Hyperbolicity and model complete fields

Jinhe Ye

University of Oxford

Oct 10, 2024

Throughout the talk, one can safely think of varieties/curves as defined by polynomial equations in projective/affine spaces working in an ambient algebraically closed field, and Zariski open subsets of varieties are varieties.

A brief history of model theory of fields

A majority of conjectures/questions in model theory of fields concern the relationship between model-theoretic “tameness” and algebraic properties.

A brief history of model theory of fields

A majority of conjectures/questions in model theory of fields concern the relationship between model-theoretic "tameness" and algebraic properties.

- Omitting certain combinatorial configurations. (stable/NIP/simple/NSOP, etc.)
- Admitting certain structural descriptions of definable sets. (strongly minimal, *o*-minimal, *C*-minimal, etc.)
- Quantifier elimination/model completeness in a reasonable language. (E.g. ACF, RCF, ACVF, *p*CF, pseudofinite fields.)

A brief history of model theory of fields

A majority of conjectures/questions in model theory of fields concern the relationship between model-theoretic "tameness" and algebraic properties.

- Omitting certain combinatorial configurations. (stable/NIP/simple/NSOP, etc.)
- Admitting certain structural descriptions of definable sets. (strongly minimal, *o*-minimal, *C*-minimal, etc.)
- Quantifier elimination/model completeness in a reasonable language. (E.g. ACF, RCF, ACVF, *p*CF, pseudofinite fields.)

Typically, the first 2 conditions follow from the last condition.

A brief history of model theory of fields

Conjecture

An infinite stable field is separably closed.

Conjecture

An infinite simple field is bounded PAC.

Conjecture

An infinite NIP field is henselian.

A brief history of model theory of fields

Definition 1

We say a field is model complete if its theory in the language of rings is model complete.

A brief history of model theory of fields

Definition 1

We say a field is model complete if its theory in the language of rings is model complete.

- Quantifier elimination implies model completeness. Quantifier elimination in the language of rings implies algebraically closedness.
- $\mathbb{C}, \mathbb{R}, \mathbb{Q}_p$ are all model complete. But the last 2 do not have quantifier elimination in the language of rings.
- Pseudofinite fields are model complete after a slight expansion of the language of rings.

Large fields

Definition 2

A field K is **large** if for any integral positive dimensional variety V over K and V has a smooth K-point, then V has a Zariski dense set of K-points. Equivalently, K is existentially closed in $K((t))$.

Large fields

Definition 2

A field K is **large** if for any integral positive dimensional variety V over K and V has a smooth K-point, then V has a Zariski dense set of K-points. Equivalently, K is existentially closed in $K((t))$.

- $\mathbb{C}, \mathbb{R}, \mathbb{Q}_p$ are large. Any field that can be equipped with a t-Henselian topology is large.
- Pseudofinite fields are large. Actually, any PAC field is large. Recall that PAC says that any geometrically integral variety over K has a K-point.
- In the above axiomatizations of largeness and PAC, it suffices to mention only curves.
- Empirically, all the fields that are “tame” are large.

Large fields and model theory

On the other hand, there are fields that are extremely wild in the sense of model theory that are still large, e.g. $\mathbb{C}((t_1, t_2))$.

On the other hand, there are fields that are extremely wild in the sense of model theory that are still large, e.g. $\mathbb{C}((t_1, t_2))$.
Several longstanding conjectures in model theory of fields have something to do with largeness as well.

Theorem 3 (Johnson-Tran-Walsberg-Y.)

Large stable fields are separably closed.

Theorem 4 (Pillay-Walsberg)

Large simple fields are bounded with projective Galois groups.

Conjecture

All the above conjectures are true assuming largeness.

Definition 5

A field K **has small Galois group** if for each $n \in \mathbb{N}$, there are only finitely many separable extensions of K of degree n.

Large fields and model theory

Definition 5

A field K **has small Galois group** if for each $n \in \mathbb{N}$, there are only finitely many separable extensions of K of degree n.

Conjecture (Junker-Koenigsmann)

Having small Galois groups + infinite imply large.

