Topological Mixing and Linear Recurrence on SMART

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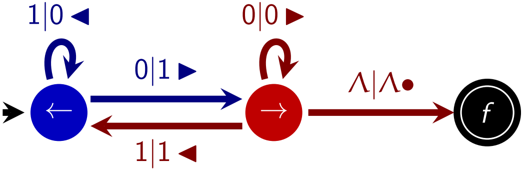
UNIVERSIDAD DEL BÍO-BÍO

Content

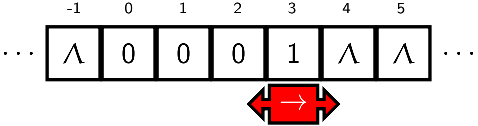
Traditional Turing Machine

Turing machine is a six-tuple: $T = (Q, \Sigma, q_0, F, \Lambda, \delta)$

Partial Transition function: $\delta : Q \times \Sigma \cup \{\Lambda\} \to Q \times \Sigma \cup \{\Lambda\} \times \{\blacktriangleleft, \bullet, \blacktriangleright\}$



Configuration: $(Q, \mathbb{Z}, (\Sigma \cup \{\Lambda\})^{\mathbb{Z}})$



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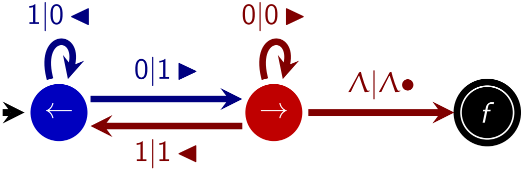

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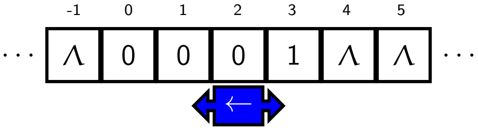

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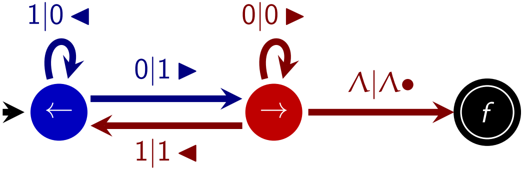

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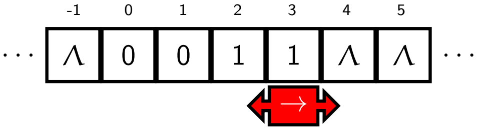

Converting to Turing Machine with Moving Tape (TMT)

Turing machine is a **four**-tuple: $T = (Q, \Sigma, \Lambda, \delta)$

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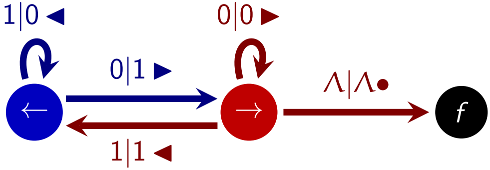

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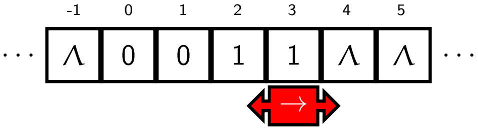

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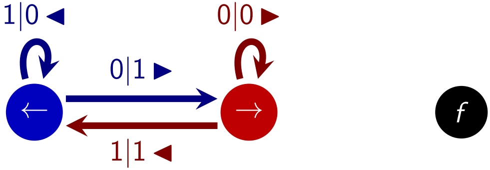
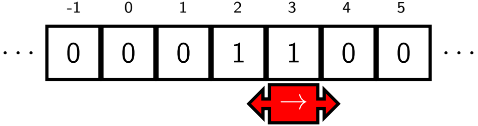


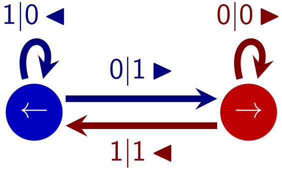
Configuration: $(Q, \mathbb{Z}, \boldsymbol{\Sigma}^{\mathbb{Z}})$



Converting to Turing Machine with Moving Tape (TMT)

Complete Turing machine: $T = (Q, \Sigma, \delta)$

Total Transition function: $\delta : Q \times \Sigma \to Q \times \Sigma \times \{◀, •, ▶\}$



