A Theorem of Infinity for $\mathbb{Z}$-*Principia Mathematica*

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Motivation

Logicism and Infinity

Principia's $\mathbb{N}$-Types

Negative (Homogeneous) Types and Typical Ambiguity

$\mathbb{Z}$PM's $\mathbb{Z}$-Types

Motivation

Russell's peculiar remark (March 1911)

> *The axiom of infinity is only necessary because of the theory of logical types, which serves to resolve the paradoxes. It is therefore possible that, by modifying the theory of types, the axiom of infinity would become unnecessary.*
> *(Russell, 1911/1992, 53)*

Volume I of *Principia* appeared in December 1910.

Volume II was mostly complete by January 1911.

Earlier (faulty) proofs of Infinity

That there are infinite classes is so evident that it will scarcely be denied. Since, however, it is capable of formal proof, we may as well prove it. A very simple proof is that suggested in the Parmenides, which is as follows. Let it be granted that there is a number 1. Then 1 is, or has Being, and therefore is a Being. But 1 and Being are two; hence there is a number 2; and so on. [...]
A better proof, analogous to the above, is derived from the fact that, if n be any finite number, then the number of numbers from 0 up to and including n is $n+1$, whence it follows that n is not the number of numbers.
(Russell, 1903/1937, §339)

Type constraints

Type constraints block such faulty proofs. No univocal typical meaning can be assigned to 'Being' or to 'n' in the proofs given.

So the usual (faulty) proofs of Infinity (which implicitly involve type-lifting) are blocked by the theory of types.

We do get strictly increasing universal classes in *Principia*

Theorem (schemata.)

$\vdash . (\exists x^t) . x^t \neq x^t$

$\vdash . (\exists x^{(t)}, y^{(t)}) . x^{(t)} \neq y^{(t)}$

$$\vdash . (\exists x^{((t))}, y^{((t))}, z^{((t))}) . x^{((t))} \neq y^{((t))} \wedge y^{((t))} \neq z^{((t))} \wedge x^{((t))} \neq z^{((t))}$$

...

So in *Principia*'s system, the universal classes of individuals are *strictly-increasing*, not just non-empty.

Since the number of individuals is strictly-increasing, we can ascend to some type t where there are at least n-many entities.

Our result

Question: Is there a modification to *Principia*'s theory of types that facilitates a (non-circular, not question-begging) proof of Infinity?

Answer: yes!

Theorem

In the system $\mathbb{Z}PM$ with the new (finitary) axiom schema $\mathbb{Z}{*}107$, we can prove Infinity.

Logicism and Infinity

Logicism

All mathematical truths are logical truths.

Whitehead and Russell aimed to prove Logicism in *Principia*:

> *The purpose of the present subsection [Volume I.A] is to set forth the first stage of the deduction of pure mathematics from its logical foundations.*
> *(Whitehead and Russell, 1925/1957, 94)*

An (alleged) problem with the proof of Logicism

Take Peano postulate #3 (Russell, 1919, 5-6):

- No two numbers have the same successor.

Principia can only prove:

- If Infin ax, then no two numbers have the same successor.

Without a theorem (or an axiom) of infinity, the third Peano postulate is not an *unconditional* theorem. But many feel that this is manifestly an unconditional truth (of pure mathematics).

A partial list

(Bruner, 1943, 6-7)—(Carnap, 1983, 44-45)—(Copi, 1971, 64, 71)—(Hatcher, 1968, 142-143)—(Ramsey, 1926/1978, 229-230)

A dilemma for Logicism

To recover the third Peano postulate unconditionally, what is needed the thesis that there are at least $\aleph_0$ individuals. But a Logicist wants to recover this from logic alone. And it seems not to be a logical truth that there are at least $\aleph_0$ individuals.

Dilemma

Either a non-logical Axiom of Infinity must be added to prove that all mathematical truths are logical ones, or else some mathematical truths remain unproven.

On either horn of this dilemma, the proof of Logicism fails.

