A Two-Cardinal Ramsey Operator on Ideals

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Overview

Introduction

Theorem of Friends and Strangers

Among any party of 6 people there are 3 people who are either all friends or all strangers.

- The above Theorem is part of *Ramsey Theory*, which studies how size can influence order.
- The collection of 3 people is called a *homogeneous set*.

Theorem of Friends and Strangers (Countable)

Among any party of countably infinite size there is an infinite subcollection that is either all friends or all strangers.

Ramsey's Theorem

Ramsey's Theorem (Ramsey, 1928)

For every function $f : [\omega]^2 \to 2$ there is an infinite $H \subseteq \omega$ such that f is constant on H (H homogeneous for f).

However the theorem for $\kappa > \omega$ cannot be proven in ZFC.

Definition

$\kappa > \omega$ is *weakly compact* if for every function $f : [\kappa]^2 \to 2$ has a κ sized homogeneous set.

Definition

$\kappa > \omega$ is a *Ramsey cardinal* if for every function $f : [\kappa]^{<\omega} \to \kappa$ with $f(a) < \min(a)$ for all $a \in [\kappa]^{<\omega}$, there is a set $H \subseteq \kappa$ of size κ homogeneous for f ($f \restriction [H]^n$ is constant for each $n < \omega$).

A One-Cardinal Ramsey Operator

Let κ be regular (If $S \subseteq \kappa$ is unbounded, $|S| = \kappa$), $I \supset [\kappa]^{<\kappa}$ an ideal on κ. This section is an overview of Feng.

Definition

- The I-positive sets: $I^+ = \{X \subset \kappa \mid X \notin I\}$
- The dual filter to I: $I^* = \{X \subseteq \kappa \mid \kappa \backslash X \in I\}$

Definition

Set $X \subseteq \kappa$ is not in $\mathscr{R}(I)$ iff for every $f : [X]^{<\omega} \to \kappa$ with $f(a) < \min(a)$ for all $a \in [X]^{<\omega}$ there is a set $H \in P(X) \cap I^+$ homogeneous for f ($f \restriction [H]^n$ is constant for all $n < \omega$).

Proposition (Equivalent Characterization of $\mathscr{R}$ Using Clubs)

Set $X \subseteq \kappa$ is not in $\mathscr{R}(I)$ iff for every $f : [X]^{<\omega} \to \kappa$ with $f(a) < \min(a)$ for all $a \in [X]^{<\omega}$ and for every club $C \subseteq \kappa$ there is a set $H \in P(X \cap C) \cap I^+$ homogeneous for f

Useful Definitions

Definition

For $S \subseteq \kappa$, $f : S \to \kappa$ is *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in S$.

Definition

I is *normal* if for every $S \in I^+$ and every regressive $f : S \to \kappa$ there is an $H \in I^+ \cap \mathscr{P}(S)$ such that $f(H)$ is constant.

Definition

Given sets $\langle S_\alpha \subseteq \kappa \mid \alpha < \kappa \rangle$ the *diagonal intersection* is defined

$$\Delta_{\alpha<\kappa} S_\alpha = \{\delta < \kappa \mid \delta \in \bigcap_{\alpha<\delta} S_\alpha\}$$

Fact: I is normal iff for every sequence $\vec{A} = \langle A_\alpha \in I^* \mid \alpha < \kappa \rangle$ we have $\Delta_{\alpha<\kappa} A_\alpha \in I^*$

$\mathscr{R}(I)$ is a Normal Ideal

Theorem (Feng, 1990)

$\mathscr{R}(I)$ is a normal ideal.

Proof.

- Without loss of generality suppose $\mathscr{R}(I)$ is non-trivial, that is $\kappa \notin \mathscr{R}(I)$.
- Assume for $X \in \mathscr{R}(I)^+$ there exists $h : X \to \kappa$ with $h(\alpha) < \alpha$ for all $\alpha \in X$.
- Towards a contradiction assume for every $\alpha < \kappa$ we have $h^{-1}(\alpha) \in \mathscr{R}(I)$.
- Thus for every $\alpha < \kappa$ fix f_α and a club C_α witnessing $h^{-1}(\alpha) \in \mathscr{R}(I)$

