# Surjective Cardinals and dually Dedekind finite sets


[Image from xkcd.com]

Online Logic Seminar
March $5^{\text{th}}$, 2026

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# Motivation

Usually, sizes of (infinite) sets are compared in the Cantorian sense of injections—where two sets are said to have the same size (cardinality) if they inject into each other.

In this talk, we will describe the dual notion of size in terms of surjections—where two sets have the same “surjective cardinality” if there are (partial) surjections both ways. We will then compare these notions in the absence of **AC**.

# Background: Injective Ordering

Given sets $X$ and $Y$, we write $|X| \leqslant |Y|$ to mean that there is an injection $X \to Y$, and $|X| = |Y|$ to mean that there is a bijection $X \to Y$.

**Proposition** **(ZF)**

For any sets $X, Y, Z$:

- $|X| \leqslant |X|$, and
- If $|X| \leqslant |Y|$ and $|Y| \leqslant |Z|$, then $|X| \leqslant |Z|$.

**Theorem [Cantor-$\cdots$-Schröder-Bernstein (CSB)]** **(ZF)**

Given sets $X$ and $Y$: if $|X| \leqslant |Y|$ and $|Y| \leqslant |X|$, then $|X| = |Y|$.

In fact, if $f : X \to Y$ and $g : Y \to X$ are the injections, then the bijection $h : X \to Y$ can be constructed so that $h \subseteq f \cup g^{-1}$.

# Background: Injective Ordering

**Theorem** **(ZF)**

Assuming **AC**, we additionally have:

1. For any $X$ and $Y$, $|X| \leqslant |Y|$ or $|Y| \leqslant |X|$.
2. Given a non-empty set $X$ of cardinals, there is a $\kappa$ in $X$ such that for any $\lambda \in X$, if $\lambda \leqslant \kappa$ then $\lambda = \kappa$.

**Theorem [Hartogs, 1915]** **(ZF)**

Assuming (1), **AC** follows. Hence, **AC** is equivalent to (1) over **ZF**.

**Question** **(ZF)**

Is **AC** equivalent to (2) over **ZF**? (We will return to this briefly.)

**Note:** **AC** is equivalent to the well-ordering principle [Zermelo, 1904]. So, in the absence of **AC**, some sets may admit no well-ordering—meaning that we cannot use initial ordinals as cardinals. The manner in which cardinals are defined as sets, however, will not matter for our purposes. (For example, use Scott's trick.)

# Background: Surjective Ordering

Given sets $X$ and $Y$, we write $|X| \leqslant^* |Y|$ to mean that there is a surjection $Y \to X$ or $X = \varnothing$. (Equivalently, $|X| \leqslant^* |Y|$ means that there is a partial surjection $Y \to X$.)

**Proposition** **(ZF)**

Given sets $X, Y, Z$:

- $|X| \leqslant^* |X|$, and
- If $|X| \leqslant^* |Y|$ and $|Y| \leqslant^* |Z|$, then $|X| \leqslant^* |Z|$.

For now, we write $|X| =^* |Y|$ to mean that $|X| \leqslant^* |Y|$ and $|Y| \leqslant^* |X|$.

# Background: Surjective Ordering

**Theorem** **(ZF)**

Assuming **AC**, we additionally have:

1. For any $X$ and $Y$, $|X| \leqslant^* |Y|$ or $|Y| \leqslant^* |X|$.
2. Given a non-empty set $X$ of cardinals, there is a $\kappa$ in $X$ such that for any $\lambda \in X$, if $\lambda \leqslant^* \kappa$ then $\lambda =^* \kappa$.

**Theorem [Sierpiński, 1948]** **(ZF)**

Assuming (1), **AC** follows. Hence, **AC** is equivalent to (1) over **ZF**.

**Question** **(ZF)**

Is **AC** equivalent to (2) over **ZF**? (We will return to this briefly.)

# Properties of $\leqslant^*$

**Proposition** **(ZF)**

$|X| \leqslant |Y|$ implies $|X| \leqslant^* |Y|$. In fact, if $f : X \to Y$ is the injecton, then the surjection $g : Y \to X$ can be constructed to be a left-inverse for $f$.

What about the converse?

Partition Principle (**PP**): $|X| \leqslant^* |Y|$ implies $|X| \leqslant |Y|$.

**PP** is so named since it is equivalent to: "for any set $X$ and equivalence relation $\sim$ on $X$, $|X/\sim| \leqslant |X|$." What can be said about **PP**?

**PP** is a consequence of **AC**; **PP** is not provable in **ZF**; and whether or not **PP** implies **AC** is one of the most frustrating open problems in set theory.

