Pseudofiniteness and measurability of the everywhere infinite forest

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1. Pseudofinite structures

2. Non-standard cardinalities of definable sets

3. Asymptotic classes of finite structures

Pseudofinite structures

Pseudofinite structures

Definition

An $\mathcal{L}$-structure M is said to be pseudofinite if any of the following equivalent properties holds:

- *Every $\mathcal{L}$-sentence σ that is true in M, is also satisfied in some finite $\mathcal{L}$-structure M_0^σ.*
- $M \models \mathrm{FIN}_{\mathcal{L}}$.
- *M is elementarily equivalent to an ultraproduct $\prod_{\mathcal{U}} M_i$ of finite $\mathcal{L}$-structures.*

Łoś' theorem: "A first-order expressible statement is true in the ultraproduct $M = \prod_{\mathcal{U}} M_i$ if and only if it is true for $\mathcal{U}$-almost all structures M_i".

Pseudofinite structures ≈ "first-order asymptotic limits of finite structures"

Examples of structures that are not pseudofinite

- The linear orders $(\mathbb{Q}, <), (\mathbb{Z}, <)$ are not pseudofinite: the sentence $\sigma := \forall x \exists y (y < x)$ is true in both structures, but is not true in any finite linear order.
- The field $(\mathbb{C}, +, \cdot)$ is not pseudofinite: the function $f(x) = x^2$ is definable and surjective, but not injective. Hence

$$(\mathbb{C}, +, \cdot) \models \forall y \exists x (x^2 = y) \wedge \exists x, y (x \neq y \wedge x^2 = y^2),$$

  but this cannot be true in any finite field.
- $(\mathbb{Z}, +)$ is not pseudofinite: the definable function $x \mapsto x + x$ is injective, but not surjective.

Examples of structures that are pseudofinite

- Every ultraproduct of finite $\mathcal{L}$-structures is pseudofinite.
- Infinite vector spaces over $\mathbb{F}_p$ are pseudofinite:
  One can simply take the ultraproduct $\prod_{\mathcal{U}} \mathbb{F}_p^n$. $\mathbb{Z}/p\mathbb{Z} \models \forall x \exists y \, (y+y=x)$ for $p >> 2$.
- $(\mathbb{Q}, +) \equiv \prod_{\mathcal{U}} (\mathbb{Z}/p\mathbb{Z}, +) \cong (\mathbb{R}, +)$
  Both are models of the complete theory DAG of torsion-free divisible abelian groups
- Pseudofinite linear orders:

Proposition

Every infinite ultraproduct of finite linear orders has order type of the form $L = \omega \oplus (I \times \mathbb{Z}) \oplus \omega^*$, where I is an $\aleph_1$-saturated dense linear order without end points.

Pseudofinite fields and abelian groups

Theorem (James Ax, 1968)

An infinite field K is pseudofinite if and only if it K is perfect, has a unique extension of degree n for each $n \in \mathbb{N}$, and is *pseudo-algebraically closed* (every absolutely irreducible variety over K has a K-rational point).

Theorem (Ershov 1963, based on Szmielew)

An abelian group G is pseudofinite if and only if for every prime p and $n < \omega$ the cardinalities

$$D(p, n; G) := |\{a \in G : p \cdot a = 0 \wedge p^n | a\}| \text{ and}$$
$$Tf(p, n; G) := |p^n G / p^{n+1} G|$$

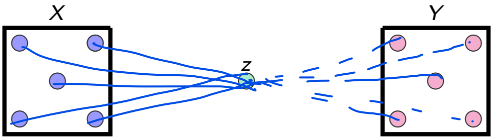
are either both infinite or both finite and equal.

Fun example: The abelian groups $(\mathbb{Z}_{p^\infty}, +)$ and $(\mathbb{Z}_{\langle p \rangle}, +)$ are not pseudofinite, but their direct sum $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{\langle p \rangle}$ is a pseudofinite group.