Available responses

Some available responses are:

1. Reject Logicism
2. Dissolve the ‘problem’ of proving Infinity
3. Solve the problem of proving Infinity

We will pursue the third option.

Remark

Because proving Logicism was *Principia*’s goal (and is our motivation), we want to only accept new *logical* principles.

***Principia*'s $\mathbb{N}$-Types**

$\mathbb{N}$-Types

Principia's type theory that starts from a lowest type. (Just as $\mathbb{N} = \{0, 1, 2, ...\}$ starts from 0.) We specify the type symbols recursively as follows:

- i is a type symbol.
- If $t_1, ..., t_n$ are type symbols, so is $(t_1, ..., t_n)$.
- Nothing else is a type symbol.

Type symbols look, for example, like this:

- $i - (i) - ((i)) - (i, i) - ((i), i) - (i, i, i) - ((i), i, ((i))) - ...$

Heterogeneous Types

Principia, like ZF set theory, proves Cantor's Power Theorem. The key to this result is *heterogeneously* typed relations (or classes), relations (or classes) between terms of different types (or classes of different places in membership chains).

For example, $y^{((i),(i))}$ is homogeneous because it is a binary relation between terms of the same type, (i).

In contrast, $z^{(((i)),(i))}$ is a heterogeneous relation from terms of type $((i))$ to terms of type (i).

Cantor's Power Theorem

Principia, like ZF set theory, proves Cantor's Power Theorem: for a class α we have $|2^\alpha| > |\alpha|$. In *Principia*'s notation:

Theorem (Cantor's Power Theorem for Classes)

∗117·66. ⊢. $\mathrm{N_0c}$‘Cl‘$\alpha > \mathrm{N_0c}$‘α, that is, for any class α, the (homogeneous) cardinal number of Cl‘α is strictly greater than the (homogeneous) cardinal number of α.

There is also this related result that *Principia* does not prove:

Theorem (Cantor's Power Theorem for Individuals)

⊢. ∼(∃β_{x^t})($\mathrm{V}_{x^{(t)}}$ sm β_{x^t}), that is, no β_{x^t} (including V_{x^t}) is similar to the universal class $\mathrm{V}_{x^{(t)}}$ (Landini, forthcoming).

Cantor's Power Theorem relies on heterogeneous types

Notice in the very statement Cantor's Power Theorem we have a relation between the cardinality of A and its power set. The power set $P(A)$ of a set A is of strictly higher type than A.

So Cantor's Power Theorem relies on a heterogeneously typed relation of cardinal similarity between classes of terms having different types (in *Principia*'s context).

Some facts about *Principia*'s $\mathbb{N}$-type system

Metatheorem

Infin ax is independent of *Principia*'s system.

The (minimal) ℕ-type hierarchy

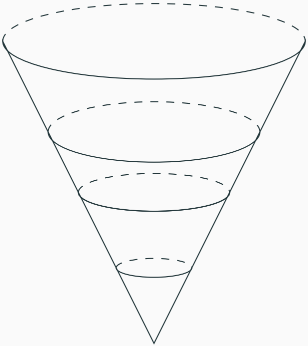

Figure 1: Simple ℕ-Type Hierarchy

Idea

What if there was no lowest type?

What if the universal classes were strictly-increasing *and* there were infinitely-many types below any given type instead of just finitely-many? Would that make a difference?

The $\mathbb{Z}$-type hierarchy

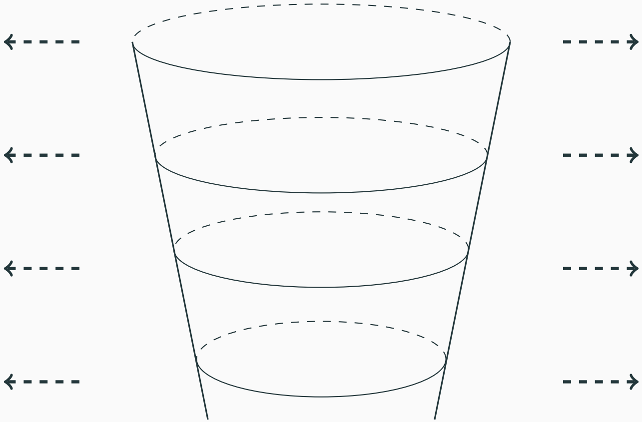

Figure 2: Simple $\mathbb{Z}$-Type Hierarchy

Negative (Homogeneous) Types and Typical Ambiguity

Wang's 1950 "Negative Types"

Hao Wang developed a system of negative types (NT).

