Abstract  In [1], Hjorth proved that for every countable ordinal $\alpha$, there exists a complete $\mathcal{L}_{\omega_1,\omega}$-sentence $\phi_\alpha$ that has models of all cardinalities less than or equal to $\aleph_\alpha$, but no models of cardinality $\aleph_{\alpha+1}$. Unfortunately, his solution yields not one $\mathcal{L}_{\omega_1,\omega}$-sentence $\phi_\alpha$, but a set of $\mathcal{L}_{\omega_1,\omega}$-sentences, one of which is guaranteed to work.

The following is new: It is independent of the axioms of ZFC which of the Hjorth sentences works. More specifically, we isolate a diagonalization principle for functions from $\omega_1$ to $\omega_1$ which is a consequence of the *Bounded Proper Forcing Axiom* (BPFA) and then we use this principle to prove that Hjorth’s solution to characterizing $\aleph_2$ in models of BPFA is different than in models of CH.

This raises the question whether Hjorth’s result can be proved in an *absolute way* and what exactly this means, which we will discuss at the end of the talk.

This is joint work with Philipp Lücke.
References

Greg Hjorth.
Knight’s model, its automorphism group, and characterizing the uncountable cardinals. 

Philipp Lücke, Ioannis Souldatos,
A lower bound for the hanf number for joint embedding. 
(Non)-Absolute Characterizations of Cardinals

Online Logic Seminar

Yiannis Souldatos

ARISTOTLE UNIVERSITY OF THESSALONIKI
History of the Problem
  Introduction
  Hjorth’s Solution

First Hjorth Construction
  Colored Version
  The Case of \( \aleph_2 \)
  A Diagonalization Property
  Forcing
  Forcing Axioms

Absolute Characterizations
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Preliminaries

\[ \beth_0 = \aleph_0 \]
\[ \beth_{\alpha+1} = 2^{\beth_\alpha} \]
\[ \beth_\lambda = \sup\{ \beth_\alpha | \alpha < \lambda \} \text{, for limit } \lambda \]

1. $L_{\omega_1,\omega} = L_{\omega,\omega} + \text{countable conjunctions} + \text{countable disjunctions}$

2. An $L_{\omega_1,\omega}$-sentence is complete if it is $\aleph_0$-categorical.

3. For every countable model $M$ there exists some complete (Scott) sentence $\phi_M$ with $M \models \phi_M$.

4. An $L_{\omega_1,\omega}$-sentence $\phi$ characterizes some cardinal $\kappa$, if $\phi$ has models in all cardinalities $[\aleph_0, \kappa]$ but no higher.

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1. In 1965 Morley proved that for each \( \alpha < \omega_1 \), there exists an \( \mathcal{L}_{\omega_1, \omega} \)-sentence \( \psi_\alpha \) that characterizes \( \beth_\alpha \).

2. The corresponding problem for \( \aleph_\alpha \) was probably known by then (but I did not find a reference).

3. In the mid-1960’s Morley and Lopez-Escobar proved:

   **Theorem**

   If \( \phi \) is an \( \mathcal{L}_{\omega_1, \omega} \)-sentence with a model of size \( \beth_{\omega_1} \), then \( \phi \) has models of any size.

4. By the mid-1970’s people were asking about characterizing cardinals by complete \( \mathcal{L}_{\omega_1, \omega} \)-sentences.

5. In 1977 Julia Knight proved that there exists a complete \( \mathcal{L}_{\omega_1, \omega} \)-sentence \( \phi_1 \) with models in \( \aleph_0 \) and \( \aleph_1 \) and no higher (\( \phi_1 \) characterizes \( \aleph_1 \)).

