Semisimplicity, Glivenko theorems, and the excluded middle

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Introduction

Semisimplicity is equivalent to the law of the excluded middle (LEM).

The semisimple companion and the Glivenko companion of a logic coincide.

…for compact propositional logics with a well-behaved negation.

Intuitionistic and classical logic

How can we obtain classical propositional logic from intuitionistic logic?

1. Classical logic is intuitionistic logic plus the LEM (the axiom $\varphi \vee \neg\varphi$).

2. Classical consequence is related to intuitionistic consequence through the Glivenko translation (the double negation translation):

$$\Gamma \vdash_{\mathrm{CL}} \varphi \iff \Gamma \vdash_{\mathrm{IL}} \neg\neg\varphi.$$

In other words, CL is the Glivenko companion of IL.

3. The algebraic models of classical logic (Boolean algebras) are precisely the semisimple models of intuitionistic logic (Heyting algebras).

In other words, CL is the semisimple companion of IL.

Modal logic S4 and S5

How can we obtain the global modal logic S5 (modal logic of equivalence relations) from the global modal logic S4 (modal logic of preorders)?

1. Modal logic S5 is modal logic S4 plus the axiom $\Box\varphi \vee \Box\neg\Box\varphi$.

This is just the LEM with disjunction $\Box x \vee \Box y$ and negation $\neg\Box x$.

2. Consequence in S5 is related to consequence in S4 through the Glivenko translation (the double negation translation):

$$\Gamma \vdash_{S5} \varphi \iff \Gamma \vdash_{S4} \neg\Box\neg\Box\varphi.$$

This is just the Glivenko translation with negation $\neg\Box x$.

3. The algebraic models of S5 are precisely the semisimple models of S4 (Boolean algebras with a topological interior operator).

Semisimplicity: algebraic definition

Each algebra $\mathbf{A}$ has a lattice of congruences $\operatorname{Con}\mathbf{A}$. An algebra $\mathbf{A}$ is **simple** if it has exactly two congruences: the equality relation ($\Delta_{\mathbf{A}}$) and the total relation ($\nabla_{\mathbf{A}}$). Equivalently, $\mathbf{A} \twoheadrightarrow \mathbf{B}$ implies that $\mathbf{B} \cong \mathbf{A}$ or $\mathbf{B}$ is a singleton.

An embedding $\iota \colon \mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{B}_i$ is **subdirect** if it covers each component $\mathbf{B}_i$ (each $\pi_i \circ \iota$ is surjective). We call $\mathbf{A}$ a **subdirect product** of the algebras $\mathbf{B}_i$.

An algebra is **semisimple** if it is a subdirect product of simple algebras. Equivalently, an algebra $\mathbf{A}$ is semisimple if $\Delta_{\mathbf{A}}$ is the intersection of all maximal non-trivial congruences of $\mathbf{A}$ (i.e. of all coatoms of $\operatorname{Con}\mathbf{A}$).

This should remind you of the (Jacobson) **radical** of a commutative ring.

Given a class of algebras K closed under subdirect products, we could more generally consider the lattice $\operatorname{Con}_{\mathsf{K}}\mathbf{A}$ of K-congruences on $\mathbf{A}$ ($\mathbf{A}/\theta \in \mathsf{K}$).

Running examples: modal algebras

A **Boolean algebra with an operator** (abbreviated **BAO** here) is a Boolean algebra with a unary operator $\Box x$ such that

$$\Box(x \wedge y) = \Box x \wedge \Box y, \qquad \Box\top = \top.$$

These algebras form a **variety** (a class of algebras defined by equations).

An **S4-algebra** (**interior algebra**) is a BAO where $\Box$ is an interior operator:

$$\Box\Box x = \Box x \leq x.$$

We use the following notation:

$$\Box_n x := x \wedge \Box x \wedge \cdots \wedge \overbrace{\Box \cdots \Box}^{n \text{ times}} x.$$

BAOs form the algebraic semantics of the basic normal modal logic K, while S4-algebras form the algebraic semantics of the modal logic S4.

Running examples: $\mathrm{FL_{ew}}$-algebras

A **residuated lattice** is an algebra which has a lattice structure $\langle L, \wedge, \vee \rangle$, a monoidal structure $\langle L, \cdot, 1 \rangle$, and multiplication has residuals $x \backslash y$ and x/y:

$$x \leq z/y \iff x \cdot y \leq z \iff y \leq x \backslash z.$$

Examples: Heyting algebra, MV-algebras, ℓ-groups, Sugihara monoids, ...

In the **commutative** case we write $x \to y$ for the common value $x \backslash y = y/x$.

