Recent progress on distance sets in the plane

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This talk

Geometric measure theory

- Quantifies the size of small/irregular sets (e.g., Cantor set, Sierpinski triangle, etc.).
- Uses fractal dimensions to measure the size of a set.
- Most prominent dimensions: Hausdorff, $\dim_H$, and packing, $\dim_P$.

Effective dimension

- Studies the intrinsic randomness of infinite objects (binary sequences, Euclidean points, etc.).
- Two main measures quantifying the amount of randomness: dim and Dim.
- Effective versions of Hausdorff and packing dimension.

We can use effective dimension to prove (non-effective) theorems in geometric measure theory. In this talk, we show how to use effective methods to improve best known bounds for *pinned distance sets*.

Kolmogorov complexity in Euclidean space

Fix a universal TM U.

- Let $r \in \mathbb{N}$, and $x \in \mathbb{R}$. The *Kolmogorov complexity of x at precision r* is

$$\begin{aligned} K_r(x) &= \text{ length of the shortest input } \pi \text{ such that } U(\pi) = d_x \\ &\approx \text{ the minimum number of bits to specify first } r \text{ bits of } x. \end{aligned}$$

  where $d_x = \frac{m}{2^r}$ is the closest dyadic rational at precision r to x.
- Can generalize this to $\mathbb{R}^n$.
- The *Kolmogorov complexity of x at precision r given y at precision t* is

$$\begin{aligned} K_{r,t}(x \mid y) &= \text{ length of the shortest input } \pi \text{ such that } U(\pi, d_y) = d_x \\ &\approx \text{ the minimum number of bits to specify first } r \text{ bits of } x \text{ if you know} \\ &\qquad \text{first t bits of } y. \end{aligned}$$

  where $d_y = (\frac{m_1}{2^t}, \dots, \frac{m_n}{2^t})$ is the closest dyadic rational at precision t to y.

Kolmogorov complexity in Euclidean space

- For every $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$, $0 \leq K_r(x) \leq nr + O(\log r)$.
    - If x is computable, then $K_r(x) = O(\log r)$
    - Almost every point satisfies $K_r(x) = r - O(\log r)$ for every $r \in \mathbb{N}$. We call these points *random*.
- Symmetry of information: For every $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $r, t \in \mathbb{N}$,

$$K_{r,t}(x, y) = K_t(y) + K_{r,t}(x \mid y) + O(\log r + \log t).$$

- We can *relativize* the definitions in the natural way to get $K^A_r(x), K^A_{r,t}(x \mid y), \ldots$ for any oracle $A \subseteq \mathbb{N}$.

Effective dimensions of points

Definition (Lutz '03, Mayordomo '03)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective Hausdorff) dimension of* x is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

Definition (Athreya et al. '07, Lutz and Mayordomo '08)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective) strong dimension of* x is

$$\mathrm{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}.$$

The effective dimensions of a point x measure the density of algorithmic information in x.

Dimension of distances

Let $x, y \in \mathbb{R}^n$. Let $d_x := \dim(x)$ and $d_y := \dim^x(y)$.

Question: What is $d := \dim^x(|x-y|)$?

1. $0 \leq d \leq \min\{1, d_y\}$.
2. If $d_y = n$, then $d = 1$.
3. This is a natural (even innocuous) question. The analogous (classical) question is one of the most fundamental open problems in geometric measure theory: Falconer's distance set conjecture.

The point-to-set principle

Theorem (J. Lutz and N. Lutz, '16)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \textit{ and}$$

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \text{Dim}^A(x).$$

- The Hausdorff and packing dimension of a *set* is characterized by the corresponding dimension of the *points* in the set.
- Allows us to use algorithmic techniques to answer questions in geometric measure theory.

Distance sets

Let $E \subseteq \mathbb{R}^n$. The distance set of E is

$$\Delta E = \{|x - y| \mid x, y \in E\}.$$

More generally, if $x \in \mathbb{R}^n$, the pinned distance of E w.r.t. x is

$$\Delta_x E = \{|x - y| \mid y \in E\}.$$

Distance sets

When E is a finite set, Erdös conjectured that $|\Delta E|$ is at least (almost) linear in terms of $|E|$.

