Limited-information strategies in Banach-Mazur games

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University of North Carolina at Charlotte

Online Logic Seminar

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The Banach-Mazur game

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EMPTY

NONEMPTY

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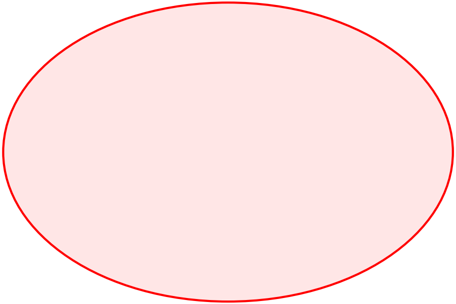
EMPTY $\quad U_0$

NONEMPTY



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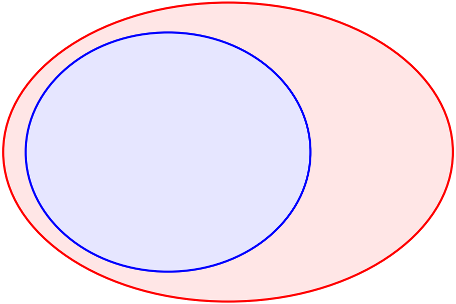
EMPTY	U_0	
	$\supseteq$	
NONEMPTY		V_0



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NONEMPTY: V_0

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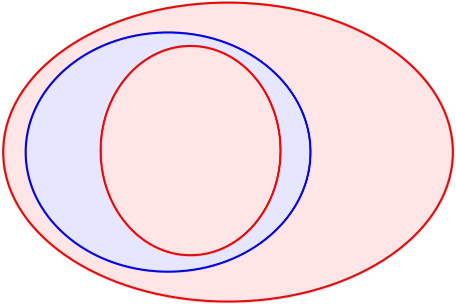

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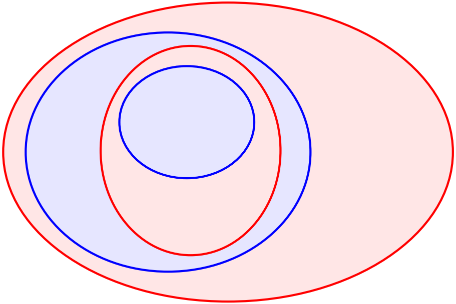

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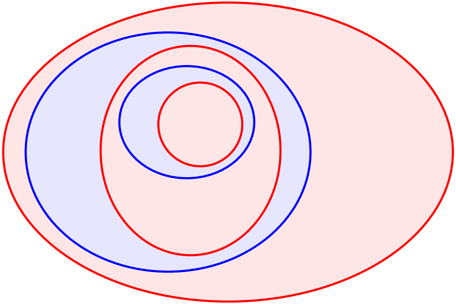

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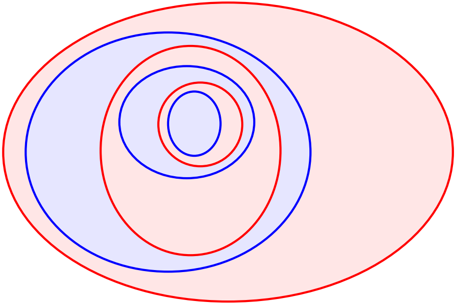

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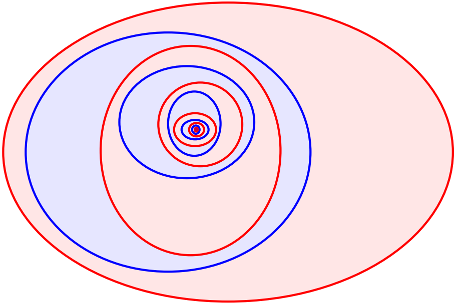

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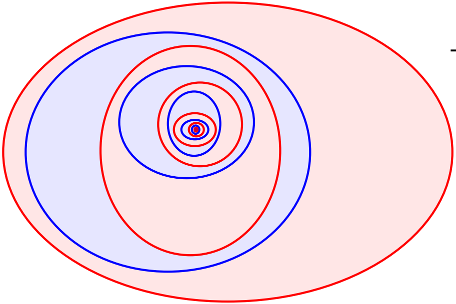

The first player (EMPTY) wins if

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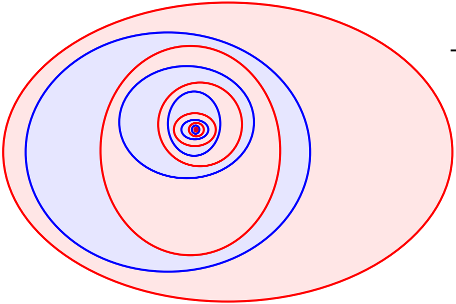

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Otherwise the second player (NONEMPTY) wins.