In a **bounded** residuated lattice we have a top $\top$ and bottom $\bot$. In an **integral** residuated lattice the monoidal unit 1 is the top element.

An $\mathbf{FL_{ew}}$**-algebra** is a bounded integral commutative residuated lattice.

Examples: Heyting algebras, MV-algebras, BL-algebras, ...

Two problems: algebraic formulation

Given a variety K, describe its semisimple algebras.

Theorem. A Heyting algebra is semisimple iff it is Boolean ($x \vee \neg x = 1$).

Theorem. An S4-algebra is semisimple iff it is an S5-algebra ($\neg\Box\neg\Box x \leq x$).

However, semisimple algebras cannot always be described equationally.

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However, semisimple algebras cannot always be described equationally.

Given a variety K, describe its semisimple subvarieties.

Theorem (Kowalski). A variety K of $\mathrm{FL}_{\mathrm{ew}}$-algebras is semisimple if and only if K validates the equation $x \vee \neg(x^n) = 1$ for some n.

Theorem (Kowalski and Kracht). A variety K of BAOs is semisimple if and only if K validates $\neg\Box\neg\Box_n x \leq x$ and $\Box_{n+1} x = \Box_n x$ for some n.

Semisimplicity and inconsistency lemmas

The paper of Kowalski on the semisimple varieties of FL_{ew}-algebras ends on the following note:

> *"[...] the argument used in the proof seems to have a certain generality to it, especially in view of its being a modification of [an analogous proof for Boolean algebras with operators]. It would be interesting to see what exactly that generality amounts to."*

We try to pinpoint what this generality consists in. A crucial tool for this purpose will be the **inconsistency lemmas** introduced by Raftery.

Logical preliminaries: syntax

Fix an algebraic signature and an infinite set of variables Var. The algebra of formulas (term algebra, absolutely free algebra) over Var is denoted **Fm**.

A **finitary logical rule** is a strict universal Horn sentence of the form

$$\mathrm{True}(\gamma_1)\ \&\ \dots\ \&\ \mathrm{True}(\gamma_n) \implies \mathrm{True}(\varphi).$$

We write this universal Horn sentence as $\Gamma \vdash \varphi$, where $\Gamma = \{\gamma_1, \dots, \gamma_n\}$.

A (possibly infinitary) **logical rule** allows for Γ to be infinite. Note that we still have a bound on how many variables may occur in Γ.

Logical preliminaries: semantics

A logical **matrix** $\langle \mathbf{A}, F \rangle$ is a pair consisting of an algebra $\mathbf{A}$ and a set $F \subseteq \mathbf{A}$ of **designated values**. Think of F as the interpretation of a predicate $\mathrm{True}(x)$.

The **logic** L determined by a class of matrices K is the strict universal Horn theory of K in the relational signature $\{\mathrm{True}(x)\}$, i.e. the set of all logical rules valid in each $\langle \mathbf{A}, F \rangle \in \mathsf{K}$. More explicitly:

$$\Gamma \vdash_{\mathrm{L}} \varphi \text{ if } v[\Gamma] \subseteq F \text{ implies } v(\varphi) \in F \text{ for each } v \colon \mathbf{Fm} \to \mathbf{A} \text{ and each } \langle \mathbf{A}, F \rangle \in \mathsf{K}.$$

A logic *simpliciter* is the logic of some class of matrices. It is **finitary** if

$$\Gamma \vdash_{\mathrm{L}} \varphi \implies \Gamma' \vdash_{\mathrm{L}} \varphi \text{ for some finite } \Gamma' \subseteq \Gamma.$$

It is **compact** if, roughly,

$$\Gamma \vdash_{\mathrm{L}} \bot \implies \Gamma' \vdash_{\mathrm{L}} \bot \text{ for some finite } \Gamma' \subseteq \Gamma.$$

Logical preliminaries: filters

A set $F \subseteq \mathbf{A}$ is an L-**filter** on $\mathbf{A}$ if $\langle \mathbf{A}, F \rangle$ is a model of L. The L-filters of $\mathbf{A}$ form a complete lattice $\mathrm{Fi}_{\mathrm{L}}\,\mathbf{A}$ (meets are intersections). It always contains the **trivial** filter $F = A$. The L-filter generated by $X \subseteq \mathbf{A}$ is denoted $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{L}} X$.

The L-filters on the algebra of formulas **Fm** are the **theories** of L, i.e. sets of formulas closed under consequence in L. These form a lattice $\mathrm{Th}\,\mathrm{L}$. The theory generated by Γ is the set of all L-consequences of Γ.

By analogy with $\Gamma \vdash_{\mathrm{L}} \varphi$ we may write $X \vdash^{\mathbf{A}}_{\mathrm{L}} a$ for $a \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{L}} X$.

