QUANTALE-VALUED MODEL THEORY AND SET THEORY

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Topological motivation – 1/25

- Quantales: Lattices introduced to deal with locales (lattices which generalize the ideal of open sets of a topological space) and multiplicative lattices of ideals from Ring Theory and Functional Analysis (e.g., C^*–algebras and von Neumann algebras).
- Flagg (1997): Topological spaces as pseudo metric spaces, distances with values in a suitable quantale built from the topology.

Model-Theoretic motivation

- Shelah – Stern (70s): Banach Spaces – behavior similar to a 2nd order logic of binary relations.
- Chang-Keisler (60s), Henson-Iovino (90s), Ben-Yaacov and others (2000s): Continuous Logic.
- Hirvonen-Hyttinen, Villaveces-Z., Boney-Z. (2000s-2010s): Metric AECs.
- Flum-Ziegler, Kucera, Pillay (80s): Model Theory for Topological algebraic structures. Leaded to Model Theory for Modules.
- Lawvere (70s): Framework in Category Theory for a logic with generalized truth values to study metric spaces.
- $\mathbb{V}$-AECs (Lieberman-Rosický-Z.): Generalization of MAECs to distances with values in a co-quantale (Flagg quantale).

Set-Theoretic motivation – 3/25

- Fitting: Models of Intuitionistic Set Theory generalizing both the universes of von Neumann and of Gödel using Intuitionistic Kripke models.
- Lano: Residuated Kripke models of Set Theory.

KEY IDEA: To consider a quantale as a set of truth values (or the range of a generalized distance to the "truth", as in Continuous Logic).

Quantale-valued Model Theory (joint with D. Reyes)

– 4/25

Definition. Let $(V, \leqslant)$ be a complete lattice. Given $\alpha, \beta \in V$, α is said to be *totally below* β ($\alpha \prec \beta$) if for every $S \subseteq V$, if $\beta \leqslant \bigvee S$ then $\alpha \leqslant \gamma$ for some $\gamma \in S$.

Definition. A *commutative, unital quantale* consists of $\mathbb{V} := (V, \star, \eta, \leqslant)$, where $(V, \leqslant)$ is a complete lattice, $(V, \star, \eta)$ is a commutative monoid and $\star$ distributes over arbitrary joins.

Definition. $\mathbb{V} := (V, +, 0, \leqslant)$ is said to be a *co-quantale* if $(V, \star, \eta, \leqslant)^{op} := (V, \star, \eta, \geqslant)$ is a commutative, unital quantale, where $\eta := 0 = \bot_{\leqslant}$ and $\star := +$.

Value co-quantales – 5/25

Definition. A co-quantale $\mathbb{V} := (V, +, 0, \leqslant)$ is said to be *constructively completely distributive* if for every $a \in V$

$$a = \bigwedge \{b \in V : a \prec b\}.$$

A *value co-quantale* is a constructively completely distributive co-quantale $\mathbb{V}$ provided that

- $0 = \perp \prec \top$
- if $\delta, \delta' \in V$ satisfy $0 \prec \delta$ and $0 \prec \delta'$, then $0 \prec \delta \wedge \delta'$.

- The 2-valued Boolean algebra $\mathbf{2} := \{0,1\}$, where $0 < 1$.
- The unit interval $\mathcal{I} = [0,1]$.
- Free locales: R a set. $\downarrow(X) = \{Y \in \mathcal{P}_{fin}(R) : Y \subseteq X\}$ $(X \in \mathcal{P}_{fin}(R))$.
  $\Omega(R) = \{p \in \mathcal{P}(\mathcal{P}_{fin}(R)) : X \in p$ implies $\downarrow(X) \subseteq p\}$
  $(\Omega(R), \cap, \mathcal{P}_{fin}(R), \supseteq)$ is a value co-quantale.
  $R := \tau$, (X, τ) a topological space.

Definition. Given $\mathbb{V}$ a co-quantale and $a, b \in V$, define
$a \dot{-} b := \bigwedge\{r \in V : r + b \geq a\}$.

Some extra conditions – 7/25

Definition. Co-divisibility (substractibility): For all $a, b \in V$, $a \leqslant b$ implies that there exists $c \in V$ such that $b = a + c$.

Definition. Dualizing element: $d \in V$ such that for all $a \in V$ we have that $a = d \dot{-} (d \dot{-} a)$.