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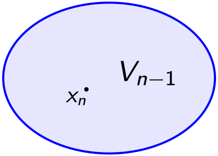

If X is countable, T_1, and has no isolated points, then EMPTY has a winning strategy in $\mathrm{BM}(X)$: Enumerate $X = \{x_n : n \in \omega\}$, and in the n^{th} round of the game choose $U_n = V_{n-1} \setminus \{x_n\}$.

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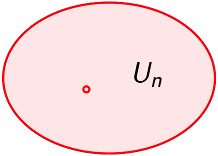

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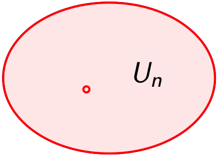

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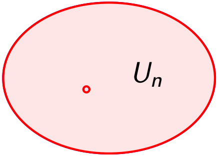

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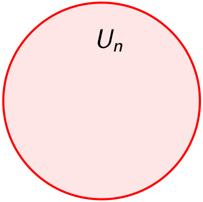

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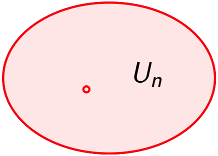

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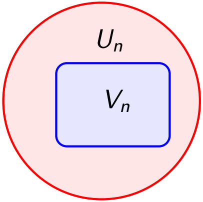

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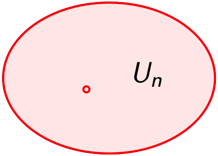

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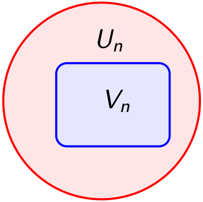
If X is a compact Hausdorff space, then NONEMPTY has a winning strategy in BM(X):

In the n^{th} round of the game, choose any nonempty open set V_n such that $\overline{V_n} \subseteq U_n$. Then $\bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} \overline{V_n} \neq \emptyset$.



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Given $k \in \omega$, a *winning k-tactic* is a winning strategy that depends only on the opponent's previous k moves. For example, the strategy for NONEMPTY in the second example on the previous slide is a winning 1-tactic.

Debs' space

Theorem (Debs; 1985)

There is a topological space X for which NONEMPTY *has a winning* 2*-tactic, but no winning* 1*-tactic.*

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Roughly, the proof that there is a 2-tactic uses topological features of the space to set up a coding mechanism, by which NONEMPTY is able to record, in each consecutive pair of her opponent's moves, the entire history of the game up to that point.

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Conjecture (Telgársky; 1987)

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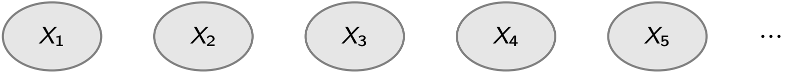

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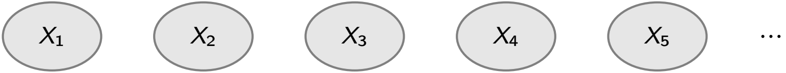

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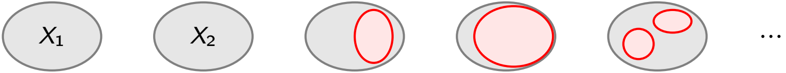

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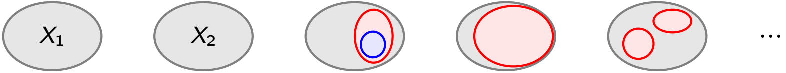

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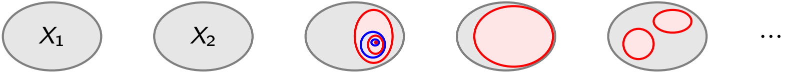

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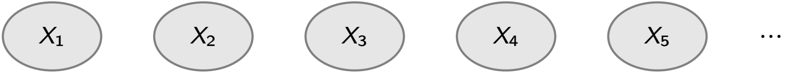

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Proof: For each k, let X_k be a space for which NONEMPTY has a winning $(k+1)$-tactic, but no winning k-tactic. Let $X = \bigsqcup_{k \in \mathbb{N}} X_k$.