Logical preliminaries: running examples

Classical logic is the logic of the class of all matrices of the form $\langle \mathbf{A}, \{\top\}\rangle$ where $\mathbf{A}$ is a Boolean algebra. The same holds for:

- intuitionistic logic and Heyting algebras,
- Łukasiewicz logic Ł and MV-algebras,
- the logic $\mathrm{FL_{ew}}$ and $\mathrm{FL_{ew}}$-algebras,
- modal logic K and BAOs,
- modal S4 and S4-algebras.

($\mathrm{FL_{ew}}$ stands for Full Lambek calculus with exchange and weakening.)

Fact. For each of these logics L and the corresponding variety K we have

$$\mathrm{Con}\,\mathbf{A} \cong \mathrm{Fi_L}\,\mathbf{A} \text{ for each algebra } \mathbf{A} \in \mathsf{K}.$$

This is a consequence of the **algebraizability** of these logics.

Example. Congruences on a BA $\longleftrightarrow$ lattice filters on a BA.

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Running examples: Łukasiewicz logic

One of the most studied non-classical logics is the infinitary Łukasiewicz logic $Ł_\infty$ and its finite-valued cousins $Ł_{k+1}$. These are extensions of FL_{ew}.

The algebra $[0,1]_Ł$ is defined by taking the unit interval with the usual lattice order and the operations

$$\neg x := 1 - x, \qquad\qquad x \oplus y := \min(x+y, 1).$$

We can define other operations in terms of these two:

$$x \odot y := \neg(\neg x \oplus \neg y) = \max(0, x+y-1),$$
$$x \to y := \neg x \oplus y = \min(1-x+y, 1).$$

Runnning examples: Łukasiewicz logic

Infinitary Łukasiewicz logic $Ł_\infty$ is the logic of $\langle [0,1]_{Ł}, \{1\} \rangle$.

Fact. $Ł_\infty$ logic is compact but not finitary:

$$\{x \to y^n \mid n \in \omega\} \vdash_{Ł_\infty} \neg x \vee y$$

The logics $Ł_{k+1}$ are defined by the submatrices $\{\frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$.

These logics do not satisfy the LEM in the form $x \vee \neg x$. However, observe that if $x := \frac{k-1}{k}$, then $x^2 = \frac{k-2}{k}, \ldots, x^{k-1} = \frac{1}{k}, x^k = \frac{0}{k}$.

In other words, $x^k \in \{0,1\}$, so $x \vee \neg(x^k)$ is a theorem.

Semisimplicity: logical definition

The lattice of L-filters on **A** is an analogue of the lattice of K-congruences. We can therefore phrase the definition of semisimplicity in terms of L-filters.

An L-filter $F \subseteq \mathbf{A}$ is **simple** if it is a maximal non-trivial L-filter on **A**, i.e. a coatom of $\mathrm{Fi}_{\mathrm{L}}\,\mathbf{A}$. (**Compare:** $\mathbf{A}/\theta$ is simple iff θ is a coatom of $\mathrm{Con}\,\mathbf{A}$.)

An L-filter $F \subseteq \mathbf{A}$ is **semisimple** if it is an intersection of simple L-filters on **A**. A model $\langle \mathbf{A}, F \rangle$ of a logic L is **(semi)simple** if F is a (semi)simple L-filter.

A logic L is **semantically semisimple** if each model of L is semisimple. It is **syntactical semisimple** if each theory of L is semisimple, i.e. if each theory is an intersection of simple theories (maximal consistent theories).

Two problems: logical formulation

Given a logic L**, describe its semisimple models.**

Theorem. A model of IL is semisimple if and only if it validates $x \vee \neg x$.

Theorem. A model of S4 is semisimple if and only if it validates $\neg\Box\neg\Box x \to x$.

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Theorem. A model of S4 is semisimple if and only if it validates $\neg\Box\neg\Box x \to x$.

Given a logic L**, describe its semisimple axiomatic extensions.**

Theorem (Kowalski). An axiomatic extension of FL_{ew} is semisimple if and only if it validates the formula $x \vee \neg(x^n)$ for some n.

Theorem (Kowalski and Kracht). An axiomatic extension of K is semisimple if and only if it validates $\neg\Box\neg\Box_n x \to x$ and $\Box_n x \to \Box_n x$ for some n.

Syntactic principles at play

We now arm ourselves with some syntactic principles:

- deduction–detachment theorems (DDTs),
- inconsistency lemmas (ILs),
- dual inconsistency lemmas (dual ILs),
- the law of the excluded middle (LEM),
- the proof by case property (PCP).

These principles come in local and global forms. Each of these syntactic principles has a certain semantic import as well (e.g. global DDT $\longleftrightarrow$ EDPC).