Definition. co-Girard: There is a dualizing element.

Continuity spaces – 8/25

Definition. $X \neq \emptyset$ a set, $\mathbb{V}$ a value co-quantale, $d : X \times X \to V$

(X, d) is a $\mathbb{V}$-continuity space iff:

- (reflexivity) for all $x \in X$, $d(x,x) = 0$, and
- (transitivity) for any $x, y, z \in X$, $d(x,y) \leqslant d(x,z) + d(z,y)$.

Definition. (X, d) a $\mathbb{V}$-continuity space, $\epsilon \in V^{+}$, $x \in X$

$B_\epsilon(x) := \{y \in X : d(x,y) < \epsilon\}$.

Definition. τ_d: Topology induced by (X, d).

Definition. $\mathbb{V}$-domain: $\mathbb{V}$-continuity space (X, d) which is T_0 and (X, τ_d^s) is compact.

Theorem (Flagg, 1997). Any topological space (X, τ) is topologically equivalent to the $\Omega(\tau)$-continuity space (X, d), where $d : X \times X \to \Omega(\tau)$ is defined by

$$d(a, b) := \{A \in \mathcal{P}_{finite}(\tau) : \text{for all } U \in A,\ a \in U \text{ implies } b \in U\}.$$

Continuous structures in this setting. — 10/25

(M, d_M): $\mathbb{V}$-continuity space with diameter $p \in \mathsf{V}$.

Continuous structure: $\mathcal{M} = ((M, d_M), (R_i)_{i \in I}, (f_j)_{j \in J}, (c_k)_{k \in K})$, where:

- $R_i : M^{n_i} \to V$: uniformly continuous mapping, with modulus of uniform continuity $\Delta_{R_i} : \mathsf{V}^+ \to \mathsf{V}^+$.

  $n_i < \omega$: arity of R_i.
- $f_j : M^{m_j} \to M$: uniformly continuous mapping with modulus of uniform continuity $\Delta_{f_j} : \mathsf{V}^+ \to \mathsf{V}^+$.

  $m_j < \omega$: arity of F_j.
- c_k is an element of M.

Some basic definitions — 11/25

L-terms and L-formulas: Defined recursively.

Connectives: $u : V^n \to V$ $(1 \leqslant n < \omega)$ a uniformly continuous mapping.

Quantifiers: $\bigvee_x$ (universal quantifier), $\bigwedge_x$ (existential quantifier).

Definition. $\mathcal{M} \preccurlyeq \mathcal{N}$: $\mathcal{M} \subseteq_L \mathcal{N}$ such that any L-formula $\varphi(x_1, \ldots, x_n)$ satisfies $\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(a_1, \ldots, a_n)$ $(a_1, \ldots, a_n \in M)$.

Tarski–Vaught test – 12/25

Theorem (Reyes–Z., 2021). Assume that $\mathbb{V}$ is a co-Girard value co-quantale. Let $\mathcal{M}$, $\mathcal{N}$ be L-structures such that $\mathcal{M} \subseteq_L \mathcal{N}$. The following are equivalent:

- $\mathcal{M} \preccurlyeq \mathcal{N}$.
- For any L-formula $\varphi(x, x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, we have that

$$\bigwedge\{\varphi^{\mathcal{M}}(c, a_1, \ldots, a_n) : c \in M\} = \bigwedge\{\varphi^{\mathcal{N}}(c, a_1, \ldots, a_n) : c \in N\}.$$

Corollary. Elementary chain property holds.

D-products and Łoś Theorem — 13/25

$\mathbb{V}$: substractible, V-domain value co-quantale provided with its symmetric topology.

Notice that (V, τ^S) is compact and Hausdorff.

D-product of $(M_i, d_{M_i})_{i \in I}$ (V-continuity spaces and D an ultraproduct on I): $(M_D, d_{M_D}) := (\prod_{i \in I} M_i, d_D)$, where $d_D((x_i)_{i \in I}, (y_i)_{i \in I}) := \lim_{i,D} d_{M_i}(x_i, y_i)$.

Theorem (Reyes-Z., 2021). For any L-formula $\phi(x_1, \ldots, x_n)$ and any tuple $((a_i^1)_{i \in I}, \ldots, (a_i^n)_{i \in I}) \in (\prod_{i \in I} M_i)^n$,

$$\phi^{\mathcal{M}_D}((a_i^1)_{i \in I}, \ldots, (a_i^n)_{i \in I}) = \lim_{i,D} \phi^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n).$$

Corollary. Compactness holds for this logic.

