptyset \iff e \in Cof,$$

and $\Sigma_{B_e} = 2^{-e}$.

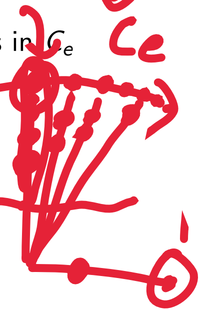
Then $B = \prod_{e\in\mathbb{N}} B_e$ has computable zero sequence (e.g., by dovetailing), $\Sigma_B = 2$, and $C_e \cap \operatorname{spec} B \neq \emptyset \iff e \in Cof$. □

Constructing the B_e

Let $(c_{e,m})_{m\in\mathbb{N}}$ be an effective sequence of rational points in C_e accumulating to the right endpoint of C_e.

Goals:

- $c_{e,m} \in \operatorname{spec} B_e \iff (\forall n \geq m)\ n \in W_e$
- Place zeros of B_e only on radii to the $c_{e,m}$ and 1



Suffices to construct $B_{e,m}$ with:

- $\operatorname{spec} B_{e,m} \subseteq \{c_{e,m}, 1\}$
- $c_{e,m} \in \operatorname{spec} B_{e,m} \iff (\forall n \geq m)\ n \in W_e$

Let $n_0 = m$. At stage s, if $n_s \in W_{e,s}$, let $n_{s+1} = n_s + 1$ and let $\arg a_s = c_{e,m}$. Otherwise, let $n_{s+1} = n_s$ and let $\arg a_s = 1$.
In either case, let $|a_s| = 2^{-s-e-m-1}$.
Then $B_{e,m}$ with zero sequence $(a_s)_{s=0}^{\infty}$ and $B_e = \prod_{m\in\mathbb{N}} B_{e,m}$ have the desired properties.

Not all Σ_2^0–closed sets are spectra

Theorem (McNicholl & Z., 2023)

There exists a Σ_2^0–closed $S \subseteq \partial\mathbb{D}$ which is not the spectrum of any computable Blaschke product.

Definition

A sequence A of points in $\mathbb{D}$ is a *computable partial sequence* if there exists $f : \mathbb{N}^2 \to \{e^{i\pi\theta} : \theta \in \mathbb{Q}\}$ such that

1. f is a computable partial function, and
2. $(m, n+1) \in \mathrm{dom}(f) \implies (m, n) \in \mathrm{dom}(f)$ and $|f(m,n) - f(m,n+1)| < 2^{-n-1}$.

Proof sketch of theorem.

Careful diagonalization argument over the computable partial sequences. □

All Π_2^0–closed sets are spectra

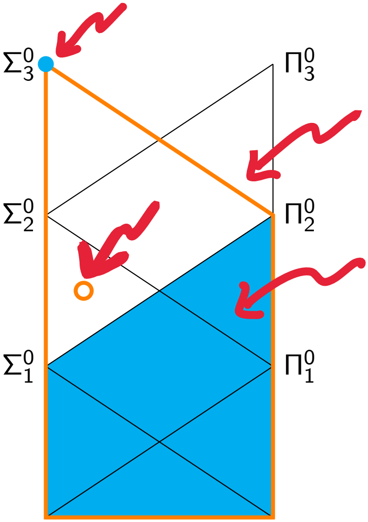
Theorem (McNicholl & Z., 2023)

If $S \subseteq \partial\mathbb{D}$ *is* Π_2^0*–closed, then there exists a computable Blaschke product with* $\operatorname{spec} B = S$.

Proof sketch.

□

Containments



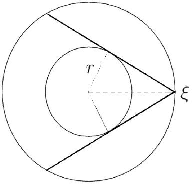
● $\subseteq$ spectra $\subseteq$ ○

Stolz regions

The following may be compared to the definitions of *cone* and *nontangential limit* in Garnett and Marshall 2005:

Definition

For $0 < r < 1$ and $\zeta \in \partial\mathbb{D}$, $S_r(\zeta)$ is the convex hull containing $\{\zeta\} \cup D_r(0)$, with ζ removed. We call $S_r(\zeta)$ the *Stolz region* at ζ with angle $2 \arcsin r$.