- In a breakthrough paper, Guth and Katz proved that $|\Delta E| \gg \frac{|E|}{\log |E|}$.
- Still an important open problem for $\mathbb{R}^n$ with $n \geq 3$.

Falconer posed an analogous question for the case that E is infinite, known as Falconer's *distance set problem*.

- If $E \subseteq \mathbb{R}^n$ has $\dim_H(E) > n/2$, then ΔE has positive measure.
- Still open in all dimensions.
- Guth, Iosevich, Ou and Wang proved that if $E \subseteq \mathbb{R}^2$ and $\dim_H(E) > 5/4$, then $\mu(\Delta E) > 0$.

Known bounds - high dimensional case

Substantial progress has been made in a slightly different direction, on the Hausdorff dimension of pinned distance sets in the plane. We assume $E \subseteq \mathbb{R}^2$ is Borel (or analytic).

- (Shmerkin) If $\dim_H(E) > 1$ and $\dim_H(E) = \dim_P(E)$, then
$$\sup_{x \in E} \dim_H(\Delta_x E) = 1.$$
- (Liu) If $\dim_H(E) = d \in (1, 5/4)$, then $\sup_{x \in E} \dim_H(\Delta_x E) \geq \frac{4}{3}d - \frac{2}{3}$.
- (S.) If $\dim_H(E) =: d > 1$, then
$$\sup_{x \in E} \dim_H(\Delta_x E) \geq \frac{d}{4} + \frac{1}{2}$$

Our results - high dimensional case

Let $E \subseteq \mathbb{R}^2$ be analytic and $1 < d < \dim_H(E)$.

- There is a subset $F \subseteq E$ of full dimension such that, for all $x \in F$,
$$\dim_H(\Delta_x E) \geq \frac{d(4-d)}{5-d}$$
  - This improves the best known bounds when $\dim_H(E) \in (1, 1.127)$.
- For all x outside a set of dimension 1
$$\dim_H(\Delta_x E) \geq \frac{\dim_P(E)+1}{2\dim_P(E)}.$$
- If $\dim_P(E) < \frac{d(3+\sqrt{5})-1-\sqrt{5}}{2}$, then for all x in a subset of full dimension $\dim_H(\Delta_x E) = 1$.
- There is a point $x \in E$ such that
$$\dim_P(\Delta_x E) \geq \frac{12-\sqrt{2}}{8\sqrt{2}} \approx 0.9356$$
  - Improves (slightly) the Keleti-Shermkin bound for packing dimension.

Known bounds - low dimensional case

We now consider the "low" dimensional case, when $\dim_H(E) \leq 1$. We assume $E \subseteq \mathbb{R}^2$ is Borel (or analytic).

- (Shmerkin and Wang) If $\dim_H(E) =: d \leq 1$ and $\dim_H(E) = \dim_P(E)$, then
$$\sup_{x\in E} \dim_H(\Delta_x E) = d.$$
- (Shmerkin and Wang) If $\dim_H(E) =: d \leq 1$, then
$$\sup_{x\in E} \dim_H(\Delta_x E) \geq \frac{d}{2} - \frac{d^2}{2(2+\sqrt{d^2+4})}.$$
- (Du, Ou, Ren and Zhang) If $\dim_H(E) = d \leq 1$, then
$$\sup_{x\in E} \dim_H(\Delta_x E) \geq \frac{5}{3}d - 1$$

Our results - low dimensional case

Theorem (Fiedler and S.)

Suppose $E \subseteq \mathbb{R}^2$ *is analytic set,* $d := \dim_H(E) \leq 1$ *and* $D := \dim_P(E)$. *Then,*

$$\sup_{x \in E} \dim_H(\Delta_x(E)) \geq d \left(1 - \frac{\alpha D - d(D+\alpha-d)}{(d+1)(\alpha D - d^2) - d^2(\alpha + D - 2d)}\right),$$

where $\alpha = \min\{1 + d, D\}$.