Global ILs and their duals were introduced by Raftery (2013). He proved the equivalence between semisimplicity and the LEM in the global case.

Global DDTs

Intuitionistic logic satisfies the equivalence

$$\Gamma, \varphi \vdash_{\mathrm{IL}} \psi \iff \Gamma \vdash_{\mathrm{IL}} \varphi \to \psi.$$

Global modal logic S4 satisfies the equivalence

$$\Gamma, \varphi \vdash_{\mathrm{S4}} \psi \iff \Gamma \vdash_{\mathrm{S4}} \Box\varphi \to \psi.$$

The $(k+1)$-valued Łukasiewicz logic $Ł_{k+1}$ satisfies the equivalence

$$\Gamma, \varphi \vdash_{Ł_{k+1}} \psi \iff \Gamma \vdash_{Ł_{k+1}} \varphi^k \to \psi.$$

These are examples of **global deduction–detachment theorems (DDTs)**: there is some set of formulas $\mathrm{I}(x, y)$ such that

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \iff \Gamma \vdash_{\mathrm{L}} \mathrm{I}(\varphi, \psi).$$

Local DDTs

Global modal logic K satisfies the equivalence

$$\Gamma, \varphi \vdash_{\mathrm{K}} \psi \iff \Gamma \vdash_{\mathrm{K}} \Box_n \varphi \to \psi \text{ for some } n \in \omega,$$

where we use the abbrevation

$$\Box_n \varphi := \varphi \wedge \Box\varphi \wedge \cdots \wedge \Box^n \varphi.$$

The logic $\mathrm{FL}_{\mathrm{ew}}$ satisfies the equivalence

$$\Gamma, \varphi \vdash_{\mathrm{FL}_{\mathrm{ew}}} \psi \iff \Gamma \vdash_{\mathrm{FL}_{\mathrm{ew}}} \varphi^n \to \psi \text{ for some } n \in \omega.$$

where we use the abbrevation

$$\varphi^n := \overbrace{\varphi \cdot \ldots \cdot \varphi}^{n \text{ times}}.$$

These are examples of **local deduction–detachment theorems (DDTs)**: there is a family $\Phi(x,y)$ of sets of formulas $\mathrm{I}(x,y)$ such that

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \iff \Gamma \vdash_{\mathrm{L}} \mathrm{I}(\varphi, \psi) \text{ for some } \mathrm{I}(x,y) \in \Phi(x,y).$$

Global inconsistency lemmas (ILs)

Intuitionistic logic satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{IL}} \bot \iff \Gamma \vdash_{\mathrm{IL}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n).$$

Global modal logic S4 satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{S4}} \bot \iff \Gamma \vdash_{\mathrm{S4}} \neg\Box(\varphi_1 \wedge \cdots \wedge \varphi_n).$$

The $(k+1)$-valued Łukasiewicz logic $Ł_{k+1}$ satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{Ł_{k+1}} \bot \iff \Gamma \vdash_{Ł_{k+1}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)^n.$$

These are examples of **global inconsistency lemmas (ILs)**: for each $n \geq 1$ there is some set of formulas $\mathrm{I}_n(x_1, \ldots, x_n)$ such that

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{L}} \bot \iff \Gamma \vdash_{\mathrm{L}} \mathrm{I}_n(\varphi_1, \ldots, \varphi_n).$$

Of course, for logics with a conjunction we can restrict to $n := 1$.

Local inconsistency lemmas (ILs)

Global modal logic K satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{K}} \bot \iff \Gamma \vdash_{\mathrm{K}} \neg\Box_n(\varphi_1 \wedge \cdots \wedge \varphi_n) \text{ for some } n \in \omega,$$

The logic $\mathrm{FL_{ew}}$ satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{FL_{ew}}} \bot \iff \Gamma \vdash_{\mathrm{FL_{ew}}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)^n \text{ for some } n \in \omega.$$

The infinitary Łukasiewicz logic Ł_∞ satisfies the same equivalence because Ł_∞ is compact and the finitary Łukasiewicz logic Ł inherits it from $\mathrm{FL_{ew}}$.