Theorem (Erdős, Rényi - 1963)

Given a fix number $r \geq 1$, $lim_{n\to\infty} \operatorname{Prob}(\mathbb{G}(n,p) \models \mathcal{A}_r) = 1$.

$$\mathcal{A}_r : \quad \forall x_1, \ldots, x_r \forall y_1, \ldots, y_r \left(\bigwedge_{1 \leq i,j \leq r} x_i \neq y_j \to \exists z \left(\bigwedge_{i \leq r} zRx_i \wedge \neg zRy_i \right) \right).$$



Theory of the *random graph*:

$$\mathrm{RG} = \{\forall x(\neg xRx), \forall x, y(xRy \to yRx)\} \cup \{\mathcal{A}_r : r \geq 1\}.$$

(Rado, 1964): There is a unique countable graph satisfying the theory RG.
$\Rightarrow$ The theory RG is complete, and the Rado graph is pseudofinite.

Questions on pseudofinite graphs

1. Which countable graphs are pseudofinite?

   Open problem: Is the generic triangle-free graph pseudofinite?

2. Which countable trees are pseudofinite?

Definition

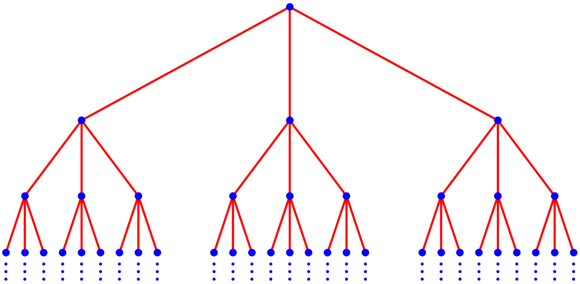
A tree is a (simple) graph without cycles. This property can be axiomatized in the language of graphs $\mathcal{L} = \{R\}$ by the theory:

$$\mathbf{Tree} = \{\forall x(\neg xRx), \forall x, y(xRy \to yRx)\}$$

$$\cup \left\{ \tau_n : \neg\exists x_1, \ldots, x_n \left(\bigwedge_{1\leq i<j\leq n} (x_i \neq x_j) \wedge \bigwedge_{i=1}^{n-1} (x_i R x_{i+1}) \wedge x_n R x_1 \right) : n \geq 3 \right\}.$$

Pseudofiniteness in countable trees

The 3-rooted tree RT_3 is not pseudofinite.



$$\sigma_{(1;3,4)} := \exists x \, [\deg(x) = 3 \wedge \forall y (y \neq x \to \deg(y) = 4)]$$

This sentence does not have finite models, due to the *Handshaking lemma*:

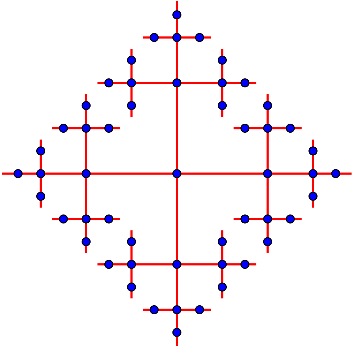
$G \models \sigma_{(1;3,4)}$

$$3 + 4(n-1) = \sum_{v \in V} \deg(v) = 2|E(G)|. = \text{even}$$

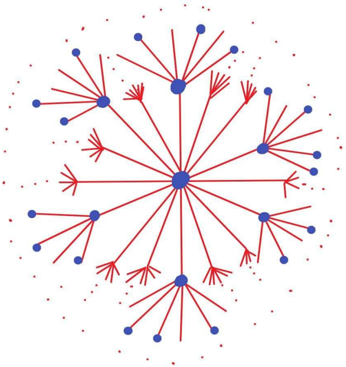
odd

The r-regular and the everywhere infinite forest

The theory $\mathcal{T}_r$ is the theory of an infinite tree Γ_r in which every vertex has degree r. The theory $\mathcal{T}_\infty$ (also known as the theory of the *everywhere infinite forest*) is the theory of an infinite tree Γ_∞ in which every vertex has infinite degree.



Γ_4



Γ_∞ (artistic representation)

Theorem (G., Robles)

The theories $\mathcal{T}_r$ and $\mathcal{T}_\infty$ are both pseudofinite.

These theories are axiomatized by the following collections of sentences:

$$\mathcal{T}_r : \textbf{Tree} \cup \{\forall x \exists^{=r} y (xRy)\}$$
$$\mathcal{T}_\infty : \textbf{Tree} \cup \{\forall x \exists^{\geq n} y (xRy) : n < \omega\}$$

Note that $\mathcal{T}_r$ and $\mathcal{T}_\infty$ cannot be realized as ultraproducts of finite trees because every finite tree has vertices of degree 1.