Corollary

Suppose $E \subseteq \mathbb{R}^2$ *is analytic set and* $d := \dim_H(E) \leq 1$. *Then,*

$$\sup_{x \in E} \dim_H(\Delta_x(E)) \geq d \left(1 - \frac{2-d}{2(1+2d-d^2)}\right).$$

Universal sets

We can generalize the problem, by considering a *pinned set* X, and a *test set* Y, and investigating

$$\sup_{x\in X} \dim_H(\Delta_x Y).$$

Let $X \subseteq \mathbb{R}^2$, and let $\mathcal{C}$ be a class of subsets of $\mathbb{R}^2$ e.g. Borel sets, analytic sets, or weakly regular sets. We say that X is *universal for pinned distances* for $\mathcal{C}$ if, for every $Y \in \mathcal{C}$ there exists some $x \in X$ such that

$$\dim_H(\Delta_x(Y)) = \min\{\dim_H(Y), 1\} \tag{1}$$

Question: Are there "small" universal sets for the class of Borel subsets of $\mathbb{R}^2$?

Universal sets

Theorem (Fiedler and S.)

Let $X \subseteq \mathbb{R}^2$ be a compact, Ahlfors-David regular set such that $\dim_H(X) > 1$. Then X is universal for pinned distances for the class of Borel sets $Y \subseteq \mathbb{R}^2$.

In particular, if $X \subseteq \mathbb{R}^2$ is a four-corner Cantor set with $\dim_H(X) > 1$, then, *for every* Borel $Y \subseteq \mathbb{R}^2$, there is a point $x \in X$ such that

$$\dim_H(\Delta_x Y) = \min\{\dim_H(Y), 1\}.$$

We can slightly weaken the assumption that X is AD-regular.

Our results (low dimensional case)

Theorem (Fiedler and S.)

Suppose $E \subseteq \mathbb{R}^2$ is analytic set, $d := \dim_H(E) \leq 1$ and $D := \dim_P(E)$. Then,

$$\sup_{x \in E} \dim_H(\Delta_x(E)) \geq d\left(1 - \frac{\alpha D - d(D + \alpha - d)}{(d+1)(\alpha D - d^2) - d^2(\alpha + D - 2d)}\right),$$

where $\alpha = \min\{1 + d, D\}$.

Corollary

Suppose $E \subseteq \mathbb{R}^2$ is analytic set and $d := \dim_H(E) \leq 1$. Then,

$$\sup_{x \in E} \dim_H(\Delta_x(E)) \geq d\left(1 - \frac{2 - d}{2(1 + 2d - d^2)}\right).$$

Pinned distance sets using effective dimension

Theorem (Fiedler, S.)

Suppose that $x, y \in \mathbb{R}^2$, $e = \frac{y-x}{|y-x|}$ *and* $0 < \sigma < 1$ *satisfy the following.*

(C1) $\dim(x), \dim(y) > \sigma$

(C2) $K^x_r(e) > \sigma r - O(\log r)$ *for all* r.

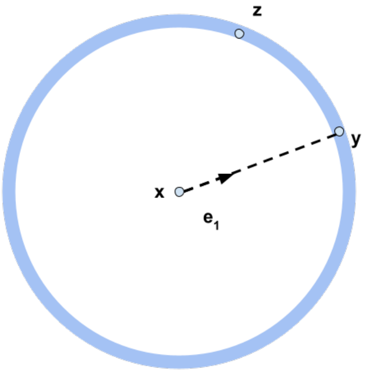
(C3) $K^x_r(y) \geq K^A_r(y) - O(\log r)$ *for all sufficiently large* r.

(C4) $K_{t,r}(e \mid y) > \sigma t - O(\log r)$ *for all sufficiently large* r *and* $t \leq r$.

