Generically Computable Linear Orderings

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Based on joint work with Wesley Calvert, Douglas Cenzer and Valentina Harizanov

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Online Logic Seminar

Summary of the talk

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1. The α-Ramsey property
2. Results for linear orderings
3. Connection with generic and coarse computability

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- Every infinite graph, has a subgraph isomorphic to K_∞ or the edgeless K_∞^c.
- Every infinite graph with finitely many colors has a monochromatic subgraph isomorphic to K_∞.
- Every infinite linear ordering has a sub-ordering isomorphic to ω or ω^*.
- Every infinite partial ordering has a sub-ordering isomorphic to ω, ω^* or an anti-chain.
- Every infinite equivalence structure has an infinite class or infinitely many classes.

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There are *many* ways to abstract Ramsey's theorem; the way below is tied tightly to the "undergraduate" examples enumerated:

Every countable structure in the class $\mathbb{K}$ has a substructure that is among the subclass $\mathbb{J} \subset \mathbb{K}$ of "simple" structures.

It is often natural and useful to ask for more than a substructure. We may need to relax our notion of simple.

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- There was already a natural way to do this: Σ_α-elementary substructures.

Definition: A substructure $\mathcal{A} \subseteq \mathcal{B}$ is Σ_α-elementary if for all $\bar{p} \in \mathcal{A}$ and $\psi \in \Sigma_\alpha$

$$\mathcal{A} \models \psi(\bar{p}) \iff \mathcal{B} \models \psi(\bar{p}).$$

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Note: Σ_α is a set of $\mathcal{L}_{\omega_1,\omega}$ formulas with $\alpha \in \omega_1$. Our results will also hold for computably infinitary formulas or first order formulas at finite levels.

The α-Ramsey property

Definition: We say a class of structures $\mathbb{K}$ has the α-Ramsey property if:

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Foreshadowing: This property of a class of structures is key to understanding when its structures have Σ_α-generically c.e. copies.

Focusing in

Results on linear orderings

The 1-Ramsey property

Theorem: [CCGH] The class of linear orderings has the 1-Ramsey property. In fact, every linear ordering has a Σ_1-elementary substructure isomorphic to ω, ω^*, ζ, $\omega \cdot \omega^*$, $\omega^* \cdot \omega$ or η.

Lemma: For linear orderings $\mathcal{A} \subseteq \mathcal{B}$ is Σ_1-elementary if the first, last, and between predicates are preserved.

Proof Sketch

Claim: Every linear ordering has a Σ_1-elementary substructure isomorphic to ω,ω^*, ζ, $\omega \cdot \omega^*$, $\omega^* \cdot \omega$ or η.

- The proof proceeds by looking at the *1-blocks* of the ordering $\mathcal{B}$, the equivalence classes of the finite-distance $\sim_1$.

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- **Case ζ:** There is a 1-block with isomorphism type ζ.

Proof Sketch (cont.)

Claim: Every linear ordering has a Σ_1-elementary substructure isomorphic to ω,ω^*, ζ, $\omega \cdot \omega^*$, $\omega^* \cdot \omega$ or η.

- **Case $\omega^* \cdot \omega$:** There is a decreasing infinite sequence of 1-blocks with isomorphism type ω.

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- **Case $\omega^* \cdot \omega$:** There is a decreasing infinite sequence of 1-blocks with isomorphism type ω.
- **Case $\omega \cdot \omega^*$:** There is an increasing infinite sequence of 1-blocks with isomorphism type ω^*.
- **Case η:** The ω blocks are well ordered and do not contain the first block, the ω^* blocks are reverse well ordered and do not contain the last block. There are no ζ blocks. Between the first ω block and the last ω^* block less than it there are only finite blocks that must be densely ordered.

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1. The set of *scattered* linear orderings has the 2-Ramsey property.
2. They must be in the ω,ω^*, ζ, $\omega \cdot \omega^*$ or $\omega^* \cdot \omega$ case from the previous theorem.
3. Each of these Σ_1-elementary substructures can be upgraded to a Σ_2-elementary substructure by adding in the first and last 1-block if they exist.
4. This argument is more technical than the previous argument, but goes through with consideration of back and forth games.