Question (Macintyre)

If K is model complete, does K have small Galois group?

Question (Junker-Koenigsmann)

If K is model complete and infinite, is K large?

Definition 5

A field K **has small Galois group** if for each $n \in \mathbb{N}$, there are only finitely many separable extensions of K of degree n.

Conjecture (Junker-Koenigsmann)

Having small Galois groups + infinite imply large.

Question (Macintyre)

If K is model complete, does K have small Galois group?

Question (Junker-Koenigsmann)

If K is model complete and infinite, is K large?

We will talk about counterexamples to the questions in joint work with Will Johnson and some generalizations with Michał Szachniewicz.

Definition 6

Let T be a theory.

- T is **inductive** if it is axiomatized by $\forall\exists$-formulas. It is equivalent to a union of a chain of models is a model.
- $M \models T$ is **existentially closed** (e.c.) if any existential formula over M that holds in $N \models T$ extending M holds in M.

Definition 6

Let T be a theory.

- T is **inductive** if it is axiomatized by $\forall\exists$-formulas. It is equivalent to a union of a chain of models is a model.
- $M \models T$ is **existentially closed** (e.c.) if any existential formula over M that holds in $N \models T$ extending M holds in M.

- The theory of fields is inductive. The existentially closed models are exactly the algebraically closed fields.
- Existentially closed models of formally real fields are the real closed fields.
- Existentially closed models of fields with a valuation are the algebraically closed valued fields.

A crash course on model companion

For an inductive theory T, we have the following:

- Any model of T embeds into an e.c. model.
- If the class of existentially closed models is an elementary class, axiomatized by T', then T' is model complete (in the language of T). In this case, we say that T has a model companion, and T' is the model companion of T.

A crash course on model companion

For an inductive theory T, we have the following:

- Any model of T embeds into an e.c. model.
- If the class of existentially closed models is an elementary class, axiomatized by T', then T' is model complete (in the language of T). In this case, we say that T has a model companion, and T' is the model companion of T.

In the previous slide: The model companion of the theory of fields is the theory of algebraically closed fields; The model companion of the theory of formally real fields is the theory of real closed fields.

Curve-excluding fields

Theorem 7 (Fermat)

$x^4 + y^4 = 1$ *has only 4 solutions in* $\mathbb{Q}$*.*

Curve-excluding fields

Theorem 7 (Fermat)

$x^4 + y^4 = 1$ *has only 4 solutions in* $\mathbb{Q}$.

Consider T_0 to be the theory of fields of characteristic 0 and $x^4 + y^4 = 1$ has only 4 solutions.

Curve-excluding fields

Theorem 7 (Fermat)

$x^4 + y^4 = 1$ *has only 4 solutions in* $\mathbb{Q}$*.*

Consider T_0 to be the theory of fields of characteristic 0 and $x^4 + y^4 = 1$ has only 4 solutions.
Note that T_0 is inductive.

Curve-excluding fields

Theorem 7 (Fermat)

$x^4 + y^4 = 1$ *has only 4 solutions in* $\mathbb{Q}$.

Consider T_0 to be the theory of fields of characteristic 0 and $x^4 + y^4 = 1$ has only 4 solutions.
Note that T_0 is inductive.

Theorem 8 (Johnson-Y.)

T_0 *has a model companion* T_1.

Corollary 9

There is a non-large model complete field.

In general, for K_0 with $\mathrm{Char}(K_0) = 0$, take any curve C over K_0 of genus ≥ 2 with only finitely many K_0-points.

Curve-excluding fields

In general, for K_0 with $\mathrm{Char}(K_0) = 0$, take any curve C over K_0 of genus ≥ 2 with only finitely many K_0-points.

Theorem 10 (Johnson-Y.)

The theory of fields K extending K_0 (naming K_0 as constants) with the $C(K) = C(K_0)$ has a model companion.

Curve-excluding fields

In general, for K_0 with $\mathrm{Char}(K_0) = 0$, take any curve C over K_0 of genus ≥ 2 with only finitely many K_0-points.

Theorem 10 (Johnson-Y.)