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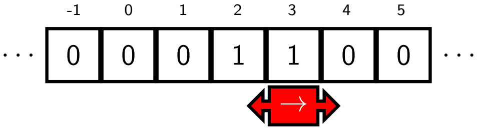

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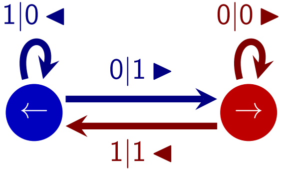

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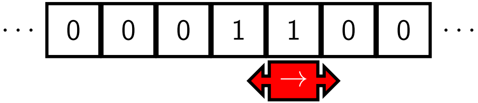

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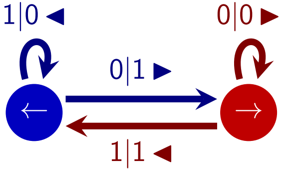
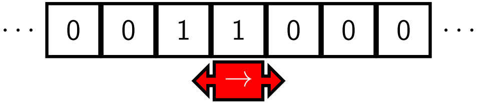


Configuration: $({}^{\omega}\boldsymbol{\Sigma}, \boldsymbol{Q}, \boldsymbol{\Sigma}^{\omega})$



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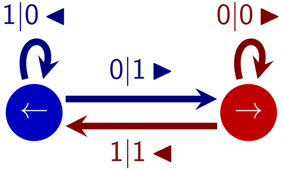

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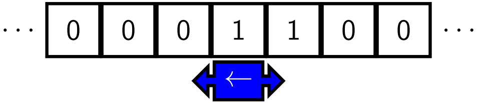

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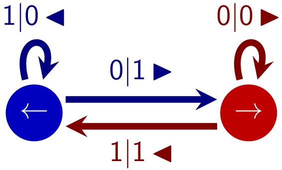

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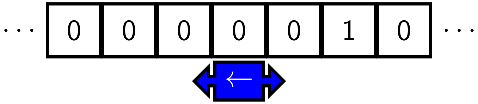
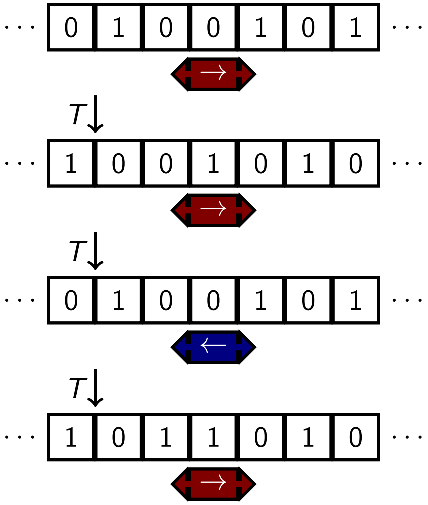


Column Factor of TMT (t-shift)



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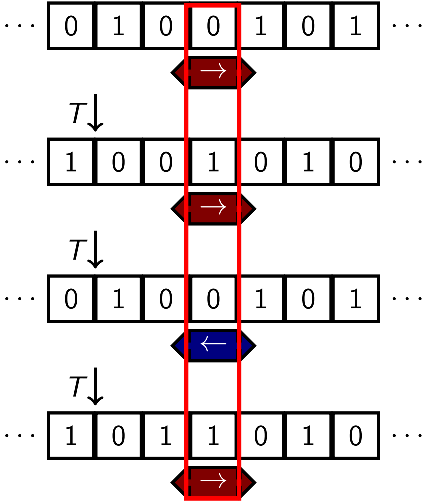