Theorem

1. *Comprehension: any (homogeneously typed) formula* ϕ *comprehends a class containing those entities satisfying* ϕ*.*
2. *Extensionality: two classes (of type* t*) are identical if and only if they contain the same members (of type* $(t-1)$*).*

Here a type index t can be any integer.

Wang's result

Question: can we prove Infinity in Negative Types?

Theorem (Wang 1952)

In the system of negative types, Infinity cannot be a theorem.

Specker's 1966 "Typical Ambiguity"

Specker explored the different meanings one might attach to "typical ambiguity" (in some sense letting the type indices go unspecified as one does in axiom schemata).

Meaning 1 If a formula T is a theorem, then T^+ is a theorem (where T^+ results from T by uniformly raising the type indices in T by 1).

Meaning 2 Given a formula T, if the type-lifted formula T^+ is a theorem, then T is a theorem.

Meaning 3 Given a formula T, $T \iff T^+$ is a theorem. (TA)

Specker's Result

The first meaning is in fact a metatheoretic fact about *Principia* and $\mathbb{Z}$PM, and also about ST and NT.

The second meaning is in general false (e.g. $\exists x^{(0)} \exists y^{(0)} (x \neq y)$).

The third meaning is a very strong fact indeed. Specker showed:

Theorem (Specker 1966)

Quine's New Foundations is equiconsistent with TS plus TA.

Some facts about New Foundations

In New Foundations (Forster, 1995, §2.1-§2.2):

1. There is one universal set V.
2. V is a complete Boolean algebra under $\subset$ (so $\mathcal{P}(V) \subset V$).
3. V is non-Cantorian (the set of singletons in V is strictly smaller than V), so Cantor's Power Theorem isn't provable.
4. V is infinite (so New Foundations proves Infinity).
5. The negation of Choice is provable.

Homogeneous ST/NT and Heterogeneous PM/$\mathbb{Z}$PM

The equivalence of ST plus TA with NF shows that ST and NT, being homogeneously typed systems, are in a logical and definite sense of a totally different character from *Principia* and $\mathbb{Z}$PM.

In ST and NT, you cannot relate cardinal numbers of different types in the object language (because instances of comprehension are homogeneously typed). You can still have universal classes in a given type with $\{x^t : x = x\}$.

Motivations for Homogeneity

There are good reasons for insisting on (implicit) homogeneity in NF; if there is to be a universal class, one cannot also have (an unrestricted) Cantor's Power Theorem.

The usual proof of Cantor's Power Theorem in ZF set theory involves assuming there is a surjection f from a set A to its power set $P(A)$. Then consider the class given by

$$\{x \in A : x \notin f(x)\}.$$

From whence we get a contradiction, showing there can be no onto function from a set to its power set.

But in NF this does not comprehend a set (and cannot do so on pain of contradiction).

Relation to *Principia*

Principia has its own features:

1. Infinity is not (dis)provable.
2. Choice is not (dis)provable.
3. There are universal classes of individuals of each type.
4. Heterogeneously typed relations can occur in instances of comprehension (and so Cantor's Power Theorem holds).

Is it possible to prove Infinity without sacrificing these other features?