The theory of fields K extending K_0 (naming K_0 as constants) with the $C(K) = C(K_0)$ has a model companion.

The proof for the general case works exactly the same way when we work with the language of rings expanded by constants for K_0. From now on, we remove the finite number of K_0 rational points of C.

Characterization of existentially closed models

Let T_0 denote the theory of fields K with characteristic 0 and $C(K) = \varnothing$.

Theorem 11 (Johnson-Y.)

$K \models T_0$ is e.c. iff the following two conditions hold:

(1) *For any finite proper extension L/K, $C(L) \neq \varnothing$*

(2) *If X is a geometrically integral variety over K, either there is a non-constant morphism $X \dashrightarrow C$ over K or $X(K) \neq \varnothing$.*

Characterization of existentially closed models

Let T_0 denote the theory of fields K with characteristic 0 and $C(K) = \varnothing$.

Theorem 11 (Johnson-Y.)

$K \models T_0$ is e.c. iff the following two conditions hold:

1. *For any finite proper extension L/K, $C(L) \neq \varnothing$*
2. *If X is a geometrically integral variety over K, either there is a non-constant morphism $X \dashrightarrow C$ over K or $X(K) \neq \varnothing$.*

Proof of $\Rightarrow$.

If K is e.c and L is a proper finite extension, then $K \not\preceq_\exists L$. It means that $L \not\models T_0$. Thus (1) is satisfied.
For X a geometrically integral variety over K, if there is no non-constant morphism $X \dashrightarrow C$ over K, it means that $C(K(X)) = \varnothing$. Thus $K \preceq_\exists K(X)$, so $X(K) \neq \varnothing$. Thus we have verified (2). □

Characterization of existentially closed models

Note that (2) is equivalent to (2').

(2') If X is a geometrically integral variety over K, either there is a non-constant morphism $X \dashrightarrow C$ over K or $X(K)$ is Zariski dense in X.

Note that (2) is equivalent to (2').

(2') If X is a geometrically integral variety over K, either there is a non-constant morphism $X \dashrightarrow C$ over K or $X(K)$ is Zariski dense in X.

We also need the following fact.

Fact

Suppose that X is an integral K-variety. Then $K \preceq_{\exists} K(X)$ if and only if $X(K)$ is Zariski dense in X.

Note that (2) is equivalent to (2').

(2') If X is a geometrically integral variety over K, either there is a non-constant morphism $X \dashrightarrow C$ over K or $X(K)$ is Zariski dense in X.

We also need the following fact.

Fact

Suppose that X is an integral K-variety. Then $K \leq_{\exists} K(X)$ if and only if $X(K)$ is Zariski dense in X.

Proof of $\Leftarrow$.

Let L/K and $L \vDash T_0$. We need to show $K \leq_{\exists} L$. WLOG, L is finitely generated over K. So $L = K(X)$ for some integral variety X/K. If X is not geometrically integral, then L contains a proper algebraic extension of K. A contradiction to (1).

Characterization of existentially closed models

Note that (2) is equivalent to (2').

(2') If X is a geometrically integral variety over K, either there is a non-constant morphism $X \dashrightarrow C$ over K or $X(K)$ is Zariski dense in X.

We also need the following fact.

Fact

Suppose that X is an integral K-variety. Then $K \preceq_{\exists} K(X)$ if and only if $X(K)$ is Zariski dense in X.

Proof of $\Leftarrow$.

Let L/K and $L \models T_0$. We need to show $K \preceq_{\exists} L$. WLOG, L is finitely generated over K. So $L = K(X)$ for some integral variety X/K. If X is not geometrically integral, then L contains a proper algebraic extension of K. A contradiction to (1).

$L \models T_0$ means there is no non-constant morphism $X \dashrightarrow C$, so $(2')$ and the fact complete the proof. □

Characterization of e.c. models in higher dimensional case

Note that in general, let $\tilde{V}$ be a geometrically integral projective variety over K_0 such that there is a subvariety $V_0 \neq V$ over K_0 with $\tilde{V}(K_0) = V_0(K_0)$. For the class of fields K such that $\tilde{V}(K) = V_0(K)$, the existentially closed ones can be characterized similarly by replacing C with $V = \tilde{V} \setminus V_0$.