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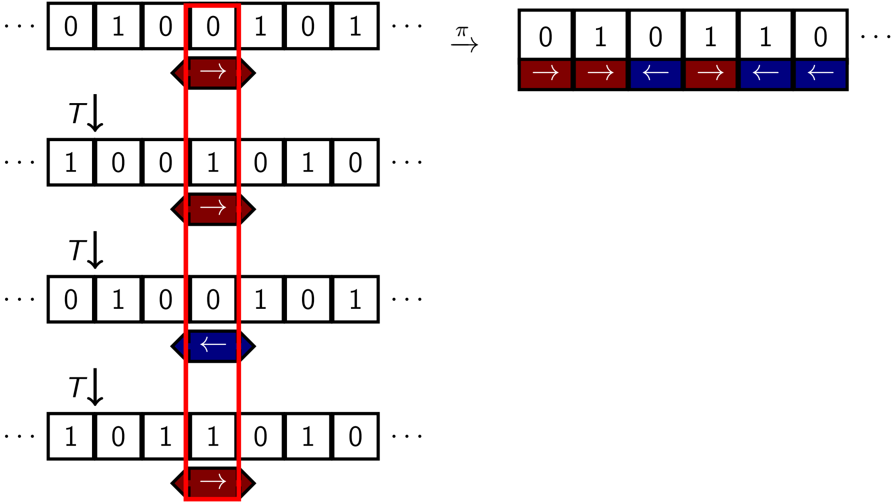

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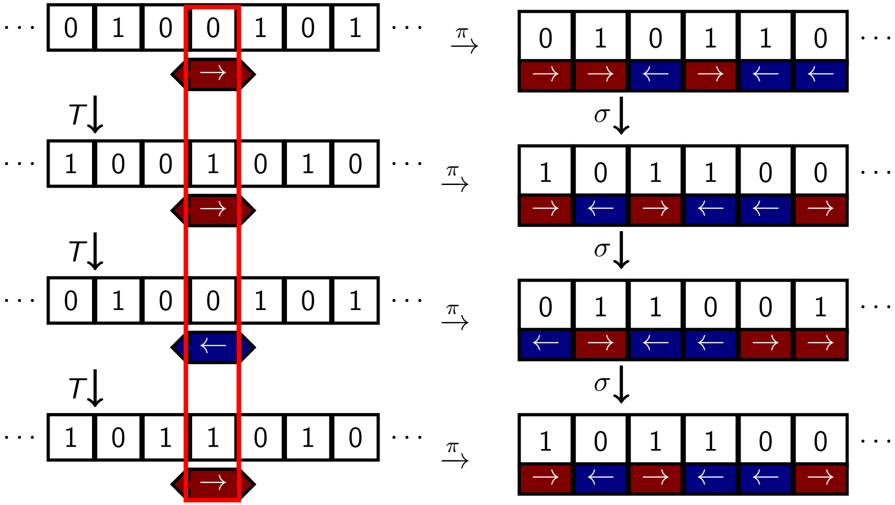

Content

Then, formally speaking...

- Considering Turing machine $T = (Q, \Sigma, \delta)$, the pair $(X = ({}^{\omega}\Sigma, Q, \Sigma^{\omega}), T)$ is a Topological Dynamical System known as *Turing Machine with moving Tape*.
- X is endowed with the cantor metric, being compact and perfect (no isolated points).
- T (to apply a Turing instruction) is a continuous map.
- Considering $\pi(u, q, v) = (q, v_0)$, the subshift known as *t-shift* (S_t, σ) is the factor $\tau : X \to S_t$, defined as $\tau(x) = (\pi(T^n(x)))_{n\in\mathbb{N}}$

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Interesting questions already answer!

- Is there an aperiodic t-shift? $(\forall w \in S_t, \forall n \in \mathbb{N} : w \neq \sigma^n(w))$
  Answer: Yes! (SMART machine)
- Is there a minimal $t-$shift? $(\forall w \in S_t, \forall v \in \mathcal{L}(S_t) : v \sqsubseteq w$[i]$)$
  Answer: Yes! (SMART machine)
- Is possible to decide if a t-shift is aperiodic and/or minimal?
  Answer: No

[i] This means v is a *subword or factor* of w: $\exists i : v = w_i w_{i+1} \ldots w_{i+|w|-1}$.