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Some remarks:

1. Hjorth’s result is in ZFC.
2. Under GCH, $\aleph_\alpha$ can be characterized by an $\mathcal{L}_{\omega_1, \omega}$-sentence iff $\alpha < \omega_1$.
3. So, Hjorth’s result is optimal in ZFC(with no extra assumptions).
4. Since Hjorth there have been similar results, e.g. characterizing $\aleph_n$, for $n \in \omega$.
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Hjorth’s Solution

- Unfortunately, Hjorth describes not one, but two constructions in his paper.
- Given some complete sentence $\phi$ which characterizes $\aleph_\alpha$, Hjorth’s first construction yields a complete sentence which characterizes either $\aleph_\alpha$ or $\aleph_\alpha + 1$.
- If the latter is the case, we are done.
- If not, then Hjorth introduces his second construction.
- If Hjorth’s first construction characterizes $\aleph_\alpha$, then Hjorth’s second construction characterizes $\aleph_\alpha + 1$.
- Notice here that the failure of the first construction to characterize $\aleph_\alpha + 1$ is used to prove that the second Hjorth construction does indeed characterize $\aleph_\alpha + 1$.
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- Unfortunately, Hjorth describes not one, but two constructions in his paper.
- Given some complete sentence $\phi$ which characterizes $\aleph_\alpha$, Hjorth’s first construction yields a complete sentence which characterizes either $\aleph_\alpha$ or $\aleph_\alpha + 1$.
- If the latter is the case, we are done.
- If not, then Hjorth introduces his second construction.
- If Hjorth’s first construction characterizes $\aleph_\alpha$, then Hjorth’s second construction characterizes $\aleph_\alpha + 1$.
- Notice here that the failure of the first construction to characterize $\aleph_\alpha + 1$ is used to prove that the second Hjorth construction does indeed characterize $\aleph_\alpha + 1$.
- In either case, there exists some $\mathcal{L}_{\omega_1, \omega}$-sentence that characterizes $\aleph_\alpha + 1$ and the induction step is complete.
- For limit stages take the disjoint union of models that characterize all the previous cardinals.
Therefore, Hjorth’s solution does not yield a single $L_{\omega_1,\omega}$-sentence $\phi_\alpha$, but a set of $L_{\omega_1,\omega}$-sentences $S_\alpha$, one of which is guaranteed to characterize $\aleph_\alpha$.

- $S_0$ and $S_1$ are singletons.
- $S_\alpha$ is finite for finite $\alpha$.
- For $\alpha = \omega$, iterating the first construction $\omega$-many times will yield a sentence that characterizes $\aleph_\omega$, regardless of what cardinal each iteration characterizes.
- So, $S_\omega$ is also a singleton.
- Similarly, $S_\lambda$ is a singleton for all limit $\lambda$ and $S_\alpha$ is finite for all $\alpha < \omega_1$.
- It was conjectured that it is independent of the axioms of ZFC which of the sentences in $S_\alpha$ characterizes $\aleph_\alpha$.
- New result: The conjecture is true.
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First Hjorth Construction

We briefly describe the first Hjorth construction.

Given: A countable model $\mathcal{M}$ which characterizes $\aleph_\alpha$.

Definition

1. Consider $\mathcal{C}$ the collection of all complete finite graphs $G$ with edges colored by elements of $\mathcal{M}$.
2. $C(a, b) = C(b, a)$ is the color assigned to $(a, b)$.
3. For $a, b \in G$, let $A^G(a, b) = \{ c \in G | C(a, c) = C(b, c) \}$ (the set of agreements).
4. $G_1 \subseteq G_2$ if $G_1, G_2$ agree on the edge colors on $|G_1|^2$ and $G_2$ introduces no new agreements, i.e., $A^{G_1}(a, b) = A^{G_2}(a, b)$ for all $a, b \in V(G_1)$. 


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Theorem (Hjorth)

\((C, \subseteq)\) satisfies the (disjoint) Amalgamation and Joint Embedding Properties (AP & JEP).

Proof...