Axioms of existentially closed models

The above characterization requires quantifying over rational functions and finite extensions, why is it first-order?

Axioms of existentially closed models

The above characterization requires quantifying over rational functions and finite extensions, why is it first-order?

- (1) is first-order, this follows from the fact that extensions of degree n are uniformly definable.
- (2) requires quantifying over the set of morphisms $X \dashrightarrow V$.

The above characterization requires quantifying over rational functions and finite extensions, why is it first-order?

- (1) is first-order, this follows from the fact that extensions of degree n are uniformly definable.
- (2) requires quantifying over the set of morphisms $X \dashrightarrow V$.

Note that the set of functions between two definable sets is not a definable set in general. For example, $\mathrm{Var}_K(\mathbb{A}^n, \mathbb{A}^1)$ is $K[X_1, \ldots, X_n]$. This can be seen as a union of definable sets indexed by the total degree in $\mathbb{N}$.

Axioms of existentially closed models

The above characterization requires quantifying over rational functions and finite extensions, why is it first-order?

- (1) is first-order, this follows from the fact that extensions of degree n are uniformly definable.
- (2) requires quantifying over the set of morphisms $X \dashrightarrow V$.

Note that the set of functions between two definable sets is not a definable set in general. For example, $\mathrm{Var}_K(\mathbb{A}^n, \mathbb{A}^1)$ is $K[X_1, \ldots, X_n]$. This can be seen as a union of definable sets indexed by the total degree in $\mathbb{N}$.

We need some bounds on the complexity of such functions. In the curve case, this is where the genus of C is ≥ 2 comes in.

Bounding complexity for curves

We need the following: For a family of varieties X, there is another definable family W such that one can identify non-constant morphism $X_a \dashrightarrow C$ with W_a.

Bounding complexity for curves

We need the following: For a family of varieties X, there is another definable family W such that one can identify non-constant morphism $X_a \dashrightarrow C$ with W_a.
If X is a family of curves, using Riemann-Hurwitz, we have that for any $f\colon X_a \dashrightarrow C$,

$$deg(f) \leq \frac{g(X_a) - 1}{g(C) - 1}.$$

This allows us to bound the complexity of a representation of f using quotients of polynomials, which makes it definable.

Bounding complexity for curves

We need the following: For a family of varieties X, there is another definable family W such that one can identify non-constant morphism $X_a \dashrightarrow C$ with W_a.
If X is a family of curves, using Riemann-Hurwitz, we have that for any $f\colon X_a \dashrightarrow C$,

$$deg(f) \leq \frac{g(X_a) - 1}{g(C) - 1}.$$

This allows us to bound the complexity of a representation of f using quotients of polynomials, which makes it definable.
In general, using sufficiently general hyperplane intersections, one can bound the complexity of the graph of $f\colon X \dashrightarrow C$ in terms of the genus of C and the (projective) degree of X and C (considering everything as embedded in $\mathbb{P}^n$).

In the general case, let L denote an algebraically closed field of characteristic 0, and X is an integral projective variety over L. We say X is

1. **bounded** if, for every normal integral projective scheme Y over L, the scheme $\underline{\mathrm{Hom}}_L(Y, X)$ is of finite type over L (in particular it is definable in ACF_0 over L);
2. 1-**bounded** if, for every smooth projective curve C over L, the scheme $\underline{\mathrm{Hom}}_L(C, X)$ is of finite type over L;
3. **groupless** if, for any connected finite type algebraic group G over L, there are no non-constant maps $G \to X$;
4. **pure** (over L) if, for any smooth variety T over L, any rational map $T \dashrightarrow X$ extends to a regular map $T \to X$.

Hyperbolicity for varieties

It is not hard to see that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Actually (1) is equivalent to (2) by Javanpekar and Kamenova. An elliptic curve is pure but not groupless. It is conjectured that groupless implies algebraic hyperbolicity (Part of Green-Griffiths-Lang).