$G = \prod_{\mathcal{U}} G_n \models T_r$ → r-regular; → $\mathrm{girth}(G_n) \to \infty$

$H = \prod G_n' \models T_\infty$ → d_n-regular, $d_n \to \infty$; → $\mathrm{girth}(G_n) \to \infty$

Proposition

Let $\mathcal{C} = \{G_n : n \in \mathbb{N}\}$ be a class of finite graphs such that:

(a) Each graph G_n is r-regular (resp. d_n-regular)

(b) $\text{girth}(G_n) \to \infty$

Then, every infinite ultraproduct M of graphs in $\mathcal{C}$ is a model of $\mathcal{T}_r$ (resp. a model of T_∞ if $d_n \to \infty$.)

One possible construction is to use so-called "lifting of graph":

G	K_{r+1}	$\longrightarrow$	$L[G]$	
d-regular			d-regular	
girth $= g$	r: 3		girth $= 2g$	6

Also, cf. **[Margulis 1982: Explicit constructions of graphs without short cycles and low density codes]**

Non-standard cardinalities of definable sets

Why study pseudofinite structures?

- If $M = \prod_{\mathcal{U}} M_i$ is an ultraproduct of finite structures, every definable set $\varphi(M^n; \overline{b})$ has a ***non-standard cardinality***

$$|\varphi(M^n; \overline{b})| = [\ |\varphi(M_i^n; \overline{b}_i)|\ ]_{\mathcal{U}} \in \mathbb{R}^{\mathcal{U}}.$$

- The counting measure on a class of finite structures can be lifted using Łoś' theorem to give notions of dimension and measure on their ultraproduct.

$$\mu(A) = \mathrm{st}\left(\frac{|A|}{|M|}\right), \quad \delta_{\mathrm{fin}}(A) = \log|A| + \mathrm{Conv}(\mathbb{Z}), \quad \delta_{\alpha}(A) = \frac{\log|A|}{\log\alpha}\ .$$

Why study pseudofinite structures?

This kind of finite/infinite connection can sometimes be used to prove qualitative properties of large finite structures.

($\rightarrow$) applications to **Extremal combinatorics**

- Freiman conjecture for non-abelian groups (Hrushovski)
- Szemerédi's Regularity Lemma (Goldbring, Towsner)
- Several results and improvements can be obtained for definable graphs:

  Stable graphs: Malliaris - Shelah / Chernikov - Starchenko.
  Finite fields: Tao / Pillay - Starchenko .

  ↳ Erdös-Hajnal conjecture for stable graphs

- Arithmetic regularity lemma:
  (Terry - Wolf / Conant - Pillay - Terry)

[Di Nasso, Goldbring, Lupini. **Non-standard methods in Ramsey Theory and combinatorial number theory**. Springer. 2019]

Definition

*A structure M is said to be **strongly minimal** if for every $M' \equiv M$ and every formula $\varphi(x, \bar{a})$ in one-variable, the set $\varphi(M'; \bar{a}) := \{b \in M' : M' \models \varphi(b; \bar{a})\}$ is either finite or cofinite.*

Examples:

- $(\mathbb{N}, =)$, $\mathcal{T}_r$ (→ "trivial" structure)
- $(\mathbb{Q}, +)$, $\mathbb{F}_p^\omega$ (→ "vector space" structure)
- $(\mathbb{C}, +, \cdot, 0, 1)$ (→ a field)

Non-examples:

- $\mathcal{E}_\infty, \mathcal{T}_\infty$ (→ "equivalence relations")
- Models of RG (→ "random-like" structure)
- $(\mathbb{R}, +, \cdot, 0, 1, <)$ (→ "order-like" structure)

Strongly minimal ultraproducts of finite structures

Theorem (Pillay, 2015)

Let $M = \prod_{\mathcal{U}} M_i$ be a **strongly minimal** ultraproduct of finite structures, and let $\alpha \in \mathbb{N}^*$ be the pseudofinite cardinality of M ($\alpha = |M|$). Then,

1. For any definable (with parameters) set $D \subseteq M^n$, there is a polynomial $P_D(x) \in \mathbb{Z}[x]$ with positive leading coefficient such that $|D| = P_D(\alpha)$. Moreover, $RM(D) = \text{degree}(P_D)$.
2. For any L-formula $\varphi(\overline{x}, \overline{y})$ there is a finite number of polynomials $P_1, \ldots, P_k \in \mathbb{Z}[x]$ and L-formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ such that:
   (a) $\{\psi_i(\overline{y}) : i \leq k\}$ is a partition of the $\overline{y}$-space.
   (b) For any $\overline{a}$, $|\varphi(M^{|x|}; \overline{a})| = P_i(\alpha)$ if and only if $M \models \psi_i(\overline{a})$.