Corollary

The collection \((C, \subseteq)\) has a “Fraissé limit”. I.e. there exists a countable structure \(F\) with the following properties:

1. \(F\) contains a countable graph \(G\) and (a copy of) \(M\).
2. (Finite Agreement) For all \(a, b \in G\), the set \(A^G_{a, b}\) is finite.
3. (Finite Closure) For every finite subset of \(G\) there exists some finite \(G_0, X \subseteq G_0\) and \(G_0 \subseteq G\). In particular, \(G_0\) is closed under \(A^G\).
4. (Finite Extension) If \(G_0, G_1\) are finite graphs with \(G_0 \subseteq G\) and \(G_0 \subseteq G_1\), then there exists an injection \(i : G_1 \hookrightarrow G\) with \(i \upharpoonright G_0 = id_{G_0}\) and \(C^G_1(a, b) = C^G(i(a), i(b))\) for all \(a, b \in G_1\).
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Remark

*The set $M$ of colors is countable when we take the Fraisse limit, but may increase in other models (up to size $\aleph_\alpha$).*

Theorem (Hjorth)

*The Scott sentence of $F$*

1. has a model of size $\aleph_\alpha$
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Proof...
Colored Version

1. Hjorth’s first construction can be modified to include vertex-colors (new elements not in $M$).
2. Amalgamation and Joint Embedding still hold.
3. The “Fraïssé limit” satisfies Finite Agreement, Finite Closure and a colored version of Finite Extension where $G_0, G_1$ are vertex-colored.
4. We will call this the colored version of Hjorth’s construction.
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Definition
Let $F^c$ be the Fraisse limit of Hjorth’s colored construction, $M$ the set of edge-colors and $N$ the set of vertex-colors. Hjorth calls any structure that satisfies the Scott sentence of $F^c$ an $(M, N)$-full structure.
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Absolute Indiscernibles

Definition
Let $M$ be a model and $X$ a (definable) subset of $M$. $X$ is a set of absolute indiscernibles (for $M$) if every permutation of $X$ extends to an automorphism of $M$.

Theorem
If $F^c$ is the (unique) countable $(M, N)$-full structure, then $N$ is a set of absolute indiscernibles.

Theorem (Hjorth)
No countable model with absolute indiscernibles can characterize $\aleph_0$.

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If $M$ characterizes $\aleph_0$, then the countable $(M, N)$-full structure characterizes $\aleph_1$ (in all models of ZFC).
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So, the first place where set theory may play a role in Hjorth’s construction is at \( \aleph_2 \).

Lemma

If CH holds and \( M \) characterizes \( \aleph_1 \), then the \((M, N)\)-full structure also characterizes \( \aleph_1 \).

Proof...

We show that there exists a model of \( \text{ZFC}(+ \text{¬CH}) \) where the \((M, N)\)-full structure characterizes \( \aleph_2 \).

Hence, it is independent of \( \text{ZFC} \) which of Hjorth’s constructions (the first or the second) characterizes \( \aleph_2 \).
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Hence, it is independent of ZFC which of Hjorth’s constructions (the first or the second) characterizes $\aleph_2$. 
The Case of $\aleph_2$

So, the first place where set theory may play a role in Hjorth’s construction is at $\aleph_2$.

**Lemma**

*If CH holds and $M$ characterizes $\aleph_1$, then the $(M, N)$-full structure also characterizes $\aleph_1$.***

**Proof...**

We show that there exists a model of $\text{ZFC}(+ \neg \text{CH})$ where the $(M, N)$-full structure characterizes $\aleph_2$.

Hence, it is independent of ZFC which of Hjorth’s constructions (the first or the second) characterizes $\aleph_2$. 
Property (\(\triangle\))

We isolated a diagonalization property that we called (\(\triangle\)).

**Definition**

1. Given a set \(X\), we say that a map \(m : [X]<\omega \mapsto [X]<\omega\) is **monotone** if \(a \subseteq m(a)\) holds for every finite subset \(a\) of \(X\).

2. (\(\triangle\)) denotes the statement:

   for every sequence \((f_\alpha : \omega_1 \mapsto \omega_1 | \alpha < \omega_1)\) and every monotone function \(m : [\omega_1]<\omega \mapsto [\omega_1]<\omega\), there exists a function \(g : \omega_1 \mapsto \omega_1\) such that for every \(a \in [\omega_1]<\omega\), there exists \(a \subseteq b \in [\omega_1]<\omega\) with the property that

   \[
   \{ \beta < \omega_1 | f_\alpha(\beta) = g(\beta) \} \subseteq m(b)
   \]

holds for all \(\alpha \in m(b)\).