It is not hard to see that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Actually (1) is equivalent to (2) by Javanpekar and Kamenova. An elliptic curve is pure but not groupless. It is conjectured that groupless implies algebraic hyperbolicity (Part of Green-Griffiths-Lang).

Definition 12

Let X, L be as before, X is **algebraically hyperbolic** if there exists $a, b \in \mathbb{R}_{\geq 0}$ and an ample line bundle $\mathcal{L}$ on X such that for every smooth curve C over L and a finite map $f\colon C \to X$ we have

$$\int_C c_1(f^*\mathcal{L}) \leq a \cdot g(C) + b,$$

where $g(C)$ is the genus of C.

Model companion for hyperbolic varieties

Let K_0 be a field of characteristic 0. Let $\tilde{V}$ be a smooth geometrically integral projective variety over K_0 with a proper closed subvariety V_0/K_0 such that $\tilde{V}(K_0) = V_0(K_0)$, let $V = \tilde{V} \setminus V_0$.

Model companion for hyperbolic varieties

Let K_0 be a field of characteristic 0. Let $\tilde{V}$ be a smooth geometrically integral projective variety over K_0 with a proper closed subvariety V_0/K_0 such that $\tilde{V}(K_0) = V_0(K_0)$, let $V = \tilde{V} \setminus V_0$.

Theorem 13 (Szachniewicz-Y.)

The class of fields K such that $V(K) = \varnothing$ has a model companion if $\tilde{V}$ is assumed to be 1-bounded.

Model companion for hyperbolic varieties

Let K_0 be a field of characteristic 0. Let $\tilde{V}$ be a smooth geometrically integral projective variety over K_0 with a proper closed subvariety V_0/K_0 such that $\tilde{V}(K_0) = V_0(K_0)$, let $V = \tilde{V} \setminus V_0$.

Theorem 13 (Szachniewicz-Y.)

The class of fields K such that $V(K) = \emptyset$ has a model companion if $\tilde{V}$ is assumed to be 1-bounded.

The idea of the proof is similar to the proof of the case when $\tilde{V}$ is a curve. Here the bound in the complexity requires some inequalities in intersection theory by Khovanskii-Teissier. The argument also recovers that 1-bounded implies bounded.

Some questions and remarks

Question 1

Is there a characterization/criterion of the non-existence of the model companion?

A substantial part of the existence of model companion can be formulated using Morley rank and degree. Here is a meta-question.

Question 2

What is a model-theoretic characterization of the definability of $\mathrm{Mor}(-, X)$ in the ω-stable/strongly minimal setting?

Some questions and remarks

Question 1

Is there a characterization/criterion of the non-existence of the model companion?

A substantial part of the existence of model companion can be formulated using Morley rank and degree. Here is a meta-question.

Question 2

What is a model-theoretic characterization of the definability of $\mathrm{Mor}(-, X)$ in the ω-stable/strongly minimal setting?

Concluding remark before we move on, $V\mathrm{XF}$ can be large. For example, it is the case when V is a smooth projective variety and $V(K_0) = \varnothing$.

Indecomposability

When C is a geometrically integral curve over K of genus ≥ 2, we have the following.

Lemma 14

Let V, W be two geometrically integral varieties over K and $f\colon V \times W \dashrightarrow C$ be a rational map, then it factors over V or W generically.

Indecomposability

When C is a geometrically integral curve over K of genus ≥ 2, we have the following.

Lemma 14

Let V, W be two geometrically integral varieties over K and $f\colon V \times W \dashrightarrow C$ be a rational map, then it factors over V or W generically.

Proof.

Assume f does not factor through V. Note that there are only finitely many non-constant rational maps $W \dashrightarrow C$ (De Franchis's theorem), then $f(x, -)$ can be chosen to be independent of x after shrinking V. □

Indecomposablity

We say that X is **indecomposable** if Lemma 14 holds when C is replaced by X.

Indecomposablity

We say that X is **indecomposable** if Lemma 14 holds when C is replaced by X.