NONEMPTY has no winning k-tactic: If NONEMPTY had a winning k-tactic for X, she would also have a winning k-tactic for X_k, because EMPTY can play $U_0 \subseteq X_k$.

Telgársky's conjecture may fail

Theorem (Brian, Dow, Milovich, and Yengulalp; 2020)

Assume GCH + $\Box$. *For every* T_3 *space* X, *if* NONEMPTY *has a winning strategy, then she has a winning 2-tactic.*

In particular, GCH + $\Box$ implies the failure of Telgársky's conjecture for T_3 spaces.

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Roughly, the proof of this theorem shows that when GCH $+ \square$ holds, it is always possible to set up a coding mechanism (much like with Debs' space, although this version is due to Fred Galvin) by which NONEMPTY is able to record, in each consecutive pair of her opponent's moves, the entire history of the game up to that point.

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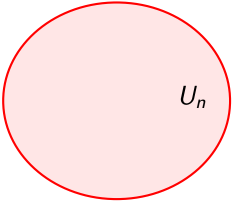

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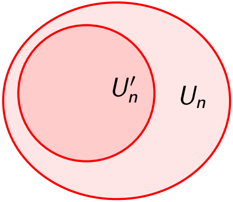

If EMPTY plays some nonempty open set U_n, choose some $U'_n \in \mathcal{B}$ with $U'_n \subseteq U_n$.

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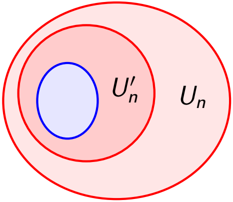

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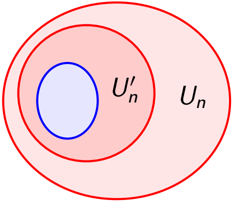

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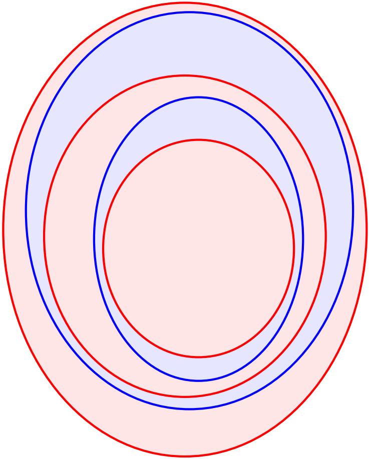

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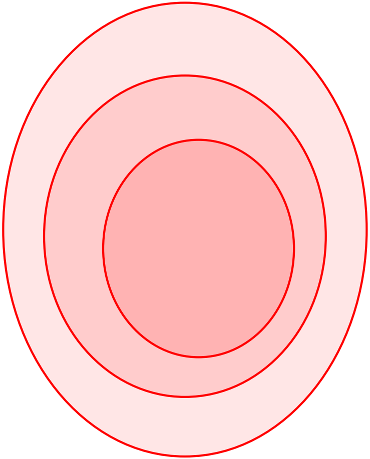

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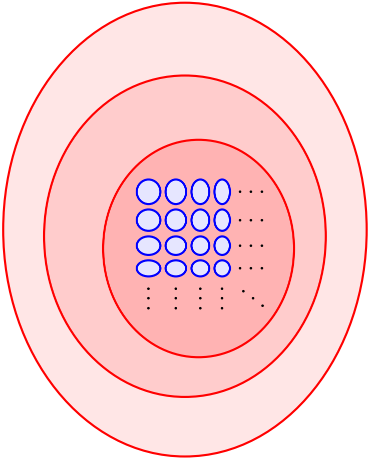

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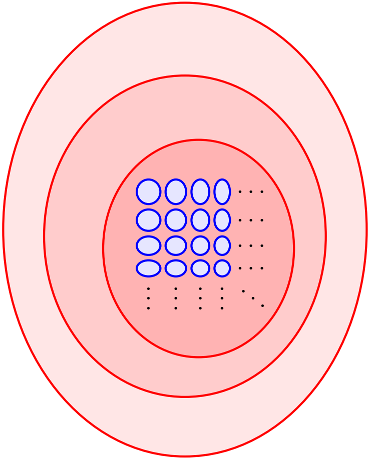