In addition, given some finite \(F \subset \omega_1\), we require that

\[
F \cap \text{range}(g) = \emptyset.
\]
Property (\(\triangle\))

We isolated a diagonalization property that we called (\(\triangle\)).

Definition

1. Given a set \(X\), we say that a map \(m : [X]^{<\omega} \mapsto [X]^{<\omega}\) is monotone if \(a \subseteq m(a)\) holds for every finite subset \(a\) of \(X\).

2. (\(\triangle\)) denotes the statement:

   - for every sequence \((f_\alpha : \omega_1 \mapsto \omega_1 | \alpha < \omega_1)\) and every monotone function \(m : [\omega_1]^{<\omega} \mapsto [\omega_1]^{<\omega}\), there exists a function \(g : \omega_1 \mapsto \omega_1\) such that for every \(a \in [\omega_1]^{<\omega}\), there exists \(a \subseteq b \in [\omega_1]^{<\omega}\) with the property that

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   - In addition, given some finite \(F \subseteq \omega_1\), we require that

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We isolated a diagonalization property that we called ($\triangle$).

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In addition, given some finite $F \subset \omega_1$, we require that

\[ F \cap \text{range}(g) = \emptyset. \]
Property (△)

We isolated a diagonalization property that we called (△).

Definition

1. Given a set \( X \), we say that a map \( m : [X]^\omega \to [X]^\omega \) is \textit{monotone} if \( a \subseteq m(a) \) holds for every finite subset \( a \) of \( X \).

2. (△) denotes the statement:

   for every sequence \( (f_\alpha : \omega_1 \to \omega_1 | \alpha < \omega_1) \) and every monotone function \( m : [\omega_1]^\omega \to [\omega_1]^\omega \), there exists a function \( g : \omega_1 \to \omega_1 \) such that for every \( a \in [\omega_1]^\omega \), there exists \( a \subseteq b \in [\omega_1]^\omega \) with the property that

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Property (⋄)

We isolated a diagonalization property that we called (⋄).

Definition

1. Given a set $X$, we say that a map $m : [X]<\omega \mapsto [X]<\omega$ is monotone if $a \subseteq m(a)$ holds for every finite subset $a$ of $X$.

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   for every sequence $(f_\alpha : \omega_1 \mapsto \omega_1 | \alpha < \omega_1)$ and every monotone function $m : [\omega_1]<\omega \mapsto [\omega_1]<\omega$, there exists a function $g : \omega_1 \mapsto \omega_1$ such that for every $a \in [\omega_1]<\omega$, there exists $a \subseteq b \in [\omega_1]<\omega$ with the property that
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In addition, given some finite $F \subset \omega_1$, we require that
   \[
   F \cap range(g) = \emptyset.
   \]
The importance of $(\Delta)$ is apparent from the following theorem.

**Theorem**

Assume that $(\Delta)$ holds and let $M$ be a countable model that characterizes $\aleph_1$. Then the countable $(M,N)$-full structure characterizes $\aleph_2$. 
Lemma

If \((\triangle)\) holds, then \(2^{\aleph_0} > \aleph_1\).

Proof...

Lemma

If \((\triangle)\) holds, then there exists a sequence \((A_\gamma | \gamma < \omega_2)\) of unbounded subsets of \(\omega_1\) with the property that for all \(\delta < \gamma < \omega_2\), the set \(A_\gamma \cap A_\delta\) is finite.

Proof...

Theorem (Baumgartner)

If \(CH\) holds and \(G\) is Add\((\omega, \omega_2)\)-generic over \(V\), then in \(V[G]\) there is no sequence \((A_\gamma | \gamma < \omega_2)\) of unbounded subsets of \(\omega_1\) with finite intersections.
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If (△) holds, then $2^\aleph_0 > \aleph_1$.

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If (△) holds, then there exists a sequence $(A_\gamma | \gamma < \omega_2)$ of unbounded subsets of $\omega_1$ with the property that for all $\delta < \gamma < \omega_2$, the set $A_\gamma \cap A_\delta$ is finite.