Theorem 15 (Kobayashi, Lazarsfeld)

If X is a smooth variety with ample cotangent bundle, then X is algebraically hyperbolic and indecomposable.

Indecomposablity

We say that X is **indecomposable** if Lemma 14 holds when C is replaced by X.

Theorem 15 (Kobayashi, Lazarsfeld)

If X is a smooth variety with ample cotangent bundle, then X is algebraically hyperbolic and indecomposable.

Algebraically hyperbolicity does not imply indecomposable: Let C be a genus 2 curve, $C \times C$ is algebraically hyperbolic but not indecomposable. However, indecomposability implies groupless.

Completions of $V\mathrm{XF}$

From now on, $V = \tilde{V} \setminus V_0$ in our setup such that $\tilde{V}$ is 1-bounded and indecomposable. We gather some facts needed in order to characterize the completions of $V\mathrm{XF}$. Recall that $\mathrm{Abs}(K) = K \cap \mathbb{Q}^{\mathrm{alg}}$. Note that a field $K \models V\mathrm{XF}_\forall$ iff $V(K) = \varnothing$.

From now on, $V = \tilde{V} \setminus V_0$ in our setup such that $\tilde{V}$ is 1-bounded and indecomposable. We gather some facts needed in order to characterize the completions of $V\mathrm{XF}$. Recall that $\mathrm{Abs}(K) = K \cap \mathbb{Q}^{\mathrm{alg}}$. Note that a field $K \models V\mathrm{XF}_\forall$ iff $V(K) = \varnothing$.

Lemma 16

Let $K_1, K_2 \models V\mathrm{XF}$ and K be a common relatively algebraically closed subfield of both K_i's, then the map $\mathrm{id} : K \to K$ is a partial elementary map between K_i's.

Completions of $V\mathrm{XF}$

From now on, $V = \tilde{V} \setminus V_0$ in our setup such that $\tilde{V}$ is 1-bounded and indecomposable. We gather some facts needed in order to characterize the completions of $V\mathrm{XF}$. Recall that $\mathrm{Abs}(K) = K \cap \mathbb{Q}^{\mathrm{alg}}$. Note that a field $K \models V\mathrm{XF}_\forall$ iff $V(K) = \varnothing$.

Lemma 16

Let $K_1, K_2 \models V\mathrm{XF}$ and K be a common relatively algebraically closed subfield of both K_i's, then the map $\mathrm{id} : K \to K$ is a partial elementary map between K_i's.

Proof Sketch.

For finitely generated K_1, K_2, we can amalgamate K_1, K_2 over K into $K_1 \otimes_K K_2 \models V\mathrm{XF}_\forall$. We can further embed into $L \models V\mathrm{XF}$. □

Completions of $V\mathrm{XF}$

Corollary 17 (Johnson-Y., Szachniewicz-Y.)

Let $K_1, K_2 \models V\mathrm{XF}$, *then* $K_1 \equiv K_2$ *iff* $\mathrm{Abs}(K_1) \cong \mathrm{Abs}(K_2)$.

Completions of $V\mathrm{XF}$

Corollary 17 (Johnson-Y., Szachniewicz-Y.)

Let $K_1, K_2 \models V\mathrm{XF}$, then $K_1 \equiv K_2$ iff $\mathrm{Abs}(K_1) \cong \mathrm{Abs}(K_2)$.

We characterize all the completions of $V\mathrm{XF}$.

Theorem 18 (Johnson-Y., Szachniewicz-Y.)

Assuming V is indecomposable, for any $F \models V\mathrm{XF}_\forall$, there is a regular extension K/F such that $K \models V\mathrm{XF}$.
Particularly, for any $F \subseteq \mathbb{Q}^{\mathrm{alg}}$ with $F \models V\mathrm{XF}_\forall$, there is $K \models V\mathrm{XF}$ with $\mathrm{Abs}(K) \cong F$.

In the case of a curve of genus at least 2, we have the following:

Corollary 19 (Johnson-Y.)