This result generalizes to **uncountably categorical** pseudofinite structures, using polynomials with rational coefficients. (A. van Abel, 2021).

Counting in strongly minimal pseudofinite structures

- In $M = (\mathbb{R}, +, 0) \cong \prod_{\mathcal{U}} (\mathbb{Z}/p\mathbb{Z}, +, 0)$, we can consider the formula

$$\varphi(x; y_1, y_2, y_3) \ : \ \exists z(z + z = x) \wedge (x = y_1 + y_2 \vee x \neq y_3).$$

We have $|\varphi(M; a_1, a_2, a_3)| = \begin{cases} \alpha & \text{if } a_1 + a_2 = a_3, \\ \alpha - 1 & \text{if } a_1 + a_2 \neq a_3. \end{cases}$

- For $M \models \mathcal{T}_r$, we can consider the formula

$$\eta(x_1, x_2; y) \ : \ (x_1 \, R \, x_2) \wedge (x_2 \neq y).$$

Then, $|\eta(M^2; a)| = r \cdot \alpha - r$, which is a polynomial with coefficients in $\mathbb{Z}$ evaluated in α. (Recall r is fixed)

$r \cdot \alpha - r \cdot 1 \in \mathbb{Z}[\alpha]$

Theorem (G., Robles)

Let $\mathcal{C} = \{G_n : n \in \mathbb{N}\}$ be a class of finite graphs such that each graph G_n is d_n-regular and both $d_n, \mathrm{girth}(G_n) \to \infty$. $T_\infty \to RM = \omega$.

Let M be an infinite ultraproduct of graphs in $\mathcal{C}$ (a model of $\mathcal{T}_\infty$) and fix the non-standard integers $\alpha = |M|$ and $\beta = [d_n]_{\mathcal{U}}$. Then for every formula $\varphi(\overline{x}, \overline{y})$ in the language of graphs there is a finite number of polynomials $p_1(X, Y), \ldots, p_k(X, Y) \in \mathbb{Z}[X, Y]$ such that:

1. For every $\overline{a} \in M^{|\overline{y}|}$, $|\varphi(M^{|\overline{x}|}, \overline{a})| = p_i(\alpha, \beta)$ for some $i \leq k$.
2. Moreover, there are formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ such that for every $\overline{a} \in M^{|\overline{y}|}$, Definability of cardinalities

$$M \models \psi_i(\overline{a}) \Leftrightarrow |\varphi(M^{|\overline{x}|}, \overline{a})| = p_i(\alpha, \beta).$$

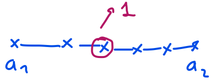
Moreover, if $D \subseteq M^r$ is definable and $\mathrm{RM}(D) = \omega \cdot n + k$ then the leading monomial of $P_D(\alpha, \beta)$ is of the form $C \cdot \alpha^n \beta^k$. Hence, ($C > 0$)

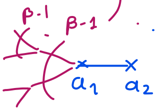
$$\delta_\ell(X) := \mathrm{st}\left(\frac{\log |X|}{\log \alpha}\right) = n.$$

Consider the theory $\mathcal{T}_\infty$ and any ultraproduct of finite graphs $M \models \mathcal{T}_\infty$,

- For the formula $\varphi(x; y_1, y_2) := D_2(x, y_1) \wedge D_3(x, y_2)$

$$|\varphi(M; a_1, a_2)| = \begin{cases} (\beta - 1)^2 & \text{if } M \models D_1(a_1, a_2) \\ 1 & \text{if } M \models D_5(y_1, y_2) \\ 0 & \text{if } M \models \neg D_1(a_1, a_2) \wedge D_5(a_1, a_2) \end{cases}$$





- For the formula $\eta(x_1, x_2; y) := (x_1 \, R \, x_2) \wedge (x_2 \neq y)$ we have

$$|\eta(M^2; a)| = \alpha \cdot \beta - \beta \ ,$$

which is a polynomial with coefficients in $\mathbb{Z}$ evaluated in α, β.