These are examples of **local inconsistency lemmas (ILs)**: for each $n \geq 1$ there is a family $\Phi_n(x_1, \ldots, x_n)$ of sets of formulas $\mathrm{I}_n(x_1, \ldots, x_n)$ such that

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{L}} \bot \iff \Gamma \vdash_{\mathrm{L}} \mathrm{I}_n(\varphi_1, \ldots, \varphi_n) \text{ for some } \mathrm{I}_n \in \Phi_n.$$

Global dual inconsistency lemmas (dual ILs)

Classical logic satisfies the equivalence

$$\Gamma \vdash_{\mathrm{CL}} \varphi \iff \Gamma, \neg\varphi \vdash_{\mathrm{CL}} \bot.$$

The $(k+1)$-ary Łukasiewicz logic satisfies the equivalence

$$\Gamma \vdash_{\text{Ł}_{k+1}} \varphi \iff \Gamma, \neg(\varphi^k) \vdash_{\text{Ł}_{k+1}} \bot.$$

The global modal logic S5 satisfies the equivalence

$$\Gamma \vdash_{\mathrm{S5}} \varphi \iff \Gamma, \neg\Box\varphi \vdash_{\mathrm{S5}} \bot.$$

These are **global dual ILs**: there is some set of formulas $\mathrm{J}(x)$ such that

$$\Gamma \vdash_{\mathrm{L}} \varphi \iff \Gamma, \mathrm{J}(\varphi) \vdash_{\mathrm{L}} \bot.$$

Local dual inconsistency lemmas (dual ILs)

The infinitary Łukasiewicz logic $Ł_\infty$ satisfies the equivalence

$$\Gamma \vdash_{Ł_\infty} \varphi \iff \Gamma, \neg(\varphi^n) \vdash_{Ł_\infty} \bot \text{ for each } n \in \omega.$$

This is a **local dual IL**: there is a family of sets of formulas $\Psi(x)$ such that

$$\Gamma \vdash_{\mathrm{L}} \varphi \iff \Gamma, \mathrm{J}(\varphi) \vdash_{\mathrm{L}} \bot \text{ for each } \mathrm{J}(x) \in \Psi(x).$$

Note the universal quantification (due to the set occuring on the left).

Fact. If L enjoys a local (global) IL w.r.t. a system of families Φ_n and a dual local IL, then it enjoys the dual local (global) IL also w.r.t. Φ_1.

The dual IL and the classical DDT

Fact. Each logic enjoys both a local (global) IL and a dual IL of any form in fact enjoys the local (global) DDT. Raftery calls this the **classical** DDT.

Example. We illustrate this for $Ł_{k+1}$ and $Ł_\infty$:

$$\begin{aligned} \Gamma, \varphi \vdash_{Ł_{k+1}} \psi &\iff \Gamma, \varphi, \neg(\psi^k) \vdash_{Ł_{k+1}} \bot \\ &\iff \Gamma \vdash_{Ł_{k+1}} \neg(\varphi \wedge \neg\psi^k)^k. \end{aligned}$$

$$\begin{aligned} \Gamma, \varphi \vdash_{Ł_\infty} \psi &\iff \Gamma, \varphi, \neg(\psi^n) \vdash_{Ł_\infty} \bot \text{ for each } n \in \omega \\ &\iff \text{for each } n \text{ there is } k \text{ such that } \Gamma \vdash_{Ł_\infty} \neg(\varphi \wedge \neg\psi^n)^k \\ &\iff \text{there is } f\colon \omega \to \omega \text{ s.t. } \Gamma \vdash_{Ł_\infty} \neg(\varphi \wedge \neg\psi^n)^{f(n)} \text{ for each } n \\ &\iff \Gamma \vdash_{Ł_\infty} \{\neg(\varphi \wedge \neg\psi^n)^{f(n)} \mid n \in \omega\} \text{ for some } f\colon \omega \to \omega. \end{aligned}$$

The law of the excluded middle (LEM)

The dual IL can be rephrased as a law of the excluded middle (LEM).

Classical logic satisfies the global LEM in the form

$$\Gamma, \varphi \vdash_{\mathrm{CL}} \psi \text{ and } \Gamma, \neg\varphi \vdash_{\mathrm{CL}} \psi \implies \Gamma \vdash_{\mathrm{CL}} \psi.$$

Global modal logic S5 satisfies the global LEM in the form

$$\Gamma, \varphi \vdash_{\mathrm{S5}} \psi \text{ and } \Gamma, \neg\Box\varphi \vdash_{\mathrm{S5}} \psi \implies \Gamma \vdash_{\mathrm{CL}} \psi.$$

The infinitary Łukasiewicz logic $Ł_\infty$ satisfies the local LEM in the form

$$\Gamma, \varphi \vdash_{Ł_\infty} \psi \text{ and } \Gamma, \neg(\varphi^n) \vdash_{Ł_\infty} \psi \text{ for each } n \implies \Gamma \vdash_{Ł_\infty} \psi.$$

A logic L has the **local LEM** w.r.t. a family $\Psi(x)$ if $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ and $\Gamma, \mathrm{J}(\varphi) \vdash_{\mathrm{L}} \psi$ for each $\mathrm{J}(x) \in \Psi(x)$ imply $\Gamma \vdash_{\mathrm{L}} \psi$, and moreover $\varphi, \mathrm{J}(\varphi) \vdash_{\mathrm{L}} \bot$ for all $\mathrm{J} \in \Psi$.

