Abstract In [1], Hjorth proved that for every countable ordinal α, there exists a complete $\mathcal{L}_{\omega_1,\omega}$- sentence ϕ_α that has models of all cardinalities less than or equal to $\aleph_\alpha$, but no models of cardinality $\aleph_{\alpha+1}$. Unfortunately, his solution yields not one $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α, but a set of $\mathcal{L}_{\omega_1,\omega}$- sentences, one of which is guaranteed to work.
The following is new: It is independent of the axioms of ZFC which of the Hjorth sentences works. More specifically, we isolate a diagonalization principle for functions from ω_1 to ω_1 which is a consequence of the *Bounded Proper Forcing Axiom* (BPFA) and then we use this principle to prove that Hjorth's solution to characterizing $\aleph_2$ in models of BPFA is different than in models of CH.
This raises the question whether Hjorth's result can be proved in an *absolute way* and what exactly this means, which we will discuss at the end of the talk.
This is joint work with Philipp Lücke.

References

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(Non)-Absolute Characterizations of Cardinals

Online Logic Seminar

Yiannis Souldatos
ARISTOTLE UNIVERSITY OF THESSALONIKI

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$$\beth_{\alpha+1} = 2^{\beth_\alpha}$$
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1. $\mathcal{L}_{\omega_1,\omega} = \mathcal{L}_{\omega,\omega}$ + countable conjunctions + countable disjunctions
2. An $\mathcal{L}_{\omega_1,\omega}$- sentence is *complete* if it is $\aleph_0$-categorical.
3. For every countable model $\mathcal{M}$ there exists some complete (Scott) sentence $\phi_\mathcal{M}$ with $\mathcal{M} \models \phi_\mathcal{M}$.
4. An $\mathcal{L}_{\omega_1,\omega}$- sentence ϕ *characterizes* some cardinal κ, if ϕ has models in all cardinalities $[\aleph_0, \kappa]$ but no higher.
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Some remarks:

1. Hjorth's result is in ZFC.
2. Under GCH, $\aleph_\alpha$ can be characterized by an $\mathcal{L}_{\omega_1,\omega}$-sentence iff $\alpha < \omega_1$.
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- Unfortunately, Hjorth describes not one, but two constructions in his paper.
- Given some complete sentence ϕ which characterizes $\aleph_\alpha$, Hjorth's first construction yields a complete sentence which characterizes either $\aleph_\alpha$ or $\aleph_{\alpha+1}$.
- If the latter is the case, we are done.
- If not, then Hjorth introduces his second construction.
- If Hjorth's first construction characterizes $\aleph_\alpha$, then Hjorth's second construction characterizes $\aleph_{\alpha+1}$.
- Notice here that the failure of the first construction to characterize $\aleph_{\alpha+1}$ is used to prove that the second Hjorth construction does indeed characterize $\aleph_{\alpha+1}$.
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- Therefore, Hjorth's solution does not yield a single $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α, but a set of $\mathcal{L}_{\omega_1,\omega}$-sentences S_α, one of which is guaranteed to characterizes $\aleph_\alpha$.
- S_0 and S_1 are singletons.
- S_α is finite for finite α.
- For $\alpha = \omega$, iterating the first construction ω-many times will yield a sentence that characterizes $\aleph_\omega$, regardless of what cardinal each iteration characterizes.
- So, S_ω is also a singleton.
- Similarly, S_λ is a singleton for all limit λ and S_α is finite for all $\alpha < \omega_1$.
- It was conjectured that it is independent of the axioms of ZFC which of the sentences in S_α characterizes $\aleph_\alpha$.
- New result: The conjecture is true.

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- Therefore, Hjorth's solution does not yield a single $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_α, but a set of $\mathcal{L}_{\omega_1,\omega}$-sentences S_α, one of which is guaranteed to characterizes $\aleph_\alpha$.
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First Hjorth Construction

We briefly describe the first Hjorth construction.

Given: A countable model $\mathcal{M}$ which characterizes $\aleph_\alpha$.

Definition

- Consider C the collection of all complete finite graphs G with edges colored by elements of M.
- $C(a, b) = C(b, a)$ is the color assigned to (a, b).
- For $a, b \in G$, let $A^G(a, b) = \{c \in G | C(a, c) = C(b, c)\}$ (the set of agreements).
- $G_1 \sqsubseteq G_2$ if G_1, G_2 agree on the edge colors on $[G_1]^2$ and G_2 introduces no new agreements, i.e. $A^{G_1}(a, b) = A^{G_2}(a, b)$ for all $a, b \in V(G_1)$.

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Theorem (Hjorth)

$(\mathcal{C}, \subseteq)$ satisfies the (disjoint) Amalgmation and Joint Embedding Properties (AP & JEP).

Proof...