The theory axiomtized by CXF *and* $\mathrm{Abs}(M) = \mathbb{Q}$ *is complete and decidable. Particularly, there is a decidable non-large field.*

Decidability and recursive axiomatizability

In the case of a curve of genus at least 2, we have the following:

Corollary 19 (Johnson-Y.)

The theory axiomtized by CXF *and* $\mathrm{Abs}(M) = \mathbb{Q}$ *is complete and decidable. Particularly, there is a decidable non-large field.*

The decidability of the partial theory CXF admits some interesting characterization as well.

Theorem 20 (Johnson-Y.)

The decidability of CXF *is equivalent to the following problem: Given* K/K_0 *finite, decide whether* $C(K)$ *is non-empty.*

Decidability and recursive axiomatizability

The axiomatization of $V\mathrm{XF}$ is not computable a priori. However, if V is algebraically hyperbolic, then the axiomatization is indeed computable.
For example, this is the case when V has ample cotangent bundle, which is indeed the case when $\tilde{V}$ comes from a sufficiently general intersection of ample line bundles in projective space. In this case, we have analogue results in terms of decidability.

Decidability and recursive axiomatizability

The axiomatization of $V\mathrm{XF}$ is not computable a priori. However, if V is algebraically hyperbolic, then the axiomatization is indeed computable.
For example, this is the case when V has ample cotangent bundle, which is indeed the case when $\tilde{V}$ comes from a sufficiently general intersection of ample line bundles in projective space. In this case, we have analogue results in terms of decidability.
However, to obtain the computable axiomatization of $V\mathrm{XF}$, one only needs a computable bound on the complexity. In theory, one could isolate other notions of hyperbolicity by imposing bounds on the complexity. But how interesting are they?

Algebraic closure

Lemma 21

Let K be a relatively algebraically closed subfield of $L \models V\mathrm{XF}$, then $K = \mathrm{dcl}(K)$.

Algebraic closure

Lemma 21

Let K be a relatively algebraically closed subfield of $L \vDash \mathrm{VXF}$, then $K = \mathrm{dcl}(K)$.

Proof.

By Lemma 16, we have an embedding:$L \otimes_K L \to F \vDash \mathrm{VXF}$. So for $b \in L \backslash K$, $\mathrm{tp}_L(b/K) = \mathrm{tp}_F(1 \otimes b/K) = \mathrm{tp}_F(b \otimes 1/K)$. So $b \notin \mathrm{dcl}(K)$. □

Corollary 22

Model-theoretic and field-theoretic acl agree in VXF (after naming K_0).

Finite extensions

A field K is Hilbertian if, for any $f \in K[X, Y]$ irreducible, there are infinitely many $a \in K$ such that $f(a, Y)$ is irreducible. Equivalently, $K(t)$ is relatively algebraically closed in an elementary extension $\tilde{K}$.

Finite extensions

A field K is Hilbertian if, for any $f \in K[X, Y]$ irreducible, there are infinitely many $a \in K$ such that $f(a, Y)$ is irreducible. Equivalently, $K(t)$ is relatively algebraically closed in an elementary extension $\tilde{K}$.

Theorem 23 (Johnson-Y., Szachniewicz-Y.)

Models of $V\mathrm{XF}$ are Hilbertian for indecomposable V.

Finite extensions

A field K is Hilbertian if, for any $f \in K[X, Y]$ irreducible, there are infinitely many $a \in K$ such that $f(a, Y)$ is irreducible. Equivalently, $K(t)$ is relatively algebraically closed in an elementary extension $\tilde{K}$.

Theorem 23 (Johnson-Y., Szachniewicz-Y.)

Models of $V\mathrm{XF}$ are Hilbertian for indecomposable V.

Proof.

Take $K \models V\mathrm{XF}$. Note there is no non-constant morphism $\mathbb{A}^1 \to V$, so $V(K(t)) = \varnothing$. By Theorem 18, there is $M/K(t)$ regular such that $M \models V\mathrm{XF}$. And $K \preceq M$ by Theorem 11. □

Finite extensions

A field K is Hilbertian if, for any $f \in K[X,Y]$ irreducible, there are infinitely many $a \in K$ such that $f(a,Y)$ is irreducible. Equivalently, $K(t)$ is relatively algebraically closed in an elementary extension $\tilde{K}$.