$\mathbb{Z}$PM's $\mathbb{Z}$-Types

$\mathbb{Z}$-Types

The type symbols of our new (heterogeneous) $\mathbb{Z}$-type theory, $\mathbb{Z}$PM, are as follows:

1. For any integer $z \in \mathbb{Z}$, z is a type symbol.
2. If $z_1, ..., z_n$ are type symbols ($n \geq 2$), then $(z_1, ..., z_n)$ is a type symbol.
3. Nothing else is a type symbol.

Another metatheorem of $\mathbb{Z}$PM

Metatheorem

For any $\mathbb{Z}$-type z, at least n-many individuals of type z exist.

$$(\exists x_1, ..., x_n) \wedge x_1^z \neq x_2^z \wedge ... \wedge x_{n-1}^z \neq x_n^z.$$

Any instance of this metatheorem for a fixed $n \in \mathbb{N}$ is provable in $\mathbb{Z}$PM's object-language as follows:

- Fix n and choose a type z.
- Pick any sufficiently low level of the $\mathbb{Z}$-type hierarchy (say, n levels below z, or $(n + (n \mod 2))/2$ levels below z).
- From that type $z - n$, iterate Cantor's Power Theorem for $\mathbb{Z}$-Individuals until you reach type z.

A problem

We cannot turn this procedure into an object-language proof because the number of levels one must descend the $\mathbb{Z}$-type hierarchy increases with n (so proof by induction will not work).

There is no meaningful object-language relation between the variable indices n in our metatheorem with $\mathbb{Z}$-type indices z.

A solution

We propose a finitary axiom schema that does not beg the question about Infinity, but suffices in the $\mathbb{Z}$PM system to prove Infinity.

$$\mathbb{Z}{*}107. \vdash (n)(n \in \mathrm{NC\,induct} \supset (\alpha)(\alpha_{x^{z+1}} \in n \supset (\exists\beta)(\beta_{x^{z}}\ \mathrm{sm}\ \alpha_{x^{z+1}}))).$$

Our new axiom schema $\mathbb{Z}{*}107$ assures us that any *finite* class of individuals in any $z+1$-type is similar to a class of individuals in the adjacent z-type below.

Accepting $\mathbb{Z}{*}107$

Remark

For a fixed $n \in \mathbb{N}$, any instance of this schema can be proved in the object-language of $\mathbb{Z}$PM and its proof will have a fixed minimal length. Thus all models of $\mathbb{Z}$PM satisfy $\mathbb{Z}{*}107$.

So if $\mathbb{Z}$PM is a logical system, then *Prin$\mathbb{Z}$ipia* plus $\mathbb{Z}{*}107$ is. But is adding $\mathbb{Z}{*}107$ question-begging?

No, because its analogue in *Principia*, $*107$, does not secure a theorem of infinity. So it is not logically equivalent to Infin ax.

A theorem of infinity for *PrinℤZipia Mathematica*

Theorem (Infinity(V_{x^z}))

$\mathbb{Z}$PM + $\mathbb{Z}$∗107. $\vdash$. $(n)(n \in \mathrm{N}_0\mathrm{Cinduct} \supset \mathrm{V}_{x^z} \notin n)$.

Proof.

Assume $(\exists n)(n \in \mathrm{N}_0\mathrm{Cinduct} \,.\, \mathrm{V}_{x^z} \in n)$.

By $\mathbb{Z}$∗107, $(\exists \beta_{x^{z-1}})(\beta_{x^{z-1}}\ \mathrm{sm}\ \mathrm{V}_{x^z})$.

This contradicts Cantor's Theorem for $\mathbb{Z}$-Individuals. □

Takeaways

- A new logical system, $\mathbb{Z}$PM plus $\mathbb{Z}$∗107, proves Infinity.
- One major criticism of the proof of Logicism in *Principia* can be rebutted in $\mathbb{Z}$PM plus $\mathbb{Z}$∗107.

We also validate Russell's insightful remark made 110 years ago:

> *The axiom of infinity is only necessary because of the theory of logical types, which serves to resolve the paradoxes. It is therefore possible that, by modifying the theory of types, the axiom of infinity would become unnecessary.*
> *(Russell, 1911/1992, 53)*

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