Corollary

The collection $(\mathcal{C}, \subseteq)$ has a "Fraisse limit". I.e. there exists a countable structure F with the following properties:

1. *F contains a countable graph G and (a copy of) $\mathcal{M}$.*
2. *(Finite Agreement) For all $a, b \in G$, the set $A^G_{a,b}$ is finite.*
3. *(Finite Closure) For every X finite subset of G there exists some finite G_0, $X \subset G_0$ and $G_0 \subseteq G$. In particular, G_0 is closed under A^G.*
4. *(Finite Extension) If G_0, G_1 are finite graphs with $G_0 \subseteq G$ and $G_0 \subseteq G_1$, then there exists an injection $i : G_1 \mapsto G$ with $i \restriction_{G_0} = id_{G_0}$ and $C^{G_1}(a, b) = C^G(i(a), i(b))$ for all $a, b \in G_1$.*

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The set M of colors is countable when we take the Fraisse limit, but may increase in other models (up to size $\aleph_\alpha$).

Theorem (Hjorth)

The Scott sentence of F

1. *has a model of size $\aleph_\alpha$*
2. *every model of size $\aleph_{\alpha+1}$ (if any) is maximal and*
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Colored Version

1. Hjorth's first construction can be modified to include vertex-colors (new elements not in M).
2. Amalgamation and Joint Embedding still hold.
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Let F^c be the Fraisse limit of Hjorth's colored construction, M the set of edge-colors and N the set of vertex-colors.

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Absolute Indiscernibles

Definition

Let M be a model and X a (definable) subset of M. X is a set of *absolute indiscernibles* (for M) if every permutation of X extends to an automorphism of M.

Theorem

If F^c is the (unique) countable (M, N)-full structure, then N is a set of absolute indiscernibles.

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No countable model with absolute indiscernibles can characterize $\aleph_0$.

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If M characterizes $\aleph_0$, then the countable (M, N)-full structure characterizes $\aleph_1$ (in all models of ZFC).

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So, the first place where set theory may play a role in Hjorth's construction is at $\aleph_2$.

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If CH *holds and* M *characterizes* $\aleph_1$, *then the* (M, N)*-full structure also characterizes* $\aleph_1$.

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We show that there exists a model of $\mathrm{ZFC}(+\neg\mathrm{CH})$ where the (M, N)-full structure characterizes $\aleph_2$.

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Property ($\triangle$)

We isolated a diagonalization property that we called ($\triangle$).

Definition

1. Given a set X, we say that a map $m : [X]^{<\omega} \mapsto [X]^{<\omega}$ is *monotone* if $a \subseteq m(a)$ holds for every finite subset a of X.
2. ($\triangle$) denotes the statement:
   for every sequence $(f_\alpha : \omega_1 \mapsto \omega_1 | \alpha < \omega_1)$ and every monotone function $m : [\omega_1]^{<\omega} \mapsto [\omega_1]^{<\omega}$, there exists a function $g : \omega_1 \mapsto \omega_1$ such that for every $a \in [\omega_1]^{<\omega}$, there exists $a \subseteq b \in [\omega_1]^{<\omega}$ with the property that

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2. (⊿) denotes the statement: for every sequence $(f_\alpha : \omega_1 \mapsto \omega_1 | \alpha < \omega_1)$ and every monotone function $m : [\omega_1]^{<\omega} \mapsto [\omega_1]^{<\omega}$, there exists a function $g : \omega_1 \mapsto \omega_1$ such that for every $a \in [\omega_1]^{<\omega}$, there exists $a \subseteq b \in [\omega_1]^{<\omega}$ with the property that

$$\{\beta < \omega_1 | f_\alpha(\beta) = g(\beta)\} \subseteq m(b)$$

holds for all $\alpha \in m(b)$.
In addition, given some finite $F \subset \omega_1$, we require that

$$F \cap range(g) = \emptyset.$$

Property (⊿)

We isolated a diagonalization property that we called (⊿).

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The importance of $(\triangle)$ is apparent from the following theorem.

Theorem

Assume that $(\triangle)$ holds and let $\mathcal{M}$ be a countable model that characterizes $\aleph_1$. Then the countable (M, N)-full structure characterizes $\aleph_2$.

Lemma

If $(\triangle)$ *holds, then* $2^{\aleph_0} > \aleph_1$.

Proof...

Lemma

If $(\triangle)$ *holds, then there exists a sequence* $(A_\gamma | \gamma < \omega_2)$ *of unbounded subsets of* ω_1 *with the property that for all* $\delta < \gamma < \omega_2$, *the set* $A_\gamma \cap A_\delta$ *is finite.*

Proof...