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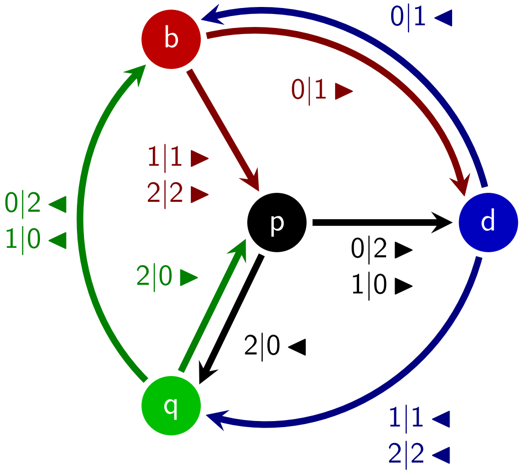
SMART machine



Known Behavior [CO**T*** '17]

- SMART machine T is minimal, aperiodic and substitutive.
- State b has the same behavior than d, but in the opposite direction. Same as p with q.
- SMART machine pushes a one to the right $\exists n \in \mathbb{N} : T^n \begin{pmatrix} 1 \underset{p}{0} 0^k \; 1 \; 0 \end{pmatrix} = \begin{pmatrix} 1 \underset{p}{0} 0^{k+1} \; 1 \end{pmatrix}$ in a recursive way.

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Decidable results [GO**T*** '15]

- *Innocuous:* Incomplete Turing machine which halts with the same tape content which it starts.
- *Embedding:* Insert a innocuous machine inside a Complete Turing machine.
- With certain cares, aperiodicity and minimality can depend on the aperiodicity and mortality[ii] of the innocuous machine, respectively.
- Every known Minimal Turing machine is an embedding of SMART machine (or BinSmart, a two symbol version of the same dynamic).

[ii] A Turing machine is *Mortal* if $(\forall x \in X)(\exists n \in \mathbb{N}) : T^n(x)$ is undef.

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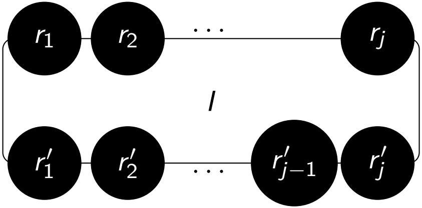

- Consider $\forall i \in \{1, ..., j\}$: r'_i: Starting states. r_i: Halting states.

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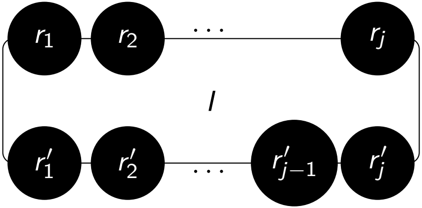
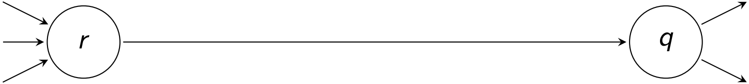


- Consider $\forall i \in \{1, ..., j\}$: r'_i: Starting states. r_i: Halting states.
- Machine I needs to be *innocuous*: $(\forall i \in \{1, ..., I\}|) : (u, r_i, v)$ evolves indefinitely, or $\exists t \in \mathbb{N} : T^t(u, r_i, v) = (u, r'_i, v)$ and it is a halting configuration.

Embedding

- Machine H:



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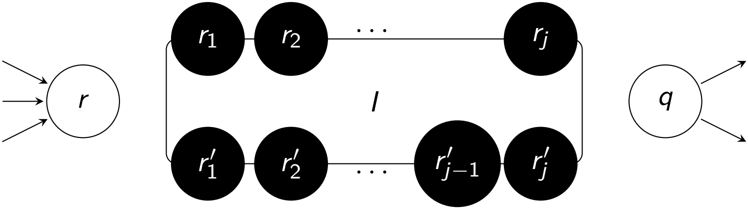

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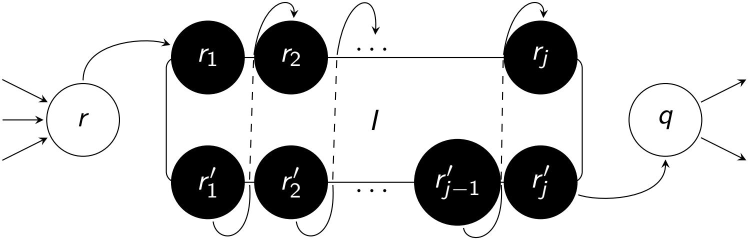

Complexity

- If machine H is Minimal, then H_I[iii] is Minimal if and only if I is mortal.
- The time machine H_I takes inside machine I before returning to machine H is bounded by $j * t_I$[iv].
- Therefore, exists an infinite amount of Minimal Turing machines, embedding mortal machines into SMART (or binSMART).