The natural strengthening of **PP** is known to be equivalent to **AC**:

**Theorem [folklore]** **(ZF)**

**AC** is equivalent to the assertion that every surjection has a right inverse.

# Properties of $\leqslant^*$

Is there a surjective analog to **CSB**?

**CSB**$^*$: if $|X| \leqslant^* |Y|$ and $|Y| \leqslant^* |X|$, then $|X| = |Y|$.

What can be said about **CSB**$^*$? **CSB**$^*$ is a consequence of **PP**, hence of **AC**. Is it provable over **ZF**?

First, we recall some definitions.

# Properties of $\leqslant^*$

**Definition**

A set $X$ is *finite* if there is $n \in \mathbb{N}$ such that $|X| = n$; *infinite* otherwise.

**Definition [Dedekind, 1888]**

A set $X$ is said to be *Dedekind finite* if it satisfies any of the following equivalent conditions:

1. Every injection $X \to X$ is a surjection
2. For any $Y \subsetneq X, |Y| < |X|$
3. There is no injection $\mathbb{N} \to X$

$X$ is said to be *Dedekind-infinite* otherwise.

Assuming **AC**, these definitions of finiteness agree. Over **ZF**, every finite set is Dedekind finite, but it is consistent that there are infinite Dedekind finite sets!

# Properties of $\leqslant^*$

**Theorem [Tarski, 1965]** **(ZF)**

If there is an infinite Dedekind finite set, then there are sets $X$ and $Y$ such that $|X| \leqslant |Y|$ (hence, $|X| \leqslant^* |Y|$), $|Y| \leqslant^* |X|$, but $|X| \neq |Y|$.

So $\mathbf{CSB}^*$ is not provable over **ZF**. As is the case with **PP**, whether or not $\mathbf{CSB}^*$ implies **AC** is open.

The natural strengthening of $\mathbf{CSB}^*$ is known to be equivalent to **AC**:

**Theorem [Banaschewski–Moore, 1990]** **(ZF)**

**AC** is equivalent to the assertion that if $f : X \to Y$ and $g : Y \to X$ are surjections, then there is a bijection $h : X \to Y$ such that $h \subseteq f \cup g^{-1}$.

# Well-foundedness

The next few slides are based on [Blass-K., 2024]. In **ZF**, there is an ambiguity in the notion of well-foundedness of cardinals:

$\mathrm{WF}^i_{b1}$ Given a non-empty set $X$ of cardinals, there is a $\kappa \in X$ such that for any $\lambda \in X$, if $\lambda \leqslant \kappa$ then $\kappa = \lambda$.

$\mathrm{WF}^i_{b2}$ If $(\kappa_i)$ is an infinite sequence of cardinals such that $\kappa_0 \geqslant \kappa_1 \geqslant \kappa_2 \geqslant \cdots$, then there is some $n \in \mathbb{N}$ such that $\kappa_n = \kappa_{n+1}$.

$\mathrm{WF}^i_{b3}$ If $(A_i)$ is an infinite sequence of sets such that $|A_0| \geqslant |A_1| \geqslant |A_2| \geqslant \cdots$, then there is some $n \in \mathbb{N}$ such that $|A_n| = |A_{n+1}|$.

$\mathrm{WF}^i_{b4}$ Given $(A_i)$, an infinite sequence of sets, and $(f_i)$, a corresponding sequence of injections $f_i : A_{i+1} \to A_i$, there is some $n \in \mathbb{N}$ such that $|A_n| = |A_{n+1}|$.

**Remark**: 1, i.e., every nonempty subset has a minimal element, is what is generally meant by an ordering being *well-founded*.

**Remark'**: In the literature, 2, 3, and sometimes both simultaneously have been referred to as "NDS". 2 and 3 could a priori be different: In **ZF**, 2 can't be deduced from 3 by just "choosing a representative" for each cardinal [Pincus, 1974].

# Well-foundedness

**Proposition** (ZF)

In **ZF**, we have:

$$\mathrm{WF}^i_{b1} \to \mathrm{WF}^i_{b2} \to \mathrm{WF}^i_{b3} \to \mathrm{WF}^i_{b4}.$$

"Sufficient conditions" for these to collapse:

**Proposition** (ZF + CC)

Assuming **CC**,

$$\mathrm{WF}^i_{b2} \longleftrightarrow \mathrm{WF}^i_{b3} \longleftrightarrow \mathrm{WF}^i_{b4}.$$

**Proposition** (ZF + DC)

Via [Spector, 1980], assuming **DC**, we have $\mathrm{WF}^i_{b_1} \longleftrightarrow \mathrm{WF}^i_{b2}$, hence

$$\mathrm{WF}^i_{b1} \longleftrightarrow \mathrm{WF}^i_{b2} \longleftrightarrow \mathrm{WF}^i_{b3} \longleftrightarrow \mathrm{WF}^i_{b4}.$$

# Even more well-foundedness principles

Why only restrict well-foundedness to the injective ordering?