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If CH holds and $G$ is Add$(\omega, \omega_2)$-generic over $V$, then in $V[G]$ there is no sequence $(A_\gamma | \gamma < \omega_2)$ of unbounded subsets of $\omega_1$ with finite intersections.
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Corollary

1. If CH holds and G is Add(ω, ω₂)-generic over V, then in V[G] the property (Δ) fails.
2. (Δ) is not a theorem of ZFC+¬CH

Question
Can we force (Δ)?

Answer
Yes!
Corollary

1. If $\text{CH}$ holds and $G$ is $\text{Add}(\omega, \omega_2)$-generic over $V$, then in $V[G]$ the property ($\Delta$) fails.

2. ($\Delta$) is not a theorem of $\text{ZFC} + \neg \text{CH}$

Question

Can we force ($\Delta$)?

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Corollary

1. *If CH holds and G is Add(ω,ω₂)-generic over V, then in V[G] the property (Δ) fails.*

2. *(Δ) is not a theorem of ZFC + ¬CH*

Question

*Can we force (Δ)?*

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1. If CH holds and G is Add(ω, ω₂)-generic over V, then in V[G] the property (∆) fails.
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1. If CH holds and $G$ is $\text{Add}(\omega, \omega_2)$-generic over $V$, then in $V[G]$ the property $(\Delta)$ fails.

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Question

Can we force $(\Delta)$?

Answer

Yes!
The following forcing notion is due to P. Larson

**Definition**

We let $\mathbb{D}$ denote the partial order defined by the following clauses:

1. A condition in $\mathbb{D}$ is a triple $p = \langle a_p, \mathcal{F}_p, \mathcal{X}_p \rangle$ such that the following statements hold:
   1.1 $a_p$ is a function from a finite subset $d_p$ of $\omega_1$ into $\omega_1$.
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2. Given conditions $p$ and $q$ in $\mathbb{D}$, we have $p \leq_{\mathbb{D}} q$ if and only if the following statements hold:
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**Definition**

We let \( \mathbb{D} \) denote the partial order defined by the following clauses:

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Theorem (Larson)

The partial order $\mathbb{D}$ is proper.
Forcing Axioms

Definition
Given a partial ordering $\mathbb{P}$ and a cardinal $\kappa$, the Forcing Axiom $FA_\kappa(\mathbb{P})$ is the following statement:

For every collection $\{l_\alpha | \alpha < \kappa\}$ of maximal antichains of $\mathbb{P}$, there exists a filter $G$ that intersects every $l_\alpha$.

If $\Gamma$ is a class of partial orderings, $FA_\kappa(\Gamma)$ is the statement that for every $\mathbb{P} \in \Gamma$, $FA_\kappa(\mathbb{P})$ holds.

Example
1. Martin’s Axiom $MA_\kappa$ is $FA_\kappa$(ccc), where $\kappa < 2^{\mathfrak{c}}$.
2. Proper Forcing Axiom $PFA$ is $FA_\kappa$(proper).
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1. Martin’s Axiom $\kappa < 2^{\aleph_0}$.
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For every collection $\{l_\alpha | \alpha < \kappa\}$ of maximal antichains of $\mathbb{P}$, there exists a filter $G$ that intersects every $l_\alpha$.
If $\Gamma$ is a class of partial orderings, $FA_\kappa(\Gamma)$ is the statement that for every $\mathbb{P} \in \Gamma$, $FA_\kappa(\mathbb{P})$ holds.

Example
1. Martin’s Axiom $MA_\kappa$, $FA_\kappa(\text{ccc})$, where $\kappa < 2^{\aleph_0}$.
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Bounded forcing axioms are defined similarly, but the size of the antichains is now bounded.