$\mathscr{R}(I)$ is a Normal Ideal

- Let $\pi : \kappa \times \kappa \to \kappa$ be a godel pairing function.
- Then the following set is a club

$$C = \Delta_{\alpha<\kappa} C_\alpha \cap \{\alpha < \kappa \mid \pi''\alpha \times \alpha \subseteq \alpha\}$$

hence $X \cap C \in \mathscr{R}(I)^+$

- Define a regressive function $f : [X \cap C]^{<\omega} \to \kappa$ as follows

$$f(\{\alpha\}) = \pi(h(\alpha), f_{h(\alpha)}(\{\alpha\}))$$

$$f(\{\alpha_1, ..., \alpha_n\}) = \begin{cases} f_{h(\alpha_1)}(\{\alpha_1, ..., \alpha_n\}) & \text{if } h(\alpha_1) = ... = h(\alpha_n) \\ 0 & \text{otherwise} \end{cases}$$

$\mathscr{R}(I)$ is a Normal Ideal

- Since $X \cap C \in \mathscr{R}(I)^+$ there is a set $H \in P(X \cap C) \cap I^+$ homogeneous for f
- Thus there is a η such that for all $\alpha \in H$ we have $f(\{\alpha\}) = \eta$
- Thus there exists a $\beta < \kappa$ such that for all $\alpha \in H$ we have $\beta < \alpha$ and $h(\alpha) = \beta$
- Recall

  $$H \subseteq \Delta_{\alpha<\kappa} C_\alpha = \{\delta < \kappa \mid \delta \in \bigcap_{\alpha<\delta} C_\alpha\}$$

  and so

  $$H \cap \{\delta < \kappa \mid \beta < \delta\} \subseteq C_\beta \cap h^{-1}(\beta)$$

- By definition of f_β, H is not homogeneous of f_β
- Thus by definition of f, H is not homogeneous on f, contradiction $\square$

Iterating the One-Cardinal Ramsey Operator

Definition

We inductively define a sequence of ideals as follows

$$\mathscr{R}^0(I) = I$$
$$\mathscr{R}^{\alpha+1}(I) = \mathscr{R}(\mathscr{R}^{\alpha}(I))$$
$$\mathscr{R}^{\alpha}(I) = \bigcup_{\beta<\alpha} \mathscr{R}^{\beta}(I) \text{ when } \alpha \text{ is a limit}$$

A Note on Infinitary Logic

Definition (Bagaria, 2019)

- Σ^1_0 or Π^1_0 formulas contain no second order quantifiers, finitely many first order quantifiers, and finitely many variables
- $\Sigma^1_{\xi+1}$ have the form $\exists X_0 \ldots \exists X_k \varphi(X_0, \ldots X_k)$ where φ is Π^1_ξ.
- $\Pi^1_{\xi+1}$ have the form $\forall X_0 \ldots \forall X_k \varphi(X_0, \ldots X_k)$ where φ is Σ^1_ξ.
- If ξ is a limit ordinal, Σ^1_ξ have the form

$$\bigvee_{\zeta<\xi} \varphi_\zeta \qquad \text{(where each } \varphi_\zeta \text{ is } \Pi^1_\zeta)$$

- If ξ is a limit ordinal, Π^1_ξ have the form

$$\bigwedge_{\zeta<\xi} \varphi_\zeta \qquad \text{(where each } \varphi_\zeta \text{ is } \Sigma^1_\zeta)$$

A Note on Indescribability

Definition

$S \subseteq \kappa$ is Π^1_ξ-*indescribable* if for every Π^1_ξ sentence and $R \subseteq V_\kappa$ with $(V_\kappa, \in, R) \models \varphi$ there is an $\alpha \in S$ such that $(V_\alpha, \in, R \cap V_\alpha) \models \varphi$

Definition

For $\xi < \kappa$ the Π^1_ξ-*indescribability ideal* on κ is given by

$$\Pi^1_\xi(\kappa) = \{X \subseteq \kappa \mid X \text{ is not } \Pi^1_\xi\text{-indescribable in } \kappa\}$$

Definition

Suppose $\vec{I} = \langle I_\alpha \mid \alpha < \kappa \rangle$ where each I_α is an ideal on α. We define an ideal on κ. $S \in \mathscr{R}^{pre}(\vec{I})^+$ iff for every $\subseteq$-regressive $f : [S]^{<\omega} \to \kappa$ and every club $C \subseteq \kappa$ there is an $\alpha \in S \cap C$ such that there is an $H \in P(S \cap C \cap \alpha) \cap I^+_\alpha$ homogeneous for f

Feng's One-Cardinal Hierarchy Results

Theorem

For all $n < \omega$ we have $\kappa \notin \mathscr{R}(\Pi^1_n(\kappa))$ if and only if