Asymptotic classes of finite structures

Multidimensional asymptotic classes

Definition (Anscombe, Macpherson, Steinhorn, Wolf)

Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures and let R be a fixed set of functions $\mathcal{C} \to \mathbb{R}^{\geq 0}$.

*We say that $\mathcal{C}$ is an R-**multidimensional asymptotic class** (or an R-**m.a.c.**) if for every formula $\varphi(\overline{x}, \overline{y})$ there are finitely many functions $h_1^\varphi, \ldots, h_k^\varphi \in R$ and formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ such that*

$$||\varphi(M^{|\overline{x}|}, \overline{a})| - h_i(M)| = o(h_i(M))$$

whenever $M \models \psi_i(\overline{a})$, as $|M| \to \infty$.

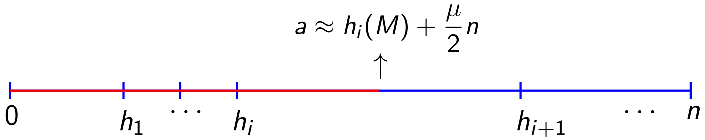
$|\varphi(M^{|x|}, \overline{a})| \approx h_i(M)$

↳ $|M| \to \infty$.

*In addition, we say that $\mathcal{C}$ is an R-m.e.c (multidimensional **exact** class) if in the condition above we have $|\varphi(M^{|\overline{x}|}, \overline{a})| = h_i(M)$.*

A non-example: finite linear orders

The class of finite linear orders is **not** a multidimensional asymptotic class.
Let $M = (\{0, 1, \ldots, n\}, <)$ and consider the formula $\varphi(x, y): \quad x < y$.
Assuming that $0 < h_1(M) < \ldots < h_k(M)$, there is $i < k$ such that $|[h_i(M), h_{i+1}(M)]| \approx \mu \cdot n$ for some $\mu > 0$.



Examples of multidimensional asymptotic classes

(1) The class of **finite fields**.

Theorem (Chatzidakis, van den Dries, Macintyre, 1992)

If $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be a formula in the language of rings, then there is a positive constant C, finitely many pairs $(d_i, \mu_i) \in \mathbb{N} \times \mathbb{Q}^{\geq 0}$ and finitely many formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ in the language of rings such that for every finite field $\mathbb{F}_q$, and every tuple $\overline{a} \in \mathbb{F}_q^m$, $\mathbb{F}_q \models \psi_i(\overline{a})$ if and only if

$|\varphi(\mathbb{F})| \approx \mu \cdot |\mathbb{F}|^d$

$$||\varphi(\mathbb{F}_q^n; \overline{a})| - \mu_i q^{d_i}| < Cq^{d_i - 1/2}.$$

$\frac{C \cdot |\mathbb{F}|^{d-1/2}}{|\mathbb{F}|^d} \to 0$

Hence, the class of finite fields is a m.a.c, with
$R = \{\mu|M|^d : (d, \mu) \in \mathbb{N} \times \mathbb{R}\} \to$ ***1-dimensional asymptotic classes***.
(when $R = \{\mu|M|^{d/N} : (d, \mu) \in \mathbb{N} \times \mathbb{R}\} \to$ ***N-dimensional asymptotic classes***.)

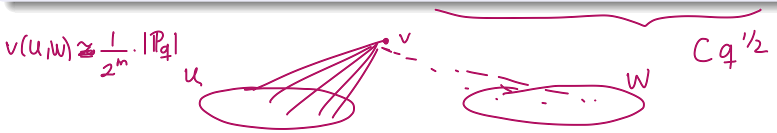
Examples of multidimensional asymptotic classes

(2) The class of **Paley graphs**: $\mathcal{C} = \{P_q : q \equiv 1 \pmod 4\}$, where P_q is the graph defined in $\mathbb{F}_q$ by stating aRb iff $a - b$ is a non-zero square.

Theorem (Bollobás, Thomason - 1985)

Let U, W be disjoint subsets of $\mathbb{F}_q$ ($q \equiv 1 \pmod 4$), such that $|U \cup W| = m$, and let S be the set of non-zero squares in $\mathbb{F}_q$. Let $v(U, W)$ be the set of elements $a \in \mathbb{F}_q$ such that $a - U \subseteq S$ and $a - W \subseteq \mathbb{F}_q \setminus S$. Then,

$$\left| |v(U, W)| - \frac{q}{2^m} \right| \leq \frac{1}{2}(m - 2 + 2^{-m+1})q^{\frac{1}{2}} + \frac{m}{2}.$$



All infinite ultraproducts of structures in this class are models of the theory RG of the random graph.