We define $\mathrm{WF}^{x}_{yk}$, where the subscript $y \in \{i, s, b\}$ (resp. the superscript $x \in \{i, s\}$) refers to the injective, surjective, or bijective comparison in the conclusion (resp. the hypothesis) of the corresponding statement, and the numerical subscripts $k \in \{1, 2, 3, 4\}$ are as before.

For example:

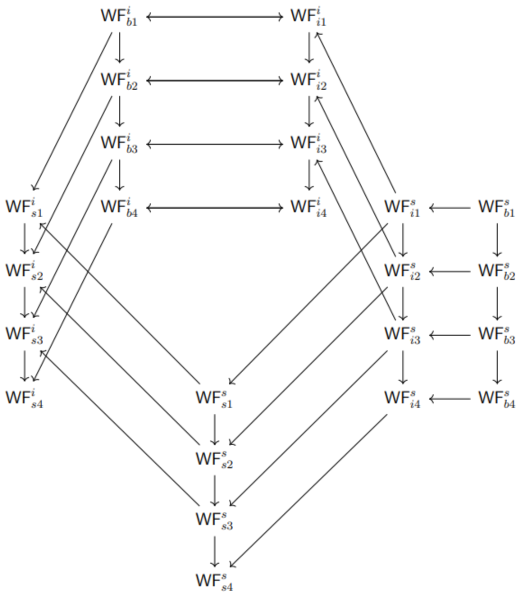
$\mathrm{WF}^{s}_{b1}$ Given a non-empty set $X$ of cardinals, there is a $\kappa \in X$ such that for any $\lambda \in X$, if $\lambda \leqslant^* \kappa$ then $\kappa = \lambda$.

$\mathrm{WF}^{s}_{s2}$ If $(\kappa_i)$ is an infinite sequence of cardinals such that $\kappa_0 \geqslant^* \kappa_1 \geqslant^* \kappa_2 \geqslant^* \cdots$, then there is some $n \in \mathbb{N}$ such that $\kappa_n \leqslant^* \kappa_{n+1}$.

$\mathrm{WF}^{s}_{b4}$ Given $(A_i)$, an infinite sequence of sets, and $(f_i)$, a corresponding sequence of surjections $f_i : A_i \to A_{i+1}$, there is some $n \in \mathbb{N}$ such that $|A_n| = |A_{n+1}|$.

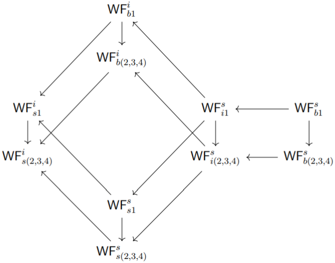
**Remark**: Even $\mathrm{WF}^{s}_{b4}$ is a strong principle, implying $\mathbf{CSB}^*$, hence **WPP**, $\mathbf{AC}^{\mathrm{WO}}$, and **DC**.

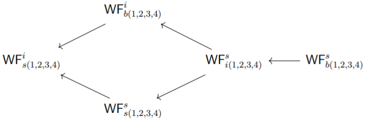
# ZF implications



# Implications additional assumptions

It is also interesting to see how this figure collapses under additional assumptions. For instance, assuming **CC** and **DC** is analogous to before:



# Implications additional assumptions

Assuming **CSB**$^*$:

$$
\begin{array}{ccccc}
\mathsf{WF}^i_{s1} & \longleftarrow & \mathsf{WF}^i_{b1} & \longleftarrow & \mathsf{WF}^s_{b1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{WF}^i_{s2} & \longleftarrow & \mathsf{WF}^i_{b2} & \longleftarrow & \mathsf{WF}^s_{b2} \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{WF}^i_{s3} & \longleftarrow & \mathsf{WF}^i_{b3} & \longleftarrow & \mathsf{WF}^s_{b3} \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{WF}^i_{s4} & \longleftarrow & \mathsf{WF}^i_{b4} & & \mathsf{WF}^s_{b4}
\end{array}
$$

Assuming **PP**, all our notions of well-foundedness are pairwise equivalent:

$$\mathsf{WF}$$

# Dual Dedekind finiteness?

$X$ is Dedekind finite (**DF**) if it satisfies any of the following equivalent conditions:

1. Every injection $X \to X$ is a surjection
2. $X$ does not inject into a proper subset (as in, for any $Y \subsetneq X, |Y| < |X|$)
3. There is no injection $\mathbb{N} \to X$

$X$ is said to be *dually Dedekind finite* (**DF**$^*$) if it satisfies these equivalent conditions:

$1^*$. Every surjection $X \to X$ is an injection

$2^*$. $X$ does not surject onto a proper superset (or, equivalently, onto $X \cup \{X\}$)

$X$ is said to be *power Dedekind finite* (**PDF**) if

$3^*$ There is no surjection $X \to \mathbb{N}$ (equivalently, there is no injection $\mathbb{N} \to \mathcal{P}(X)$)

Assuming **AC**, these definitions of finiteness agree. Over **ZF**,

**Theorem [Truss, 1974]** **ZF**

The following (consistently irreversible) implications hold: **PDF** $\implies$ **DF**$^*$ $\implies$ **DF**.