Theorem (Iovino, 2001). There is no logic for analytic structures that extends properly "Continuous Logic" and satisfies both the compactness theorem and the elementary chain property.

Notice that $\mathbb{V} := ([0,1], +, 0, \leqslant)$ satisfies all the required conditions in our previous setting.

Also, in this way we obtain a new approach for Continuous Logic.

Definition. We say that $\mathbb{Q} = (\mathbb{Q}, \wedge, \vee, \cdot, \rightarrow, 1, 0)$ is a commutative integral Quantale (or equivalently a complete residuated lattice) if:

- $(\mathbb{Q}, \wedge, \vee, 1, 0)$ is a complete bounded lattice with 1 as top element and 0 as bottom element.
- $(\mathbb{Q}, \cdot, 1)$ is a commutative monoid.
- For all $x, y_i \in \mathbb{Q}$ with $i \in I$, $x \cdot \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \cdot y_i)$

  and $\rightarrow$ is defined as $x \rightarrow y := \bigvee\{z \in \mathbb{Q} : x \cdot z \leq y\}$.

Modal Residuated Logic – 16/25

We consider two types of conjunctions: weak conjunction ($\wedge$) – strong conjunction ($\&$).

- $\varphi \equiv \psi := (\varphi \to \psi) \& (\psi \to \varphi)$,
- $\sim \varphi := \varphi \to \bot$,
- $\top := \sim \bot$.

Residuated formulas (R–formulas): By recursion.

Modal Residuated formulas: Same symbols as in Residuated Logic together with an unary connective of possibility $\Diamond$.

Idea: To interpret $\Diamond$ by using a quantic nucleus.

$MR-\mathcal{L}$–formulas: By recursion.

Quantic nucleus – 17/25

Closure operator: $\gamma : \mathbb{Q} \to \mathbb{Q}$ such that for every $x, y \in \mathbb{Q}$:

- (Expansivity) $x \leqslant \gamma(x)$.
- (Idempotency with respect to $\circ$) $\gamma(\gamma(x)) = \gamma(x)$.
- (Monotonicity) If $x \leqslant y$, then $\gamma(x) \leqslant \gamma(y)$.

Quantic nucleus: for every $x, y \in \mathbb{Q}$ $\gamma(x) \cdot \gamma(y) \leqslant \gamma(x \cdot y)$.

Respection of implications: $\gamma(x \to y) = 1$ if and only if $\gamma(x) \to \gamma(y) = 1$.

Idempotency with respect to products: for every $x \in \mathbb{Q}$ $\gamma(x^2) = \gamma(x)$.

Modal Residuated Kripke Models (Ono) —

Residuated Kripke $\mathcal{L}$-model (or R-Kripke $\mathcal{L}$-model: $\mathcal{A} = (\mathbb{P}, \leqslant, \Vdash, \mathcal{D})$ such that

- $\mathbb{P} = (\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty)$ is a complete SO-commutative monoid.
- Given $p_i, q \in \mathbb{P}$, with $i \in I$ and φ an atomic $\mathcal{L}_{\mathcal{A}}$-sentence, then
  - If $\bigwedge_{i \in I} p_i \leqslant q$ and for each $i \in I$ $\mathcal{A} \Vdash_{p_i} \varphi$, then $\mathcal{A} \Vdash_q \varphi$.
  - $\mathcal{A} \Vdash_\infty \varphi$ for every atomic $R - \mathcal{L}_{\mathcal{A}}$-sentence φ.
  - $\mathcal{A} \Vdash_p \perp$ if and only if $p = \infty$.