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4. $Sh(A)$ does not always have a Σ_2-elementary substructure with a computable copy, e.g. if A is Σ^0_3 immune.
5. This follows from the fact that the set of block sizes of an ordering is Σ^0_3 in the presentation of that ordering.

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Answer: No.

Some Theorems

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Note: This is new behavior for structures in general; these results also apply to structurally complete classes like graphs and groups.

Proof idea for Σ_2

- We construct a continuous embedding $\Phi : \mathbb{LO} \to \mathbb{LO}$ such that $\mathcal{B}$ is well founded if and only if $\Phi(\mathcal{B})$ has an infinite, computable Σ_2-elementary substructure.

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- We define $\Phi(\mathcal{B}) = (Sh(A) + \omega) \cdot \mathcal{B}$ for a Σ^0_3 immune set A.
- If $\mathcal{B}$ is ill-founded it definitely has $\omega \cdot \omega^*$ as a Σ_2-elementary substructure.
- If $\mathcal{B}$ is well founded, a technical argument shows that any substructure must have blocks of size n for infinitely many $n \in A$.

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- There are many technical elements, many actually developed in (G., Rossegger) to analyze Scott sentences of linear orderings, that allow us to bootstrap the argument in the Σ_2 case to these higher cases.

Spilling the beans

Connections to Generic and Coarse Computability

Generic and Coarse Computability

Definition: Let $S \subseteq \mathbb{N}$.

1. The density of S up to n, denoted by $\rho_n(S)$, is given by

$$\rho_n(S) = \frac{|S \cap \{0, 1, 2, \ldots, n\}|}{n+1}.$$

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A set A is said to be *coarsely computable* if and only if there is a *total* computable function ϕ that agrees with χ_A on a set of asymptotic density 1.

Generically and Coarsely Computable structures

- A substructure $\mathcal{B}$ of $\mathcal{A}$, with universe B, is a *computably enumerable (c.e.) substructure* if the set B is c.e., each relation is c.e. and the graph of each function is c.e.

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- $\mathcal{A}$ is Σ_α*-generically c.e.* if there is a dense c.e. set D such that the substructure $\mathcal{D}$ with universe D is a c.e. substructure and also a Σ_α elementary substructure of $\mathcal{A}$.

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- A structure $\mathcal{A}$ is Σ_α*-coarsely c.e.* if there exist a dense set D and a c.e. structure $\mathcal{E}$ such that the substructure $\mathcal{D}$ with universe D is a Σ_α elementary substructure of both $\mathcal{A}$ and $\mathcal{E}$.

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Lemma: For a linear ordering $\mathcal{A}$, $\mathcal{A}$ has a Σ_α-generically c.e. copy if and only if it has a Σ_α elementary substructure that is isomorphic to a computable structure.

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Theorem:[CCGH] The class of linear orderings with a $\Sigma_{\alpha+2}$-generically c.e. copy is $\boldsymbol{\Sigma}^1_1$-complete.

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Theorem:[Nadel] The set of models with computable copies in any $\mathcal{L}_{\omega_1,\omega}$ elementary class is Borel. If the class is an $\mathcal{L}_{c,\omega}$ elementary class, then this set is $\mathbf{\Sigma}^0_{\omega_1^{ck}+3}$ at worst.

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We even see this difference at the lowest interesting level of our hierarchy:

Theorem:[CCGH] The class of successor linear orderings with a Σ_1-generically c.e. copy is $\mathbf{\Sigma}^1_1$-complete.

Thank you!

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- We lack an exact structural characterization for having coarse copies.
- It acts quite differently: $Sh(A)$ always has a Σ_2 coarsely computable copy.
- We can bootstrap this example by considering $\zeta^\beta \cdot Sh(A)$; coarse computability and generic computability do not coincide at any computable level.

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Our canonical example of an ordering with no Σ_2 coarsely computable copy is the η representation of A for A not Σ^3_0:

$$K_A := \eta + a_0 + \eta + a_1 + \eta + \cdots .$$

This takes the place of $Sh(A)$ and generally leads to far more technical arguments.

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This takes the place of $Sh(A)$ and generally leads to far more technical arguments.
The main idea is that any 2-elementary substructure of an η representation is isomorphic to that η representation.
Once you get an η representation of a sufficiently complex A as a sub-ordering you can extract A by listing block sizes starting at a certain point; in general this takes about one more jump to execute than the generic arguments.