- *The ideals $\mathscr{R}^{pre}(\Pi^1_n(\kappa))$ and $\Pi^1_{n+2}(\kappa)$ are nontrivial*
- *The ideal generated by $\mathscr{R}^{pre}(\Pi^1_n(\kappa))$ and $\Pi^1_{n+2}(\kappa)$ is a nontrivial normal ideal, and in this case*

$$\mathscr{R}(\Pi^1_n(\kappa)) = \overline{\mathscr{R}^{pre}(\Pi^1_n(\kappa)) \cup \Pi^1_{n+2}(\kappa)}$$

Theorem

If $n, m < \omega$ and $\kappa \in \mathscr{R}^{n+1}(\Pi^1_m(\kappa))^+$ then for all $i < n$, $X \in \mathscr{R}^i(\Pi^1_m(\kappa))^+$ it follows that the set

$$\{\alpha < \kappa \mid X \cap \alpha \in \mathscr{R}^i(\Pi^1_m(\alpha))^+\}$$

is contained in $\mathscr{R}^{i+1}(\Pi^1_m(\kappa))^*$ but not in $\mathscr{R}^i(\Pi^1_m(\kappa))^*$

A Note on Canonical Sequences

Feng was able to extend these results using canonical sequences.

Definition

For $f, g : \kappa \to \kappa$ we say $f < g$ iff

$$\{\alpha < \kappa \mid f(\alpha) < g(\alpha)\}$$

contains a club (with $\leq$ defined similarly).

Definition

Sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ of functions $f_\alpha : \kappa \to \kappa$ is *canonical* if

- For all $\alpha, \beta < \kappa^+, \alpha < \beta$ implies $f_\alpha < f_\beta$
- For all sequences $\langle g_\alpha \mid \alpha < \kappa^+ \rangle$ of functions $g_\alpha : \kappa \to \kappa$ such that $\alpha, \beta < \kappa^+, \alpha < \beta$ implies $g_\alpha < g_\beta$ we have $f_\alpha \leq g_\alpha$

Note: There are canonical sequences at each regular cardinal $\kappa > \omega$

The One-Cardinal Hierarchy

Theorem

If $\alpha < \kappa^+$, $\xi < \kappa$ and $\kappa \in \mathscr{R}^{\alpha+1}(\Pi^1_\xi(\kappa))^+$ then for all $\beta < \alpha$ and for all $X \in \mathscr{R}^\beta(\Pi^1_\xi(\kappa))^+$ it follows that the set

$$\{\gamma < \kappa \mid X \cap \gamma \in \mathscr{R}^{f_\beta(\gamma)}(\Pi^1_\xi(\gamma))^+\}$$

is contained in $\mathscr{R}^{\beta+1}(\Pi^1_\xi(\kappa))^*$ but not in $\mathscr{R}^\beta(\Pi^1_\xi(\kappa))^*$

Thus iterating $\mathscr{R}$ up to $\alpha < \kappa^+$ on forms a proper hierarchy on a very large class of ideals.

Background

Let κ be inaccessible, λ a regular cardinal, $A \subseteq \lambda$ with $|A| \geq \kappa$.

Definition

$P_\kappa A$ are the subsets of A of size $< \kappa$

Definition

For $x, y \in P_\kappa A$ let $x \sqsubset y$ iff $x \subseteq y$ and $|x| < |y \cap \kappa|$ (equivalently iff $x \in P_{\kappa_y} y$ where $\kappa_y = |y \cap \kappa|$.)

Definition

$S \subseteq P_\kappa A$ is unbounded in $P_\kappa A$ if for every $x \in P_\kappa A$ there is a $y \in S$ such that $x \sqsubset y$

Proposition

The collection $I_{\kappa,A} = \{X \subseteq P_\kappa A \mid X \text{ is not unbounded}\}$ is a nontrivial ideal on $P_\kappa A$

Background

Let $I \supseteq I_{\kappa,A}$ be an ideal on $P_\kappa A$. Notice

- $I^+_{\kappa,A}$ is the set of unbounded subsets of $P_\kappa A$, and
- $I^*_{\kappa,A} = \{\hat{x} \mid x \in P_\kappa A\}$ where $\hat{x} = \{y \in P_\kappa A \mid x \sqsubset y\}$ is the filter dual to $I_{\kappa,A}$.
- I^+ and I^* are defined similarly.