Modal Residuated Kripke Models (Ono) — 19/25

- $\mathcal{A} \Vdash_p (\varphi \& \psi)$, if and only if, there are $q, r \in \mathbb{P}$ such that $p \geq q \cdot r$, $\mathcal{A} \Vdash_q \varphi$ and $\mathcal{A} \Vdash_r \psi$.
- $\mathcal{A} \Vdash_p (\varphi \vee \psi)$, if and only if, there are $q, r \in \mathbb{P}$ such that $p \geq q \wedge r$, and both $(\mathcal{A} \Vdash_q \varphi$ or $\mathcal{A} \Vdash_q \psi)$ and $(\mathcal{A} \Vdash_r \varphi$ or $\mathcal{A} \Vdash_r \psi)$ hold.
- $\mathcal{A} \Vdash_p (\varphi \wedge \psi)$, if and only if, $\mathcal{A} \Vdash_p \varphi$ and $\mathcal{A} \Vdash_p \psi$.
- $\mathcal{A} \Vdash_p (\varphi \to \psi)$, if and only if, for all $q, r \in \mathbb{P}$ if $\mathcal{A} \Vdash_q \varphi$ and $p \cdot q \leq r$, then $\mathcal{A} \Vdash_r \psi$.
- $\mathcal{A} \Vdash_p \exists x \varphi(x)$ if and only there exists an index set I such that for every $i \in I$, there exist $d_i \in D$ and $q_i \in \mathbb{P}$ such that $\bigwedge_{i \in I} q_i \leq p$ and $\mathcal{A} \Vdash_{q_i} \varphi(d_i)$.
- $\mathcal{A} \Vdash_p \forall x \varphi(x)$, if and only if, for all $b \in D$, $\mathcal{A} \Vdash_p \varphi(b)$.

Conucleus: $\delta : \mathbb{P} \to \mathbb{P}$ ($(\mathbb{P}, \leqslant, \cdot)$ complete SO-monoid) such that for all $p, q, p_i \in \mathbb{P}$ $(i \in I)$

- $\delta(p) \leqslant p$.
- If $p \leqslant q$, then $\delta(p) \leqslant \delta(q)$.
- $\delta(\delta(p)) = \delta(p)$.
- $\delta(p \cdot q) \leqslant \delta(p) \cdot \delta(q)$.
- $\delta(\bigwedge_{i \in I} p_i) = \bigwedge_{i \in I} \delta(p_i)$

Complete modal SO-commutative monoid: $\mathbb{P} = (\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty, \delta)$ such that

- $(\mathbb{P}, \leqslant, \wedge, \cdot, 1, \infty)$ is a complete SO-commutative monoid.
- δ is a conucleus on $(\mathbb{P}, \leqslant, \cdot)$.

$\mathcal{A} \Vdash_p \Diamond\varphi$: There exists $q \in \mathbb{P}$ such that $\mathcal{A} \Vdash_q \varphi$ and $\delta(q) \leqslant p$.

Strongly hereditary sets – 21/25

Strongly hereditariness: $\emptyset \neq A \subseteq \mathbb{P}$ such that for all $c_i \in A$ and $d \in \mathbb{P}$ for $i \in I$, if $\bigwedge_{i \in I} c_i \leqslant d$, then $d \in A$.

$$\mathbb{P}^* := \{A \subseteq \mathbb{P} : A \text{ is strongly hereditary}\}$$

Fact (Moncayo-Z.). The operation $\gamma_\delta : \mathbb{P}^* \to \mathbb{P}^*$ defined as

$$\gamma_\delta(A) := \{p \in \mathbb{P} : \text{there is } q \in A \text{ such that } \delta(q) \leqslant p\}$$

is a quantic nucleus on $(\mathbb{P}^*, \subseteq, \cdot)$.

If there is no ambiguity, we denote $\gamma := \gamma_\delta$.

Codifying subsets - 22/25

In classical logic: $A \subseteq B$ is codified by its characteristic function $\chi_A : B \to \{0,1\}$: $a \in A$ iff $\chi_A(a) = 1$ (for all $a \in B$).

$\mathbb{P}^*$-subset of $\mathcal{A}$: function f such that $Dom(f) \subseteq D$ and $Ran(f) \subseteq \mathbb{P}^* = \{A \subseteq \mathbb{P} : A \text{ is strongly hereditary}\}$

Extensional: $f : D \to \mathbb{P}^*$ such that for each $g, h \in D$

$$f(g) \cdot \{p \in \mathbb{P} : \mathcal{A} \Vdash_p (g = h)\} \subseteq f(h).$$

$$(g = h) := \Diamond \sim (\exists x) \sim (x \in g \to x \in h))\&(\Diamond \sim (\exists x) \sim (x \in h \leftarrow x \in g)).$$