[iii] Machine I embedded into machine H
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Questions not answered yet

- Is there a minimal t-shift which is topological mixing? $(\forall u, v \in \mathcal{L}(S_t), \exists n \in \mathbb{N}, \forall N > n, \exists w \in \mathcal{L}(S_t) : |uw| = N \wedge uwv \in \mathcal{L}(S_t))$
- Is top. mixing decidable in t-shifts?[v]
- Is weak mixing decidable in minimal t-shifts? $((S_t \times S_t, \sigma \times \sigma)$ is transitive[vi].)

[v] An easy example of a non-minimal top. mixing t-shift is the full shift machine.

[vi] $(\exists w \in S_t, \forall v \in \mathcal{L}(S_t) : v \sqsubseteq w)$

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Considerations before the proof

- We center in top. mixing in TMT, as t-shift is indeed a factor.
- Let us declare $\overset{*}{X} = ({}^*\Sigma \times Q \times \Sigma^*)$. In this way, the open sets in X are defined by elements $u \in \overset{*}{X}$, as:

$$[u] = \{x \in X : u \sqsubseteq x^{\text{vii}}\}$$

- The *return time* in a Turing machine T for a given $u \in \overset{*}{X}$ is $R_T(u) = max\{n : (\forall w \sqsupseteq u) n = min\{m > 0 : T^m(w) \sqsupseteq u\}\}$. If T is minimal, then $R_T(u)$ is finite for all u. This concept can also be applied in subshifts.

[vii] This means that, if $u = (v, r, v')$, then $(\exists w \in^\omega \Sigma, \exists w' \in^\omega \Sigma)$ such that $x = (wv, r, v'w')$

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- The *return time* in a Turing machine T for a given $u \in \overset{*}{X}$ is $R_T(u) = max\{n : (\forall w \sqsupseteq u) n = min\{m > 0 : T^m(w) \sqsupseteq u\}\}$. If T is minimal, then $R_T(u)$ is finite for all u. This concept can also be applied in subshifts.

[vii] This means that, if $u = (v, r, v')$, then $(\exists w \in^\omega \Sigma, \exists w' \in^\omega \Sigma)$ such that $x = (wv, r, v'w')$

Considerations before the proof

- We center in top. mixing in TMT, as t-shift is indeed a factor.
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Main Theorem

Theorem

SMART machine has a topological Mixing TMT.

Proof Sketch

Topological Mixing: $(\forall u, v)(\exists n)(\forall N > n)(\exists u' \sqsupseteq u) : T^N(u') \sqsupseteq v$

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Let us use: $\forall u, v \in ({}^*\Sigma \times Q \times \Sigma^*)$, $\exists k$ big enough such that:

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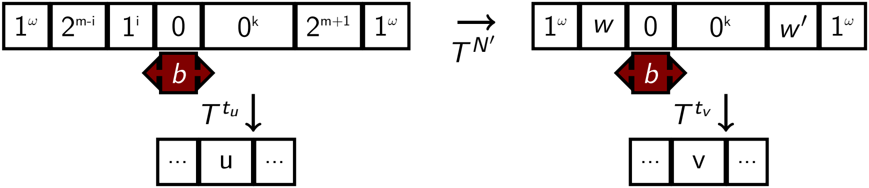
Now, $m = R_T(, b, 00^k)$, $\forall N' > J(k, m)$[viii], $\exists i \leq m$ and $w, w' \in \Sigma^*$:



[viii] $J(k, m) = \frac{3^{k+3}}{4}(3^{2m} - 1) - m$

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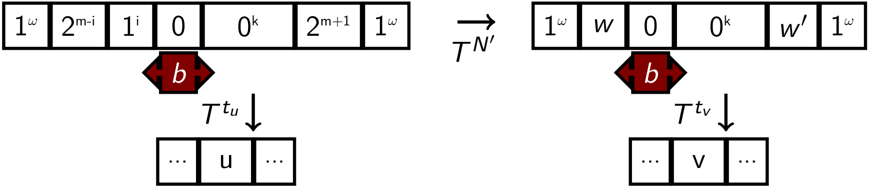

Therefore, $n = J(k, m)$, $N = N' - t_u + t_v$ and
$u' = T^{t_u}(1^\omega 2^{m-i} 1^i, b, 00^k 2^{m+1} 1^\omega)$.

[viii] $J(k, m) = \frac{3^{k+3}}{4}(3^{2m} - 1) - m$

Content

Theorem

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Every Mixing notion is undecidable in t-shifts.

Proof Sketch

- Consider full shit machine $H = (\{r_0\}, \{0, 1, 2\}, \forall i : \delta(r_0, i) = (r_0, i, \blacktriangleright))$.
- Consider I any binary innocuous machine.
- Replace instruction $\delta(r_0, 2) = (r_0, 2, \blacktriangleright))$ to form the embedding H_I.
- As machine H_I moves freely to the right with symbol 0 (or 1), then H_I is top. mixing if and only if I is aperiodic[ix].

[ix] We are referring to aperiodic seeing a finite amount of tape. If it is not, H_I is not even transitive

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Content

Linearly Recurrent

- $x \in \Sigma^{\mathbb{N}}$ is *linearly recurrent* if $\exists K \in \mathbb{N}, \forall u \sqsubseteq x : R_x(u) \leq K|u|$.
- A minimal subshift S is said to be *linearly recurrent* if it posses an element $w \in S$ which is linearly recurrent.
- In particular, if a minimal subshift S is linearly recurrent, then $\forall w \in S : w$ is linearly recurrent.

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Linearly Recurrent in t-shifts

- Every minimal subshift product of a substitution is linearly recurrent [Durant].
- Weak Mixing is decidable in linearly recurrent subshifts [Durant].
- SMART machine has a minimal and substitutive t-shift [CO**T*** 17'], therefore it is linearly recurrent.
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Linear Recurrence Lemma

Lemma

Embedding a Mortal Turing machine on a Linearly Recurrent Turing machine is Linearly Recurrent.

Proof Sketch

- Consider minimal and linearly recurrent machine H.
- Consider I any Innocuous mortal machine with j pairs.
- Consider any $u \in \mathcal{L}(S_{H_I})$ and consider $w \sqsupseteq u$ any extension of u that does not repeat u (u is prefix of w and $|w|_u = 1$). As H_I is minimal, we know that $|w|$ is always finite.

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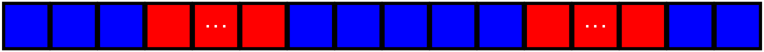
$w =$ 

As we stated, the time it takes inside I (in red) is bounded by $j * t_I$ in every call.

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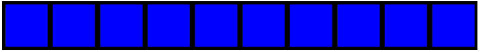
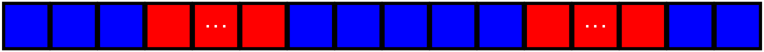
$w' =$ 

If we take out the red parts, we get $w' \in \mathcal{L}(S_H)$. We know that every possible w' is bounded by some $K * |u'|$ (u without red parts).

Proof Sketch

- Consider minimal and linearly recurrent machine H.
- Consider I any Innocuous mortal machine with j pairs.
- Consider any $u \in \mathcal{L}(S_{H_I})$ and consider $w \sqsupseteq u$ any extension of u that does not repeat u (u is prefix of w and $|w|_u = 1$). As H_I is minimal, we know that $|w|$ is always finite.

$w =$ 

Therefore, every $w \leq K * |u| * j * t_I = (K * j * t_I) * |u|$. Ergo, H_I is linearly recurrent.

Conjecture

Conjecture

Weak Mixing is decidable in minimal t-shifts

Content

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CONICYT
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A Ω

**Dirección General de Investigación,
Desarrollo e Innovación**

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