Theorem 23 (Johnson-Y., Szachniewicz-Y.)

Models of $V\mathrm{XF}$ are Hilbertian for indecomposable V.

Proof.

Take $K \models V\mathrm{XF}$. Note there is no non-constant morphism $\mathbb{A}^1 \to V$, so $V(K(t)) = \varnothing$. By Theorem 18, there is $M/K(t)$ regular such that $M \models V\mathrm{XF}$. And $K \preceq M$ by Theorem 11. □

Hilbertian fields do not have small Galois group.

Corollary 24

There is a model complete field that does not have small Galois group.

Finite extensions

Lemma 25

Let V be indecomposable, $K \models V\mathrm{XF}_\forall$, and L/K be a proper finite extension. Suppose X is a geometrically integral variety over L. Consider the Weil restriction of $W = \mathrm{Res}_{L/K} X$. Then there is no non-constant rational map over $W \dashrightarrow V$.

Proof Sketch.

Let $L = K[X]/P(X)$ for some irreducible polynomial P over K. Over $F = K^{alg}$, W_F is a finite product of X_F's indexed by the roots of P. If there is a non-constant rational map defined over K, by Lemma 14, it defines a root of $P(X)$. There is no K-definable root of $P(X)$. □

Finite extensions

From the above, we get the following.

Corollary 26 (Johnson-Y., Szachniewicz-Y.)

Every proper finite extension L of $K \models V\mathrm{XF}$ is PAC for indecomposable V.

Proof.

Let X be a geometrically integral variety over L, by above and (2), $W(K) \neq \varnothing$, hence $X(L) \neq \varnothing$. □

Finite extensions

From the above, we get the following.

Corollary 26 (Johnson-Y., Szachniewicz-Y.)

Every proper finite extension L of $K \models V\mathrm{XF}$ is PAC for indecomposable V.

Proof.

Let X be a geometrically integral variety over L, by above and (2), $W(K) \neq \varnothing$, hence $X(L) \neq \varnothing$. □

This recovered a result by Srinivasan.

Corollary 27 (Srinivasan)

There is a virtually large yet non-large field.

From the above, we get the following.

Corollary 26 (Johnson-Y., Szachniewicz-Y.)

Every proper finite extension L of $K \models \mathrm{VXF}$ is PAC for indecomposable V.

Proof.

Let X be a geometrically integral variety over L, by above and (2), $W(K) \neq \varnothing$, hence $X(L) \neq \varnothing$. □

This recovered a result by Srinivasan.

Corollary 27 (Srinivasan)

There is a virtually large yet non-large field.

Axiom (2) is sufficient to guarantee the above corollary holds, so Axiom (1) actually follows from Axiom (2) for indecomposable V.

Classification theory

We next study $V\mathrm{XF}$ in terms of classification theory in the sense of Shelah.

For PAC fields, not having small Galois group implies TP_2, and finite separable extensions of Hilbertian fields are Hilbertian.

Theorem 28

$V\mathrm{XF}$ has TP_2 for indecomposable V.

For this page, we work in $\mathcal{CXF}$ only.
We say that a formula $\varphi(x; y)$ has SOP_n for $n \geq 3$ if there is sequence $(a_i)_{i \in \mathbb{N}}$ such that $\varphi(a_i; a_j)$ holds iff $i < j$ but the partial type

$$\{\varphi(x_1; x_2), \ldots, \varphi(x_{n-1}; x_n), \varphi(x_n; x_1)\}$$

is inconsistent.

Theorem 29

$\mathcal{CXF}$ is NSOP_4, and there is a completion of it that is strictly NSOP_4.

Proof.

Let's first assume the NSOP_4 claim. Let $C := x^4 + y^4 + z^4 = 0$ and $\mathrm{Abs}(K) = \mathbb{R} \cap \mathbb{Q}^{alg}$. Let $\varphi(x; y)$ be saying $x - y$ is a non-zero 4th power. It has SOP_3. □

Thank you for your attention.