Then $\dim^x(|x-y|) \geq \sigma \left(1 - \frac{\alpha D - \sigma(D+\alpha-\sigma)}{(\sigma+1)(\alpha D - \sigma^2) - \sigma^2(\alpha + D - 2\sigma)}\right)$, *where* $D = \max\{D_x, D_y\}$ *and* $\alpha = \min\{D, 1+\sigma\}$.

1. Most of the work is in proving this theorem. We then use the point-to-set principle to conclude the classical theorem on the Hausdorff dimension of pinned distance sets.
2. Note that we assume the direction from x to y is of high dimension given x or y.

Let $x, y \in \mathbb{R}^2$. We want to lower bound $\dim^x(|x-y|)$.

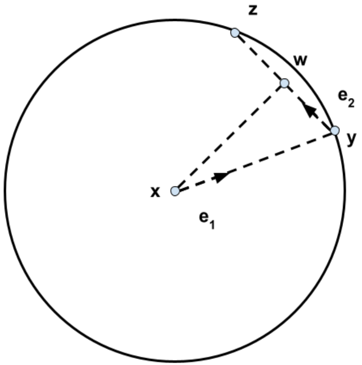


- From the definition of dimension, it is natural to fix a precision $r \in \mathbb{N}$, and give a lower bound on $K_r^x(|x-y|)$.
- By the symmetry of information, this is equivalent to prove an upper bound on

$$K_r^x(y \mid |x-y|) \approx K_r^x(y) - K_r^x(|x-y|).$$

- Intuitively: Given x as an oracle, and knowing (an approximation of) $|x-y|$, how hard is it to compute (an approximation of) y?
- Enumerate rationals z such that $K_r(z) \leq K_r(y)$ and $||x-y| - |x-z|| < 2^{-r}$. Suffices to bound the dyadic covering number of such z's.

Goal: Bound $K_r^x(y \mid |x-y|)$. Can enumerate rationals z such that $K_r(z) \leq K_r(y)$ and $||x-y| - |x-z|| < 2^{-r}$.

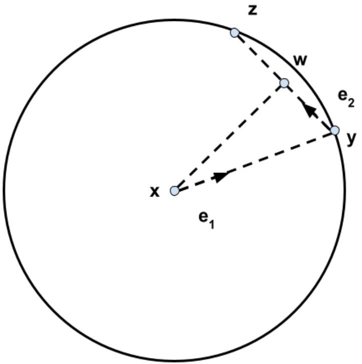


For any $e \in \mathcal{S}^1$ and $x \in \mathbb{R}^2$, $p_e x = e \cdot x$.

1. $|p_{e_1} y - p_{e_1} z| \lesssim |y - z|^2$
2. $p_{e_2} x = p_{e_2} w$

- Reduce this to *projections*.
- If there is such a z with $|z - y| < 2^{-r/2}$, then $p_{e_1} y \approx p_{e_1} z$.
- That is, y and z determine (an approximation of) e_1: $K_{r-t}(e_1 \mid y) \lesssim K_r(z \mid y)$.
- If there are many such z's then the direction e_1 is compressible.
- But we assumed that the direction was of high dimension.

Goal: Bound $K_r^x(y \mid |x-y|)$. Can enumerate rationals z such that $K_r(z) \leq K_r(y)$ and $||x-y| - |x-z|| < 2^{-r}$.



For any $e \in \mathcal{S}^1$ and $x \in \mathbb{R}^2$, $p_e x = e \cdot x$.

1. $|p_{e_1} y - p_{e_1} z| \lesssim |y-z|^2$
2. $p_{e_2} x = p_{e_2} w$

- If there is such a z then $p_{e_2} x \approx p_{e_2} w$.
- That is, y and z determine a line containing x.
- Moreover, this line has some randomness, since it is close to $e_1^{\perp}$.
- Given this, we can bound $K_{r-t}(x \mid y, z)$.
- If there are many such z's then the point x is compressible.
- But we assumed that the pinned point x was of high dimension.

The End

Thank you!