The proof by cases property (PCP)

Intuitionistic logic, as well as $\mathrm{FL_{ew}}$, satisfies the equivalence

$$\Gamma, \varphi_1 \vdash_{\mathrm{IL}} \psi \;\&\; \Gamma, \varphi_2 \vdash_{\mathrm{IL}} \psi \iff \Gamma, \varphi_1 \vee \varphi_2 \vdash_{\mathrm{IL}} \psi.$$

The global modal logic S4 satisfies the equivalence

$$\Gamma, \varphi_1 \vdash_{\mathrm{S4}} \psi \;\&\; \Gamma, \varphi_2 \vdash_{\mathrm{S4}} \psi \iff \Gamma, \Box\varphi_1 \vee \Box\varphi_2 \vdash_{\mathrm{S4}} \psi.$$

The global modal logic K satisfies the equivalence

$$\Gamma, \varphi_1 \vdash_{\mathrm{K}} \psi \;\&\; \Gamma, \varphi_2 \vdash_{\mathrm{K}} \psi \iff \Gamma, \Box_n\varphi_1 \vee \Box_n\varphi_2 \vdash_{\mathrm{K}} \psi \text{ for some } n \in \omega.$$

These are global and local instances of the **proof by cases property** (PCP).

The LEM and the PCP

The global LEM can be stated in axiomatic form, given a global PCP or DDT.

Example. S5 enjoys the global LEM:

$$\Gamma, \varphi \vdash_{S5} \psi \ \& \ \Gamma, \neg\Box\varphi \vdash_{S5} \psi \implies \Gamma \vdash_{S5} \psi.$$

By the global PCP this is equivalent to

$$\Gamma, \Box\varphi \vee \neg\Box\varphi \vdash_{S5} \psi \implies \Gamma \vdash_{S5} \psi.$$

Taking $\Gamma := \emptyset$, $\varphi := x$, and $\psi := \Box x \vee \Box\neg\Box x$ yields: $\emptyset \vdash_{S5} \Box x \vee \Box\neg\Box x$.

Similarly, applying the global DDT yields

$$\Gamma \vdash_{S5} \Box\varphi \to \psi \ \& \ \Gamma \vdash_{S5} \neg\Box\varphi \to \psi \implies \Gamma \vdash_{S5} \psi.$$

By a similar argument, we obtain: $\emptyset \vdash_{S5} (\Box x \to y) \to ((\neg\Box x \to y) \to y)$.

The dual IL and the LEM

Fact. The local dual IL and the LEM w.r.t. the same family are equivalent.

Proof. Assume for the sake of simplicity that the family is $\neg_n x$ for $n \in \omega$.

(LEM $\implies$ dual IL) If $\Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \bot$ for each n, then in particular $\Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \varphi$ for each n. Moreover, $\Gamma, \varphi \vdash_{\mathrm{L}} \varphi$, so $\Gamma \vdash_{\mathrm{L}} \varphi$ by the LEM.

(Dual IL $\implies$ LEM) If $\Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \psi$ for each n, then $\Gamma, \neg_n \varphi, \neg_m \psi \vdash_{\mathrm{L}} \bot$ for each m, n by the dual IL, so $\Gamma, \neg_m \psi \vdash_{\mathrm{L}} \varphi$ for each m again by the dual IL.

If $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$, then $\Gamma, \neg_m \psi \vdash_{\mathrm{L}} \psi$ for each m by Cut. But $\psi, \neg_m \psi \vdash_{\mathrm{L}} \bot$, so $\Gamma, \neg_m \psi \vdash_{\mathrm{L}} \bot$ for each $m \in \omega$ by Cut and $\Gamma \vdash_{\mathrm{L}} \psi$ by the dual IL.

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Semisimplicity and the LEM

Fact. Each compact logic with the dual local IL is semisimple.

Proof. If $\Gamma \nvdash_{\mathrm{L}} \varphi$, then $\Gamma, \neg_n \varphi \nvdash_{\mathrm{L}} \bot$ for some $n \in \omega$. By compactness $\Gamma, \neg_n \varphi$ extends to a simple L-theory Δ. Then $\Delta \vdash_{\mathrm{L}} \neg_n \varphi$, therefore $\Delta \nvdash_{\mathrm{L}} \varphi$.

Fact. Each semisimple logic with the local IL has the dual local IL.