$\mathcal{P}^{\mathbb{P}^*}(D)$: collection of mappings f which are γ-regular and extensional $\mathbb{P}^*$-subsets of $\mathcal{A}$

von Neumann-like hierarchy

$\mathcal{V}^{\mathbb{P}*}_{\alpha+1} := (\mathbb{P}, \leqslant, \delta, \Vdash, \mathcal{R}^{\mathbb{P}*}_{\alpha+1})$ where $\mathcal{R}^{\mathbb{P}*}_{\alpha+1} := \mathcal{R}^{\mathbb{P}*}_{\alpha} \cup \mathcal{P}^{\mathbb{P}*}(\mathcal{R}^{\mathbb{P}*}_{\alpha})$

If $p \in \mathbb{P}$ and $f, g \in \mathcal{R}^{\mathbb{P}*}_{\alpha+1}$, then:

- If $f, g \in \mathcal{R}^{\mathbb{P}*}_{\alpha}$, then $\mathcal{V}^{\mathbb{P}*}_{\alpha+1} \Vdash_p (f \in g)$, if and only if, $\mathcal{V}^{\mathbb{P}*}_{\alpha} \Vdash_p (f \in g)$.
- If $f \in \mathcal{R}^{\mathbb{P}*}_{\alpha}$ and $g \in \mathcal{R}^{\mathbb{P}*}_{\alpha+1} \setminus \mathcal{R}^{\mathbb{P}*}_{\alpha} = \mathcal{P}^{\mathbb{P}*}(\mathcal{R}^{\mathbb{P}*}_{\alpha})$, then $\mathcal{V}^{\mathbb{P}*}_{\alpha+1} \Vdash_p (f \in g)$, if and only if, $p \in g(f)$.
- If $f \in \mathcal{R}^{\mathbb{P}*}_{\alpha+1} \setminus \mathcal{R}^{\mathbb{P}*}_{\alpha} = \mathcal{P}^{\mathbb{P}*}(\mathcal{R}^{\mathbb{P}*}_{\alpha})$, then $\mathcal{V}^{\mathbb{P}*}_{\alpha+1} \Vdash_p (f \in g)$, if and only if, $p \in \bigvee_{h \in dom(g)} P_h$,

  where $P_h := g(h) \cdot (P_{f \subseteq h} \cdot P_{h \subseteq f})$ and

$$P_{f \subseteq h} := \bigcap_{x \in \mathcal{R}^{\mathbb{P}*}_{\alpha}} (f(x) \to \{q \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha} \Vdash_q \sim\sim \Diamond(x \in h)\})$$

$$P_{h \subseteq f} := \bigcap_{x \in \mathcal{R}^{\mathbb{P}*}_{\alpha}} (f(x) \leftarrow \{q \in \mathbb{P} : \mathcal{V}^{\mathbb{P}*}_{\alpha} \Vdash_q \sim\sim \Diamond(x \in h)\}).$$

The main theorem. – 24/25

γ-dense: $x \in \mathbb{Q}$ such that $\gamma(x) = 1_{\mathbb{Q}}$.

$\mathcal{F}_\gamma$: Collection of all γ-dense elements of $\mathbb{P}^*$.

Fact. $A \approx_{\mathcal{F}_\gamma} B$, if and only if, $A \to B \in \mathcal{F}_\gamma$ and $B \to A \in \mathcal{F}_\gamma$ defines an equivalence relation. Also, we have that $\mathbb{H} := \mathbb{P}^*/\mathcal{F}_\gamma = \mathbb{P}^*/_{\approx_{\mathcal{F}_\gamma}} = \{|A| : A \in \mathbb{P}^*\}$ is a complete Heyting algebra.

Theorem (Moncayo-Z.). For every ordinal α, there exist a bijection between $R_\alpha^{\mathbb{P}^*}$ and $R_\alpha^{\mathbb{H}}$ (where if $f \in R_\alpha^{\mathbb{P}^*}$, f' denotes the image of f via this bijection) such that for every MR-$\mathcal{L}_\in$-formula with no universal quantifiers $\varphi(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in R_\alpha^{\mathbb{P}^*}$,

$$|\{p \in \mathbb{P} : \mathcal{V}_\alpha^{\mathbb{P}^*} \Vdash_p \varphi(a_1, \ldots, a_n)\}| = [\![\varphi(a_1', \ldots, a_n')]\!]_\alpha^{\mathbb{H}}.$$

REFERENCES. – 25/25

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THANK YOU VERY MUCH!

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