# Dually Dedekind Finite sets

It has been known that if $A$ and $B$ are **DF** (or **PDF**), then so is $A \times B$ and $A \sqcup B$ [Truss, 1974]. What about **DF**$^*$?

**Theorem [Mao–Shen, 2025]**

It is consistent with **ZF** that there exists a family $(A_n)_{n\in\omega}$ of sets such that, for all $n \in \omega$, $A_n^n$ is dually Dedekind finite whereas $A_n^{n+1}$ is dually Dedekind infinite.

# How bad can the situation be?

Finally, if you still had some hope that the ordering will be somewhat nice in the absence of **AC**, we state a classical theorem and a recent analog that shed light on how bad the situation can really be.

**Theorem [Jech, 1966; Takahashi 1967]**

Let $(P, \preceq)$ be a partial pre-order. Then there is a model $\mathcal{U}$ of **ZF**, and sets $(A_p)_{p \in P}$ in $\mathcal{U}$ such that

$$p \preceq q \iff |A_p| \leqslant |A_q|.$$

**Theorem [Shen–Zhou, 2025]**

Let $(P, \preceq, \preceq^*)$ be a set equipped with two partial pre-orders such that $\preceq^* \subseteq \preceq$. Then there is a model $\mathcal{U}$ of **ZF**, and sets $(A_p)_{p \in P}$ in $\mathcal{U}$ such that

$$p \preceq q \iff |A_p| \leqslant |A_q| \quad \text{and} \quad p \preceq^* q \iff |A_p| \leqslant^* |A_q|$$

These are both proven with permutation models (which are really models of **ZFA**), and the results are transferred using the Jech-Sochor Embedding Theorem.

Given an infinite cardinal $\kappa$ in the **ZFC** model where $P$ lives, the construction can be modified so that so that $\mathbf{DC}_\kappa$ (dependent choice of length $\kappa$) holds in $\mathcal{U}$.

# Open Questions

In [Shen–Zhou, 2025], the authors ask:

1. Is there a model $\mathcal{U}$ in which $(P, \preceq, \preceq^*)$ can be embedded so that each $A_p$ is Dedekind finite? Power Dedekind finite?

The question of whether or not cardinal well-foundedness implies **AC** breaks up into a long list of questions with the extremes being:

2. $\mathsf{WF}^s_{b1} \implies$ **AC**? $\mathsf{WF}^i_{s4} \implies$ **AC**? $\mathsf{WF}^s_{s4} \implies$ **AC**?

The **ZF** figure also suggests the following questions:

3. Are there other implications between the WF principles? Are any implications provably irreversible?

Note that if there were principles $\mathsf{WF}_a$ and $\mathsf{WF}_b$ with $\mathsf{WF}_a \implies \mathsf{WF}_b \not\Longrightarrow \mathsf{WF}_a$, then $\mathsf{WF}_b \not\Longrightarrow$ **AC**.

Since **DC** causes the figure to collapse vertically and **PP** causes said figure to collapse completely, one may ask

4. Is there a choice principle that causes a horizontal collapse but not a vertical one?

Such a principle would have to be weaker than **PP** and incomparable with **DC**.

In [Harrison-Trainor–K., 2025], Harrison-Trainor and I gave a complete axiomatization of choiceless cardinality comparison. In [Jin–Shen, 2025], the authors ask:

5. What is the corresponding complete axiomatization for surjective cardinals?

Thank you for your attention!

A generating set for all the references:

1. Paul Howard and Jean E. Rubin. *Consequences of the Axiom of Choice*. Mathematical Surveys and Monographs, volume 59, American Mathematical Society, 1998.
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5. Jiaheng Jin and Guozhen Shen. A note on surjective cardinals. *Journal of symbolic logic*, 2025. doi.org/10.1017/jsl.2025.10113.
6. Guozhen Shen and Wenjie Zhou. On ordering of surjective cardinals, 2025. arxiv.org/abs/2507.04028.
7. Ruihuan Mao and Guozhen Shen. A note on dual Dedekind finiteness, *Logic Journal of the IGPL* 33(5), 2025.