Proof. If $\Gamma \nvdash_{\mathrm{L}} \varphi$, then $\Gamma \subseteq \Delta \nvdash_{\mathrm{L}} \varphi$ for some simple L-theory Δ. Then $\Delta, \varphi \vdash_{\mathrm{L}} \bot$ by the simplicity of Δ and $\Delta \vdash_{\mathrm{L}} \neg_n \varphi$ for some $n \in \omega$ by the local IL. But $\Gamma \subseteq \Delta$, so $\Gamma, \neg_n \varphi \nvdash_{\mathrm{L}} \bot$ for some $n \in \omega$.

Semisimplicity and the LEM

Theorem. The following are equivalent for each compact logic with the local (global) IL w.r.t. some family Φ:

- syntactic semisimplicity,
- the dual local (global) IL,
- the local (global) LEM,
- the dual local (global) IL w.r.t. Φ_1,
- the local (global) LEM w.r.t. Φ.

"Syntactic semisimplicity" is not the algebraist's notion of semisimplicity. However, we can obtain the algebraist's notion of semisimplicity (semantic semisimplicity) with some technical but routine extra work.

Semantic semisimplicity and the LEM

Theorem. The following are equivalent* for each compact logic with the local (global) IL w.r.t. some family Φ:

- syntactic semisimplicity,
- the dual local (global) IL,
- the local (global) LEM,
- semantic semisimplicity,
- the semantic dual local (global) IL w.r.t. Φ,
- the semantic local (global) LEM w.r.t. Φ.

* Terms and conditions apply: each set in Φ_1 is finite and $|\Phi_n| \leq |\mathrm{Var}\,\mathrm{L}|$ for each n.

In the global case this theorem is due to Raftery (his assumptions are somewhat stronger than required).

The semantic dual local IL or LEM is the dual local IL or LEM for arbitrary algebras in place of **Fm**: $\Gamma \vdash_{\mathrm{L}} \varphi$ is replaced by $X \vdash^{\mathbf{A}}_{\mathrm{L}} a$.

Recall that $X \vdash^{\mathbf{A}}_{\mathrm{L}} a$ means: a lies in the L-filter generated by X on **A**.

Application: semisimple extensions of $\mathrm{FL}_{\mathrm{ew}}$

Theorem. An axiomatic extension of $\mathrm{FL}_{\mathrm{ew}}$ is semisimple if and only if it validates $x \vee \neg_n x$ for some $n \in \omega$, where $\neg_n := \neg(x^n)$.

Proof. Each axiomatic extension L of $\mathrm{FL}_{\mathrm{ew}}$ inherits the local IL of $\mathrm{FL}_{\mathrm{ew}}$. The semisimplicity of L is thus equivalent to the following local LEM:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad (\forall n \in \omega)\, \Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \psi}{\Gamma \vdash_{\mathrm{L}} \psi}$$

The LEM holds if $\varphi \vee \neg_n \varphi$ is a theorem for some $n \in \omega$ by the PCP:

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \text{ and } \Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \psi \implies \Gamma, \varphi \vee \neg_n \varphi \vdash_{\mathrm{L}} \psi \implies \Gamma \vdash_{\mathrm{L}} \psi.$$

Conversely, if the LEM holds, we choose suitable Γ, φ, ψ:

$$\varphi := x, \qquad \psi := y, \qquad \Gamma := \bigcup_{n \in \omega} \{\neg_n x \to y\} \cup \{x \to y\}.$$

The LEM yields $\Gamma \vdash_{\mathrm{L}} y$. By finitarity there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathrm{L}} y$. That is, $x \to y, \neg_n x \to y \vdash_{\mathrm{L}} y$. Substituting $x \vee \neg_n x$ for y yields $\emptyset \vdash_{\mathrm{L}} x \vee \neg_n x$.

Other results in the vicinity

We can also use this method to describe the semisimple varieties of BAOs. The proof is slightly more complicated, since we only have the local PCP.

Two similar results lie beyond the immediate reach of our method. We hope to be able to extend it to cover these.

Theorem (Kowalski & Ferreirim, unpublished). A variety K of integral commutative residuated lattices is semisimple if and only if K satisfies the equation $x \vee (x^n \to y) = 1$ for some $n \in \omega$.

Obstacle. No negation. The total theory Fm is not compact (fin. generated).

Theorem (Werner & Wille, 1970). A variety K of commutative rings is semisimple if and only if K satisfies $x = x^n$ for some $n \geq 2$.

Obstacle. A parametrized global IL rather than a local one.

Glivenko theorems and semisimple companions

Our main theorem also has applications to Glivenko theorems.