Definition

Given $S \subseteq P_\kappa A$ we definite

$$[S]^{<\omega}_{\sqsubset} = \bigcup_{n<\omega} \{(x_1, ..., x_n) \in S^n \mid x_1 \sqsubset ... \sqsubset x_x\}$$

Definition

A function $[S]^{\leq\omega}_{\sqsubset} \to P_\kappa A$ is $\sqsubset$-regressive if $f(x_1, ..., x_n) \sqsubset x_1$ for all $(x_1, ..., x_n) \in [S]^{\leq\omega}_{\sqsubset}$

A Two-Cardinal Ramsey Operator

Definition

$S \in \mathscr{R}_\sqsubset(I)^+$ iff every $\sqsubset$-regressive function $f : [S]^{<\omega}_\sqsubset \to P_\kappa A$ has a homogeneous set $H \subseteq S$ in I^+

An equivalent characterization of $\mathscr{R}_\sqsubset(I)$ uses weak clubs

Proposition

$S \in \mathscr{R}_\sqsubset(I)^+$ iff for every $\sqsubset$-regressive function $f : [S]^{<\omega}_\sqsubset \to P_\kappa A$ and every weak club C in $P_\kappa A$ there is a set $H \subseteq S \cap C$ in I^+ homogeneous for f

Definition

I is *strongly normal* if for all $S \in I^+$ and all $\sqsubset$-regressive $f : S \to P_\kappa A$ there is a $T \in P(S) \cap I^+$ homogeneous for f

$\mathscr{R}_\sqsubset(I)$ is a Strongly Normal Ideal

Theorem (Cody, White, 2022)

$\mathscr{R}_\sqsubset(I)$ is a strongly normal ideal on $P_\kappa A$.

Proof.

- Without loss of generality suppose $\mathscr{R}_\sqsubset(I)$ is nontrivial, that is $P_\kappa A \notin \mathscr{R}_\sqsubset(I)$
- Suppose $X \in \mathscr{R}_\sqsubset(I)^+$ and $h : X \to P_\kappa A$ is $\sqsubset$-regressive, that is $h(x) \sqsubset x$ for all $x \in X$
- Towards a contradiction assume for each $y \in P_\kappa A$ we have $h^{-1}(y) \in \mathscr{R}_\sqsubset(I)$
- Thus for every $y \in P_\kappa A$ fix a $\sqsubset$-regressive f_y and a weak club C_y witnessing $h^{-1}(y) \in \mathscr{R}_\sqsubset(I)$

$\mathscr{R}_\sqsubset(I)$ is a Strongly Normal Ideal

- Let $\pi : P_\kappa A \times P_\kappa A \to P_\kappa A$ be a pairing function
- Then the following set is a weak club

$$C = \Delta_\sqsubset \{C_y | y \in P_\kappa A\} \cap \{\pi'' P_{\kappa_x} x \times P_{\kappa_x} x \subseteq P_{\kappa_x} x\}$$

  hence $X \cap C \in \mathscr{R}_\sqsubset(I)^+$
- Define a $\sqsubset$-regressive $f : [X \cap C]^{\leq\omega}_\sqsubset \to P_\kappa A$ by letting

$$f(\{x\}) = \pi(h(x), f_{h(x)}(\{x\}))$$

$$f(x_1, ..., x_n) = \begin{cases} f_{h(x_1)}(x_1, ..., x_n) & \text{if } h(x_1) = ... = h(x_n) \\ 0 & \text{otherwise} \end{cases}$$

$\mathscr{R}_\sqsubset(I)$ is a Strongly Normal Ideal

- Since $X \cap C \in \mathscr{R}_\sqsubset(I)^+$, there is an $H \in P(X \cap C) \cap I^+$ which is homogeneous for f.
- Thus there is $z \in P_\kappa A$ such that $f(\{x\}) = z$ for all $x \in H$.
- Thus there is a $y \in P_\kappa A$ such that $h(x) = y \sqsubset x$ for all $x \in H$
- By definition of diagonal intersection we have

$$H \cap \{a \in P_\kappa A \mid y \sqsubset a\} \subseteq C_y \cap h^{-1}(y)$$

- By definition of f_y it follows that H is not homogeneous for f_y
- But this implies there is $x_1, x_2 \in H$ such that $\pi(y, f_y(\{x_1\}) \neq \pi(y, f_y(\{x_2\}))$
- Hence $f(\{x_1\}) \neq f(\{x_2\})$ and so H is not homogeneous for f, contradiction. $\square$

Two-Cardinal Inductive Definitions

Definition (Two-Cardinal Cumulative Hierarchy up to κ)

$$V_0(\kappa, A) = A$$
$$V_{\alpha+1}(\kappa, A) = P_\kappa(V_\alpha(\kappa, A)) \cup V_\alpha(\kappa, A)$$
$$V_\alpha(\kappa, A) = \bigcup_{\beta<\alpha} V_\beta(\kappa, A) \text{ for } \alpha \text{ a limit}$$