**Definition**

Given a partial ordering $\mathbb{P}$ and a cardinal $\kappa$, the Bounded Forcing Axiom $BFA_\kappa(\mathbb{P})$ is the following statement:

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If $\Gamma$ is a class of partial orderings, $BFA_\kappa(\Gamma)$ is the statement that for every $\mathbb{P} \in \Gamma$, $BFA_\kappa(\mathbb{P})$ holds.
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Generic $\Sigma_1$-Absoluteness

**Definition**

If $\Gamma$ is a class of posets, $\Sigma_1(X)$-absoluteness for $\Gamma$ is the following statement:

For every poset $P \in \Gamma$, every $\Sigma_1$-formula $\phi(x_1, \ldots, x_n)$, and every $a_1, \ldots, a_n \in X$,

$$\phi(a_1, \ldots, a_n) \text{ iff } V^{r.o.}(P) \models \phi(\check{a}_1, \ldots, \check{a}_n)$$

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Forcing Axioms and Generic Absoluteness

Forcing axioms are equivalent to generic $\Sigma_1$-absoluteness

Theorem

Let $\mathbb{P}$ be a partial ordering and $\kappa$ an infinite cardinal of uncountable cofinality. Then the following are equivalent:

1. $\text{BFA}_\kappa(\mathbb{P})$
2. $\Sigma_1(P(\kappa))$-absoluteness for $\mathbb{P}$.
3. $\Sigma_1(H(\kappa^+))$-absoluteness for $\mathbb{P}$.

Corollary

The following statements are equivalent:

1. $\text{BPF A}$ holds.
2. If $\varphi(v)$ is a $\Sigma_1$-formula, $z$ is an element of $H(\omega_2)$, $\mathbb{P}$ is a proper forcing and $p$ is a condition in $\mathbb{P}$ with $p \Vdash \varphi(z)$, then $\varphi(z)$ holds.
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Theorem
BPFA implies that $(\Delta)$ holds.

Idea of the Proof Fix a sequence of functions 
$\vec{f} = (f_\alpha : \omega_1 \mapsto \omega_1 | \alpha < \omega_1)$, a finite subset $F$ of $\omega_1$ and a monotone function $m : [\omega_1]^{<\omega} \mapsto [\omega_1]^{<\omega}$.

Let $G$ be $\mathbb{D}$-generic over the ground model $V$. Work in $V[G]$ and define $g = \bigcup \{a_p | p \in G\}$.

Then $g : \omega_1 \mapsto \omega_1$ with $F \cap \text{range}(g) = \emptyset$ and $g$ satisfies the desired finite intersection property with all $f_\alpha$'s.

Since this statement can be formulate by a $\Sigma_1$-formula with parameters $\vec{f}, F, m \in H(\omega_2)^V$, we can use BPFA to conclude the given statement also holds in $V$. 
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BPFA implies that $(\triangle)$ holds.

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We can actually do better (i.e. reduce the consistency strength)

**Theorem**

(Δ) can be forced over a model of CH with a proper forcing \( \mathbb{P} \) that satisfies the \( \aleph_2 \)-chain condition.

**Idea of the Proof** The proper forcing \( \mathbb{P} \) is a “matrix version” of Larson’s forcing \( \mathbb{D} \).
Absolute Characterizations

Summary:

- Hjorth proved that there exists a countable model $M$ which characterizes $\aleph_1$ in all models of ZFC.
- Using $M$ he constructed a countable $(M, N)$-full structure $S$.
- $S$ characterizes $\aleph_1$ in models of CH and $\aleph_2$ in models of BPFA.
- One may ask if our results for $\aleph_2$ generalize to higher cardinalities, e.g. $\aleph_3$.
- To prove this one would have to extend our results for functions $f : \omega_1 \mapsto \omega_1$ to functions $f : \omega_2 \mapsto \omega_2$ (which is considerably harder).
- However, the main question here should be different.
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1. In $\mathbb{L}$, there exists a unique code $c$ for a complete $\mathcal{L}_{\alpha^+, \omega}$-sentence $\psi_\alpha$ such that $\Phi(\alpha, c)$ holds.
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Fact (Shoenfield absoluteness)

$\Sigma^1_3$-statements are upwards absolute between transitive models of set theory with the same ordinals.

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Is there a $\Sigma^1_3$-formula $\Phi(v_0, v_1)$ in the language of second-order arithmetic with the property that the axioms of ZFC prove that the following statements hold:

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1. For every real $a$, there is a unique real $b$ such that $\Phi(a, b)$ holds.