The Glivenko theorem for intuitionistic logic states that:

$$\Gamma \vdash_{\mathrm{CL}} \varphi \iff \Gamma \vdash_{\mathrm{IL}} \neg\neg\varphi.$$

The Glivenko theorem for S4 states that:

$$\Gamma \vdash_{\mathrm{S5}} \varphi \iff \Gamma \vdash_{\mathrm{S4}} \neg\Box\neg\Box\varphi.$$

It is no coincidence that $\neg\varphi$ occurs in the inconsistency lemma for IL, while $\neg\Box\varphi$ occurs in the inconsistency lemma for S4.

Semisimple companions

Definition. The semisimple companion $\alpha(\mathrm{L})$ of a logic L is the logic determined by the semisimple models of L.

Example. The semisimple companion of IL is CL, of S4 is S5.

Theorem. If a logic L enjoys the **global** IL, then a model of L is semisimple if and only if it is a model of the semisimple companion $\alpha(\mathrm{L})$.

Theorem. If a logic L enjoys the **global** IL and either the PCP or the DDT, then $\alpha(\mathrm{L})$ is the extension of L by the axiomatic form of the LEM.

Example. The LEM has the axiomatic form $\Box x \vee \Box\neg\Box x$ given the PCP and the IL of S4. Therefore the semisimple models of S4 are precisely the models of S5, i.e. the semisimple S4-algebras are precisely the S5-algebras.

Glivenko theorems

The semisimple companion $\alpha(\mathrm{L})$ of a compact logic L is equivalently the largest extension of L with the same antitheorems as L (same $\Gamma \vdash_{\mathrm{L}} \bot$).

Theorem. In a compact logic with a local IL, $\Gamma \vdash_{\alpha(\mathrm{L})} \varphi$ if and only if

$$\varphi, \Delta \vdash_{\mathrm{L}} \bot \implies \Gamma, \Delta \vdash_{\mathrm{L}} \bot \text{ for each } \Delta \subseteq \mathrm{Fm}.$$

The global inconsistency lemma now yields:

$$\Delta \vdash_{\mathrm{L}} \neg\varphi \implies \Gamma, \Delta \vdash_{\mathrm{L}} \bot \text{ for each } \Delta \subseteq \mathrm{Fm}$$

This is equivalent to $\Gamma, \neg\varphi \vdash_{\mathrm{L}} \bot$, and applying the inconsistency lemma again yields $\Gamma \vdash_{\mathrm{L}} \neg\neg\varphi$. The theorem can therefore be rephrased as:

Theorem. There is a Glivenko connection between L and $\alpha(\mathrm{L})$ whenever L is compact with the global IL. (The local IL yields a more complicated link.)

Glivenko theorems: examples

The Glivenko theorems for intuitionistic logic and S4 are special cases:

$$\Gamma \vdash_{CL} \varphi \iff \Gamma \vdash_{IL} \neg\neg\varphi,$$
$$\Gamma \vdash_{S5} \varphi \iff \Gamma \vdash_{S4} \neg\Box\neg\Box\varphi.$$

The semisimple companion of Hájek's basic fuzzy logic BL is the infinitary Łukasiewicz logic $Ł_\infty$ (the logic of semisimple MV-algebras), so

$$\Gamma \vdash_{Ł_\infty} \varphi \iff \Gamma \vdash_{BL} \neg_{f(n)}\neg_n\varphi \text{ for some } f\colon \omega \to \omega.$$

If Γ is finite, there is a global Glivenko theorem due to Cignoli & Torrens:

$$\Gamma \vdash_{Ł_\infty} \varphi \iff \Gamma \vdash_{Ł} \varphi \iff \Gamma \vdash_{BL} \neg\neg\varphi.$$

This is beyond the scope of our method. Conversely, the Glivenko theorem for S4 lies beyond the immediate scope of the method of Cignoli & Torrens.

Glivenko theorems: other approaches

The existing approaches to Glivenko theorems due to Cignoli & Torrens and Galatos & Ono are quite different and incomparable in strength:

- both consider Glivenko theorems relative to a certain term,
- in particular, Cignoli & Torrens assume that the double negation is a homomorphism onto an algebra of regular elements,
- neither requires a connection between negation and inconsistency,
- both only consider global Glivenko theorems.

Fundamentally, our notion of negation is related to antitheorems (although there are some tricks we can use to get around this).

Conclusion

Inconsistency lemmas and dual inconsistency lemmas capture important properties of negation, analogous to the deduction theorem.

The dual IL, or equivalently the LEM, is equivalent to semisimplicity, under some mild conditions (including compactness and the IL).

This can be exploited to provide simple proof of theorems of the form: a subvariety of a given variety is semisimple if and only if ...

It also enable us to draw Glivenko-like connections between logics and their semisimple companions.

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Thank you for your attention!