Theorem (Baumgartner)

If CH *holds and* G *is Add*(ω, ω_2)*-generic over* V, *then in* $\mathrm{V}[G]$ *there is no sequence* $(A_\gamma | \gamma < \omega_2)$ *of unbounded subsets of* ω_1 *with finite intersections.*

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Corollary

1. *If* CH *holds and* G *is* $Add(\omega, \omega_2)$*-generic over* V*, then in* $\mathrm{V}[G]$ *the property* $(\measuredangle)$ *fails.*
2. $(\measuredangle)$ *is not a theorem of* $\mathrm{ZFC}+\neg\mathrm{CH}$

Question

Can we force $(\measuredangle)$*?*

Answer

Yes!

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The following forcing notion is due to P. Larson

Definition

We let $\mathbb{D}$ denote the partial order defined by the following clauses:

1. A condition in $\mathbb{D}$ is a triple $p = \langle a_p, \mathscr{F}_p, \mathscr{X}_p \rangle$ such that the following statements hold:
   1.1 a_p is a function from a finite subset d_p of ω_1 into ω_1.
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2. Given conditions p and q in $\mathbb{D}$, we have $p \leq_{\mathbb{D}} q$ if and only if the following statements hold:
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Theorem (Larson)

The partial order $\mathbb{D}$ is proper.

Forcing Axioms

Definition

Given a partial ordering $\mathbb{P}$ and a cardinal κ, the Forcing Axiom $FA_\kappa(\mathbb{P})$ is the following statement:
For every collection $\{I_\alpha | \alpha < \kappa\}$ of maximal antichains of $\mathbb{P}$, there exists a filter G that intersects every I_α.
If Γ is a class of partial orderings, $FA_\kappa(\Gamma)$ is the statement that for every $\mathbb{P} \in \Gamma$, $FA_\kappa(\mathbb{P})$ holds.

Example

- Martin's Axiom MA_κ: $FA_\kappa(ccc)$, where $\kappa < 2^\omega$
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Bounded Forcing Axioms

Bounded forcing axioms are defined similarly, but the size of the antichains is now bounded.

Definition

Given a partial ordering $\mathbb{P}$ and a cardinal κ, the Bounded Forcing Axiom $BFA_\kappa(\mathbb{P})$ is the following statement:
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Bounded Forcing Axioms

Bounded forcing axioms are defined similarly, but the size of the antichains is now bounded.

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Generic Σ_1-Absoluteness

Definition

If Γ is a class of posets, $\Sigma_1(X)$-absoluteness for Γ is the following statement:

For every poset $\mathbb{P} \in \Gamma$, every Σ_1-formula $\phi(x_1, \ldots, x_n)$, and every $a_1, \ldots, a_n \in X$,

$$\phi(a_1, \ldots, a_n) \text{ iff } V^{r.o.(\mathbb{P})} \models \phi(\check{a}_1, \ldots, \check{a}_n)$$

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Forcing Axioms and Generic Absoluteness

Forcing axioms are equivalent to generic Σ_1-absoluteness

Theorem

Let $\mathbb{P}$ be a partial ordering and κ an infinite cardinal of uncountable cofinality. Then the following are equivalent:

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Corollary

The following statements are equivalent:

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BPFA *implies that* $(\triangle)$ *holds.*

Idea of the Proof Fix a sequence of functions $\vec{f} = (f_\alpha : \omega_1 \mapsto \omega_1 | \alpha < \omega_1)$, a finite subset F of ω_1 and a monotone function $m : [\omega_1]^{<\omega} \mapsto [\omega_1]^{<\omega}$.

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Then $g : \omega_1 \mapsto \omega_1$ with $F \cap range(g) = \emptyset$ and g satisfies the desired finite intersection property with all f_α's.

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We can actually do better (i.e. reduce the consistency strength)

Theorem

($\triangle$) can be forced over a model of CH with a proper forcing $\mathbb{P}$ that satisfies the $\aleph_2$-chain condition.

Idea of the Proof The proper forcing $\mathbb{P}$ is a “matrix version” of Larson’s forcing $\mathbb{D}$.

Absolute Characterizations

Summary:

- Hjorth proved that there exists a countable model M which characterizes $\aleph_1$ in all models of ZFC.
- Using M he constructed a countable (M, N)-full structure S.
- S characterizes $\aleph_1$ in models of CH and $\aleph_2$ in models of BPFA.
- One may ask if our results for $\aleph_2$ generalize to higher cardinalities, e.g. $\aleph_3$.
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2. *What does it mean to have an absolute characterizations?*

(1) is open. We suggest some answers for (2)

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Fact (Shoenfield absoluteness)

Σ^1_3-statements are upwards absolute between transitive models of set theory with the same ordinals.

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Is there a Σ^1_3-formula $\Phi(v_0, v_1)$ in the language of second-order arithmetic with the property that the axioms of ZFC prove that the following statements hold:

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