2. If $\alpha$ is a countable ordinal, $c$ is a code for a complete $L_{\omega_1, \omega}$-sentence that characterizes $\mathbb{N}_\alpha$ and $d$ is a real with the property that $\Phi(c, d)$ holds, then $d$ is a code for a complete $L_{\omega_1, \omega}$-sentence that characterizes $\mathbb{N}_{\alpha+1}$. 
Theorem (Woodin)

The existence of a proper class of Woodin cardinals implies that the theory of $\mathcal{L}(\mathbb{R})$ with real parameters is generically absolute.

Question

Is there a formula $\Phi(v_0, v_1)$ in the language of set theory with the property that the theory $\text{ZFC} + \text{There exists a proper class of Woodin cardinals}$ proves the following statements hold:

1. For every real $a$, there is a unique real $b$ such that $\Phi(a, b)$ holds in $\mathcal{L}(\mathbb{R})$.

2. If $\alpha$ is a countable ordinal, $c$ is a code for a complete $\mathcal{L}_{\omega_1, \omega}$-sentence that characterizes $\aleph_\alpha$ and $d$ is a real with the property that $\Phi(c, d)$ holds in $\mathcal{L}(\mathbb{R})$, then $d$ is a code for a complete $\mathcal{L}_{\omega_1, \omega}$-sentence that characterizes $\aleph_{\alpha+1}$. 
Theorem (Woodin)

The existence of a proper class of Woodin cardinals implies that the theory of $L(\mathbb{R})$ with real parameters is generically absolute.

Question

Is there a formula $\Phi(v_0, v_1)$ in the language of set theory with the property that the theory $\text{ZFC} + \text{There exists a proper class of Woodin cardinals}$ proves the following statements hold:

1. For every real $a$, there is a unique real $b$ such that $\Phi(a, b)$ holds in $L(\mathbb{R})$.

2. If $\alpha$ is a countable ordinal, $c$ is a code for a complete $L_{\omega_1, \omega}$-sentence that characterizes $\aleph_\alpha$ and $d$ is a real with the property that $\Phi(c, d)$ holds in $L(\mathbb{R})$, then $d$ is a code for a complete $L_{\omega_1, \omega}$-sentence that characterizes $\aleph_{\alpha+1}$. 
Theorem (Woodin)

The existence of a proper class of Woodin cardinals implies that the theory of $L(\mathbb{R})$ with real parameters is generically absolute.

Question

Is there a formula $\Phi(v_0, v_1)$ in the language of set theory with the property that the theory $\text{ZFC} + \text{There exists a proper class of Woodin cardinals}$ proves the following statements hold:

1. For every real $a$, there is a unique real $b$ such that $\Phi(a, b)$ holds in $L(\mathbb{R})$.

2. If $\alpha$ is a countable ordinal, $c$ is a code for a complete $\mathcal{L}_{\omega_1, \omega}$-sentence that characterizes $\mathbb{N}_\alpha$ and $d$ is a real with the property that $\Phi(c, d)$ holds in $L(\mathbb{R})$, then $d$ is a code for a complete $\mathcal{L}_{\omega_1, \omega}$-sentence that characterizes $\mathbb{N}_{\alpha + 1}$. 
Theorem (Woodin)

The existence of a proper class of Woodin cardinals implies that the theory of $L(\mathbb{R})$ with real parameters is generically absolute.

Question

Is there a formula $\Phi(v_0, v_1)$ in the language of set theory with the property that the theory ZFC + There exists a proper class of Woodin cardinals proves the following statements hold:

1. For every real $a$, there is a unique real $b$ such that $\Phi(a, b)$ holds in $L(\mathbb{R})$.

2. If $\alpha$ is a countable ordinal, $c$ is a code for a complete $\mathcal{L}_{\omega_1, \omega}$-sentence that characterizes $\aleph_\alpha$ and $d$ is a real with the property that $\Phi(c, d)$ holds in $L(\mathbb{R})$, then $d$ is a code for a complete $\mathcal{L}_{\omega_1, \omega}$-sentence that characterizes $\aleph_{\alpha+1}$. 
Thank you!

Questions?
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