Definition(Iterating the Two-Cardinal Ideal Operator)

$$\mathscr{R}^0_\sqsubset(I) = I$$
$$\mathscr{R}^{\alpha+1}_\sqsubset(I) = \mathscr{R}_\sqsubset(\mathscr{R}^\alpha_\sqsubset(I))$$
$$\mathscr{R}^\alpha_\sqsubset(I) = \bigcup_{\beta<\alpha} \mathscr{R}^\beta_\sqsubset(I) \text{ when } \alpha \text{ is a limit}$$

Important Definitions

Definition

$S \subseteq P_\kappa A$ is Π^1_ξ-*indescribable* in $P_\kappa A$ if whenever $(V_\kappa(\kappa, A), \in, R) \models \varphi$ where $R \subseteq V_\kappa(\kappa, A)$ and φ is a Π^1_ξ sentence, there is an $x \in S$ such that

$$x \cap \kappa = |x \cap \kappa| \text{ and } (V_{\kappa_x}(\kappa_x, x), \in, R \cap V_{\kappa_x}(\kappa_x, x)) \models \varphi$$

Definition

The Π^1_ξ-*indescribability ideal* on $P_\kappa A$ is the collection

$$\Pi^1_\xi(\kappa, A) = \{X \subseteq P_\kappa A \mid X \text{ is not } \Pi^1_\xi\text{-indescribable in } P_\kappa A\}$$

Two-Cardinal Hierarchy Results

Definition

Suppose κ is regular, $\kappa \leq |A|$. Further suppose $\vec{I} = \langle I_x \mid x \in P_\kappa A \rangle$ where for every $x \in P_\kappa A$, I_x is an ideal on $P_{\kappa_x} x$. We define an ideal on $P_\kappa A$. $S \in \mathscr{R}^{pre}_{\sqsubset}(\vec{I})^+$ iff for every $\sqsubset$-regressive $f : [S]^{\leq\omega}_{\sqsubset} \to P_\kappa A$ and every weak club $C \subseteq P_\kappa A$ there is some $x \in S \cap C$ such that there is an $H \in P(S \cap C \cap P_{\kappa_x} x) \cap I_x^+$ homogeneous for f.

Theorem (Cody, White, 2022)

For all $n < \omega$, we have $P_\kappa A \notin \mathscr{R}_{\sqsubset}(\Pi^1_n(\kappa, A))$ if and only if

- The ideals $\mathscr{R}^{pre}_{\sqsubset}(\Pi^1_n(\kappa, A))$ and $\Pi^1_{n+2}(\kappa, A)$ are nontrivial
- The ideal generated by $\mathscr{R}^{pre}_{\sqsubset}(\Pi^1_n(\kappa, A))$ and $\Pi^1_{n+2}(\kappa, A)$ is a nontrivial strongly normal ideal, and in this case

$$\mathscr{R}_{\sqsubset}(\Pi^1_n(\kappa, A)) = \overline{\mathscr{R}^{pre}_{\sqsubset}(\Pi^1_n(\kappa, A)) \cup \Pi^1_{n+2}(\kappa, A)}$$

Two-Cardinal Hierarchy Results

Note: There are some additional facts about canonical functions used to prove the following.

Theorem (Cody, White, 2022)

If $\xi < \kappa$, $\alpha < |A|^+$, then for all $\beta < \alpha$ and for all $X \in \mathscr{R}^{\beta}_{\sqsubset}(\Pi^1_\xi(\kappa, A))^+$ it follows that the set

$$\{x \in P_\kappa A \mid X \cap P_{\kappa_x} x \in \mathscr{R}^{f_\beta(x)}_{\sqsubset}(\Pi^1_\xi(\kappa_x, x))^+\}$$

is contained in $\mathscr{R}^{\beta+1}_{\sqsubset}(\Pi^1_\xi(\kappa, A))^*$ but not $\mathscr{R}^{\beta}_{\sqsubset}(\Pi^1_\xi(\kappa, A))^*$

Open Question

- It can be shown that there exist cardinals $\kappa \leq \lambda$ such that $\mathscr{I}^2_{\sqsubset}(I_{\kappa,\lambda})$ being nontrivial is strictly stronger in consistency strength than the existence of cardinals $\kappa \leq \lambda$ for which $\mathscr{I}_{\sqsubset}(\Pi^1_\beta(\kappa,\lambda))$ is nontrivial for all $\beta < \kappa$
- It is not known if the analogous result holds for the $\mathscr{R}_{\sqsubset}$ operator

Works Cited

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