Examples of multidimensional exact classes

(3) (AMSW) The class $\mathcal{C}_d$ of graphs with bounded degree d is m.e.c.

↳ acyclic degree $= d$ ⟶ T_d

(4) (AMSW) The class of all finite abelian groups is m.e.c.

$\mathcal{C} = \{(\mathbb{Z}/n\mathbb{Z}, +) : n < \omega\}$ 1-dimensional asymptotic class

(5) (G., Robles) Suppose $\mathcal{C} = \{G_n : n \in \mathbb{N}\}$ is a class of regular graphs such that both $\text{degree}(G_n), \text{girth}(G_n) \to \infty$. Then $\mathcal{C}$ is a multidimensional exact class.

In this case, $R = \{p(|G_n|, \text{degree}(G_n)) : p(X, Y) \in \mathbb{Z}[X, Y]\}$

Ultraproducts of asymptotic classes

Theorem (Macpherson, Steinhorn)

1. If every ultraproduct of a class $\mathcal{C}$ is strongly minimal, then $\mathcal{C}$ is a 1-dimensional asymptotic class.
2. Every infinite ultraproduct of structures in a 1-dimensional asymptotic class is supersimple of SU-rank 1.

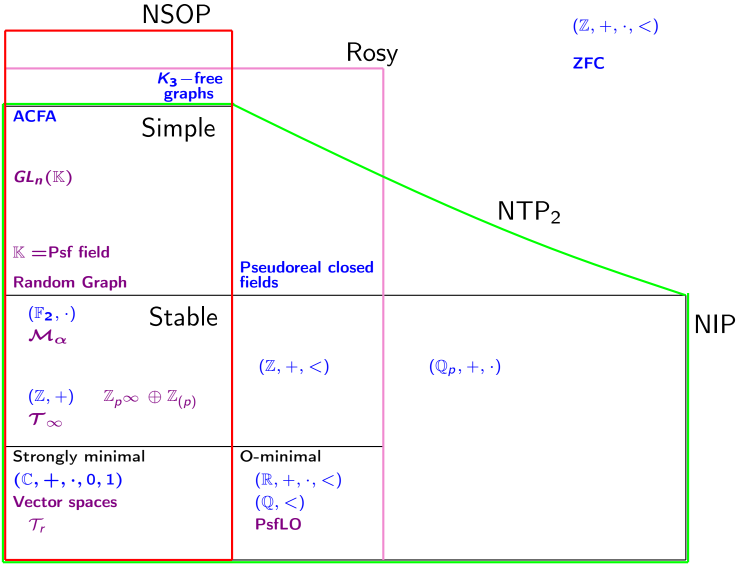
Similarly, every ultraproduct of an N-dimensional asymptotic class is supersimple of finite SU-rank ($\leq N$).

Idea: Each instance of dividing for formulas in one variable is witnessed by a drop of dimension. In ultraproducts of asymptotic classes, there are only finitely many possible dimensions.

Theorem (Anscombe, Macpherson, Steinhorn, Wolf)

Every infinite ultraproduct of a multidimensional asymptotic class is NSOP.

Model theory = geography of tame mathematics



A more detailed map at www.forkinganddividing.com (made by Gabriel Conant)

Underlying philosophy:

A geography of tame fragments and tame classes of finite structures may yield some insight into finite model theory and more applications to finite (extremal) combinatorics.

Final remarks and questions

- There are open problems about the pseudofiniteness of certain important structures:

  e.g. Triangle-free generic graph, the rational Urysohn sphere, etc.

- What can be said about the infinite ultraproducts of notable classes of finite structures?

  e.g. Fraïssé classes, Ramsey classes, etc.

- Which nice classes of graphs satisfy the conditions described to obtain models of $\mathcal{T}_r, \mathcal{T}_\infty$ as ultraproducts?

  Ramanujan graphs, expanders, etc.

- In famous examples of pseudofinite structures, what can we say about the classes of finite structures approximating them?

  For instance, is $\mathcal{M}_\alpha$ elementarily equivalent to an ultraproduct of graphs in a multidimensional asymptotic class?

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