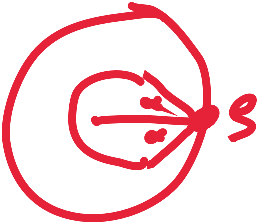


Figure: The Stolz region $S_r(\xi)$ (from Fulmer 2014).

Such a region is given explicitly by

$$S_r(\zeta) = \left\{ z \in \mathbb{D} : \frac{|\zeta - z|}{1 - |z|} \leq \frac{1 + r}{1 - r} \right\}$$

Nontangential limits of modulus 1

Definition
A sequence of points $\{a_n\}_{n=0}^{\infty}$ in $\mathbb{D}$ is said to approach $\zeta \in \partial\mathbb{D}$ *nontangentially* if

1. $\lim_{n\to\infty} a_n = \zeta$, and
2. $(\exists r \in (0,1))\ (\forall n \in \mathbb{N})\ a_n \in S_r(\zeta)$.

Definition
A function $f : \mathbb{D} \to \mathbb{C}$ has *nontangential limits of modulus* 1 at $\zeta \in \partial\mathbb{D}$ if, for all sequences $\{a_n\}_{n=0}^{\infty}$ which approach ζ non-tangentially, $\lim_{n\to\infty} f(a_n)$ exists and is of modulus 1.

inner : has radial limits of mod 1 a.e.

→ : nontangential

The Frostman condition

Definition (as stated in Matheson 2007)

A Blaschke product B with zero sequence $\{a_n\}_{n=0}^{\infty}$ satisfies the *Frostman condition* at $\zeta \in \partial\mathbb{D}$ if

$$\sigma_B(\zeta) := \sum_{n=0}^{\infty} \frac{1 - |a_n|}{|\zeta - a_n|}$$

is finite. The function $\sigma_B(\zeta)$ is the *Frostman indicator* of B.
A Blaschke product is *uniform Frostman* if its *Frostman constant*

$$\sigma_B := \sup_{\zeta \in \partial\mathbb{D}} \sigma_B(\zeta)$$

is finite.

We note that $1 - |a_n| \leq \frac{1-|a_n|}{|\zeta - a_n|}$, so satisfaction of the Frostman condition implies that a sequence is Blaschke.

Frostman's theorem

Theorem (Frostman 1942)

A Blaschke product B has nontangential limits of modulus 1 at $\zeta \in \partial\mathbb{D}$ if and only if B satisfies the Frostman condition at ζ.

This reveals an interesting parallel between the Frostman condition and the Blaschke condition:

- Just as Blaschke requires $\sum_{n\in\mathbb{N}} 1 - |a_n| < \infty$ so that $1 - |a_n| \to 0$ "rapidly,"
- Frostman similarly requires $\sum_{n\in\mathbb{N}} \frac{1-|a_n|}{|\zeta - a_n|} < \infty$ so that $\frac{1-|a_n|}{|\zeta-a_n|} \to 0$ (so all the zeros never lie in one Stolz region) "rapidly."

Thus, if a Blaschke product is uniform Frostman, then we may say it "has nontangential limits of modulus 1" (at all points of $\partial\mathbb{D}$).

Uniform Frostman Blaschke Products

Theorem (Matheson 2007)

If B is a uniform Frostman Blaschke product, then $\operatorname{spec} B$ *is nowhere dense in* $\partial\mathbb{D}$.

Theorem (Matheson 2007)

If F is a nowhere closed dense subset of $\partial\mathbb{D}$ and $\epsilon > 0$, then there exists a uniform Frostman Blaschke product B with $\operatorname{spec} B = F$ *and $\sigma_B < 1 + \epsilon$.*

Computable Uniform Frostman Blaschke Products

Theorem (McNicholl & Z., 2023)

If $S \subseteq \partial\mathbb{D}$ is computably closed and nowhere dense, then for every $k \in \mathbb{N}$, there is a computable (uniform Frostman) Blaschke product f with $\operatorname{spec}(f) = S$ and $\sigma_f < 1 + 2^{-k}$.

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