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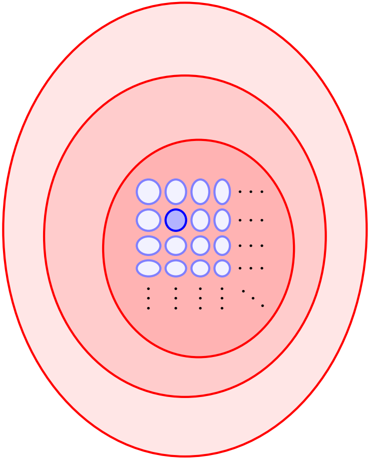

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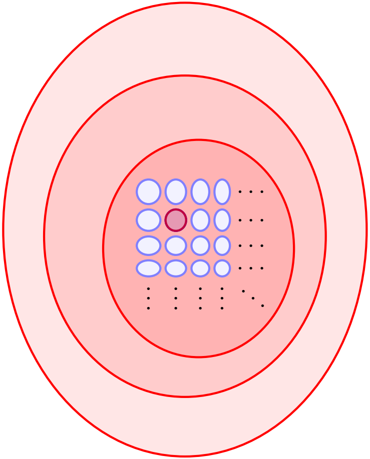

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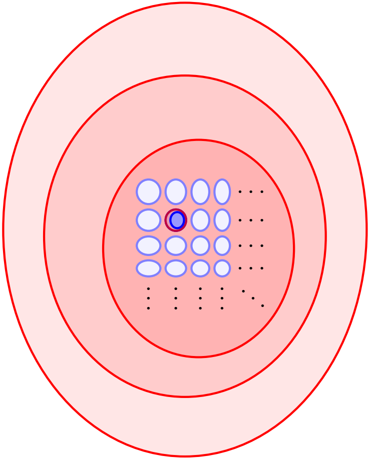

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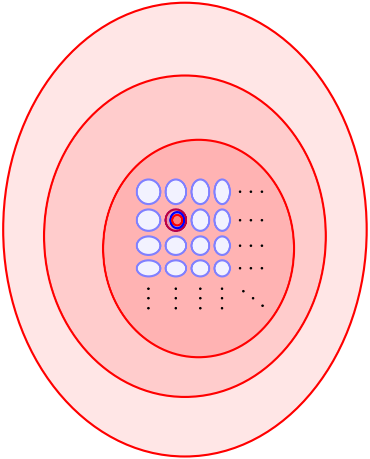

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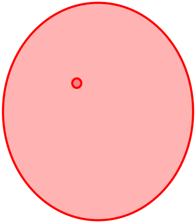

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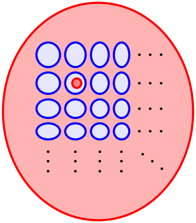

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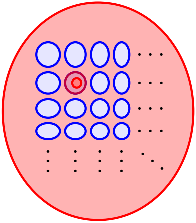

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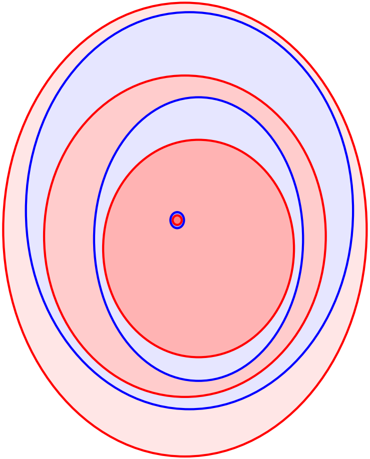

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From this, NONEMPTY deduces the rest of the game's history.

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$\triangledown(X)$: There is a pseudo-basis $\mathcal{B}$ for X such that for every $U \in \mathcal{B}$, there is a collection $\mathcal{S}$ of disjoint nonempty open subsets of U such that $|\{V \in \mathcal{B} \,:\, U \subseteq V\}| \leq |\mathcal{S}|$.

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If this statement is true for some space X, then, via coding, *If* NONEMPTY *has a winning strategy in* BM(X)*, then she has a winning* 2*-tactic.* In particular, no such space can witness Telgársky's conjecture.

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There are T_2 spaces X that fail to satisfy $\triangledown(X)$. But we do not know if such spaces can witness Telgársky's conjecture.

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Recall the main theorem under discussion:

Theorem (Brian, Dow, Milovich, and Yengulalp; 2020)

Assume GCH + $\Box$. *For every* T_3 *space* X, *if* NONEMPTY *has a winning strategy, then she has a winning* 2*-tactic. In particular,* GCH + $\Box$ *implies the failure of Telgársky's conjecture for* T_3 *spaces.*

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What about the first implication?

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just as before. $\mathbb{D}$ is dense in $\mathbb{P}$ by the same argument as before. Also as before, for any given $p_\alpha \in \mathbb{P}$, our definition of $\mathbb{D}$ ensures that $\{d \in \mathbb{D} : p_\alpha \text{ extends } d\} \subseteq \{p_\beta : \beta \leq \alpha\}$.

GCH + □ and magic enumerations of posets

Roughly, GCH + □ enables us to use a similar argument for larger $\mathbb{P}$.

Lemma (Soukup and Soukup; 2017)

Assume GCH + □, *and let H denote a "sufficiently large" fragment of the set-theoretic universe. If* $\mathbb{P}$ *is any ccc poset, there is an enumeration* $\{p_\alpha : \alpha < \kappa\}$ *of* $\mathbb{P}$ *such that, for every* $\alpha < \kappa$, *there is a countable collection* $\mathcal{N}$ *of countably closed elementary submodels of H such that* $\mathbb{P} \cap \bigcup \mathcal{N}$ *is a co-countable subset of* $\{p_\xi : \xi < \alpha\}$.

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just as before. $\mathbb{D}$ is dense in $\mathbb{P}$ by the same argument as before. Also as before, for any given $p_\alpha \in \mathbb{P}$, our definition of $\mathbb{D}$ ensures that $\{d \in \mathbb{D} : p_\alpha \text{ extends } d\} \subseteq \{p_\beta : \beta \leq \alpha\}$. The lemma is used to prove this set is countable.

GCH + $\Box$ and magic enumerations of posets

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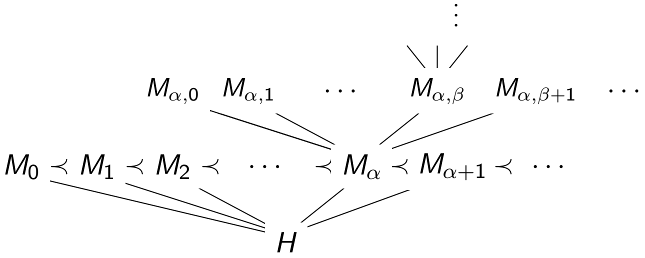

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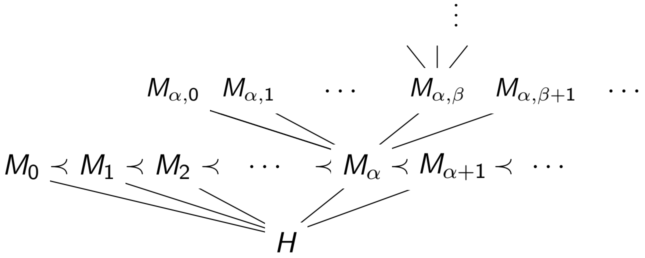

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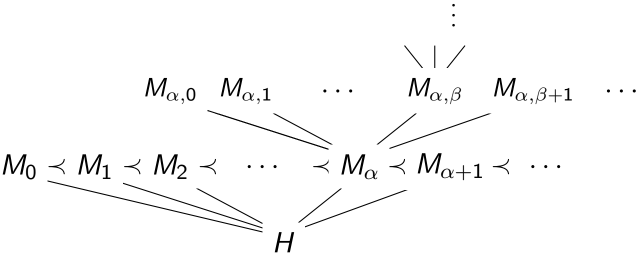

The lemma on the previous slide only enables us to prove $\triangledown$ for ccc posets. Our proof that GCH + $\square$ implies $\triangledown$ uses a generalization of sage Davies trees with stronger closure properties, and this gives us a similar lemma for posets with larger antichains.

The independence of $\triangledown$

The principle $\triangledown$ is independent of ZFC:

Theorem (Brian, Dow, Milovich, and Yengulalp; 2020)

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Proof idea: Adding lots of Cohen reals to a model of GCH $+ \square$ gives a model of $\triangledown + \neg$CH.

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Given the existence of a huge cardinal, there is a forcing extension satisfying GCH *but not* $\triangledown$. *Therefore* GCH *does not imply* $\triangledown$ *(unless huge cardinals are inconsistent).*

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We also show (again, modulo a huge cardinal) it is consistent with GCH that $\triangledown$ fails for the measure algebra of weight $\aleph_\omega$.

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What other methods are there for reducing arbitrary winning strategies to winning limited information strategies? Are there other robust coding mechanisms that do not require $\triangledown$?

The end

Thank you for listening