Cohesive Powers of Linear Orders

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Cohesive sets

Let

$$\vec{A} = (A_0, A_1, A_2, \dots)$$

be a countable sequence of subsets of $\mathbb{N}$.

Then there is an infinite set $C \subseteq \mathbb{N}$ such that, for every i:

$$\text{either } C \subseteq^* A_i$$
$$\text{or } C \subseteq^* \overline{A_i}.$$

C is called cohesive for $\vec{A}$, or simply $\vec{A}$-cohesive.

Definition

If $\vec{A}$ is the sequence of computable sets, then C is called r-cohesive.

If $\vec{A}$ is the sequence of c.e. sets, then C is called cohesive.

Skolem's countable non-standard model of true arithmetic

Skolem (1934):

Let C be cohesive for the sequence of arithmetical sets.
(Such a C is also called arithmetically indecomposable.)

Consider arithmetical functions $f, g\colon \mathbb{N} \to \mathbb{N}$. Define:

$$
\begin{array}{lcl}
f =_C g & \text{if} & C \subseteq^* \{n : f(n) = g(n)\} \\
f < g & \text{if} & C \subseteq^* \{n : f(n) < g(n)\} \\
(f + g)(n) & = & f(n) + g(n) \\
(f \times g)(n) & = & f(n) \times g(n)
\end{array}
$$

Let $[f] = \{g : g =_C f\}$ denote the $=_C$-equivalence class of f.

Form a structure $\mathcal{M}$ with domain $\{[f] : f \text{ arithmetical}\}$ and

$$[f] < [g] \text{ if } f < g; \qquad [f] + [g] = [f + g]; \qquad [f] \times [g] = [f \times g].$$

Then $\mathcal{M}$ models true arithmetic!

Effectivizing Skolem's construction

Tennenbaum wanted to know:

What if we did Skolem's construction, but

- used computable functions $f\colon \mathbb{N} \to \mathbb{N}$ in place of arithmetical functions;
- only assumed that C is r-cohesive?

Do we still get models of true arithmetic?

Feferman-Scott-Tennenbaum (1959):

It is not even possible to get models of Peano arithmetic in this way.

Lerman (1970) has further results in this direction:

If you only consider co-maximal sets C, then the structure you get depends only on the many-one degree of C.

(Co-maximal means co-c.e. and cohesive.)

Cohesive products

Let L be a computable language, $(\mathcal{A}_n \mid n \in \mathbb{N})$ be a uniformly computable sequence of L-structures, $|\mathcal{A}_i| \subseteq \mathbb{N}$ and $C \subseteq \mathbb{N}$ be cohesive. The cohesive product of $(\mathcal{A}_n \mid n \in \mathbb{N})$ over C is the L-structure $\Pi_C \mathcal{A}_n$ defined as follows.

Let D be the set of partial computable functions φ such that $\forall n(\varphi(n)\downarrow \to \varphi(n) \in |\mathcal{A}_n|)$ and $C \subseteq^* \mathsf{dom}(\varphi)$.

$$\begin{array}{lcl} \varphi =_C \psi & \text{if} & C \subseteq^* \{n : \varphi(n) = \psi(n)\} \\ R(\psi_0, \dots, \psi_{k-1}) & \text{if} & C \subseteq^* \{n : R^{\mathcal{A}_n}(\psi_0(n), \dots, \psi_{k-1}(n))\} \\ F(\psi_0, \dots, \psi_{k-1})(n) & = & f^{\mathcal{A}_n}(\psi_0(n), \dots, \psi_{k-1}(n)) \end{array}$$

Let $[\varphi]$ denote the $=_C$-equivalence class of φ.

Let $\Pi_C \mathcal{A}_n$ be the structure with domain $\{[\varphi] : \varphi \in D\}$ and

$$\begin{array}{lcl} R([\psi_0], \dots, [\psi_{k-1}]) & \text{if} & R(\psi_0, \dots, \psi_{k-1}) \\ F([\psi_0], \dots, [\psi_{k-1}]) & = & [F(\psi_0, \dots, \psi_{k-1})]. \end{array}$$

Cohesive powers

Dimitrov (2009):

If $\mathcal{A}_n = \mathcal{A}$ is the same fixed computable structure $\mathcal{A}$ for every n, the cohesive product $\Pi_C \mathcal{A}_n$ is called the cohesive power of $\mathcal{A}$ over C and is denoted $\Pi_C \mathcal{A}$.

Cohesive products by co-c.e. cohesive sets also have the helpful property that every member of the cohesive product has a total computable representative.

A computable structure $\mathcal{A}$ always naturally embeds into its cohesive powers.

$$\kappa : x \mapsto \text{ the constant function } x.$$

- If $\mathcal{A}$ is finite and C is cohesive, then every partial computable function $\varphi : \mathbb{N} \to |\mathcal{A}|$ with $C \subseteq^* \mathsf{dom}(\varphi)$ is eventually constant on C, and hence $\mathcal{A} \cong \Pi_C \mathcal{A}$.
- If $\mathcal{A}$ is an infinite computable structure, then every cohesive power $\Pi_C \mathcal{A}$ is countably infinite.

Uniformly n-decidable structures

- A computable structure is a structure having a computable atomic diagram (0-decidable).
- A decidable structure is a structure having a computable elementary diagram.
- An n-decidable structure is a structure having a computable Σ_n-elementary diagram.
- A sequence $(\mathcal{A}_i \mid i \in \mathbb{N})$ of L-structures is uniformly computable, uniformly decidable, or uniformly n-decidable if the respective sequence of atomic, elementary, or Σ_n-elementary diagrams is uniformly computable.

Łoś theorem for n-decidable structures

Theorem

Let L be a computable language, let $(\mathcal{A}_i \mid i \in \mathbb{N})$ be a sequence of uniformly n-decidable L-structures, $|\mathcal{A}_i| \subseteq \mathbb{N}$, and let C be cohesive. Then for any $[\varphi_0], \ldots, [\varphi_{m-1}] \in |\Pi_C \mathcal{A}_i|$

1. if $\Phi(v_0, \ldots, v_{m-1})$ is a Σ_{n+2} formula, then

   $$\Pi_C \mathcal{A}_i \models \Phi([\varphi_0], \ldots, [\varphi_{m-1}]) \to C \subseteq^* \{i \mid \mathcal{A}_i \models \Phi(\varphi_0(i), \ldots, \varphi_{m-1}(i))\}$$

2. if $\Phi(v_0, \ldots, v_{m-1})$ is a Π_{n+2} formula, then

   $$C \subseteq^* \{i \mid \mathcal{A}_i \models \Phi(\varphi_0(i), \ldots, \varphi_{m-1}(i))\} \to \Pi_C \mathcal{A}_i \models \Phi([\varphi_0], \ldots, [\varphi_{m-1}])$$

3. if $\Phi(v_0, \ldots, v_{m-1})$ is a Δ_{n+2} formula, then

   $$C \subseteq^* \{i \mid \mathcal{A}_i \models \Phi(\varphi_0(i), \ldots, \varphi_{m-1}(i))\} \leftrightarrow \Pi_C \mathcal{A}_i \models \Phi([\varphi_0], \ldots, [\varphi_{m-1}])$$

Łoś theorem for n-decidable structures

Dimitrov : For cohesive powers of a computable structure the fundamental theorem of cohesive powers holds.

1. Łoś's theorem holds for Σ_2 sentences and Π_2 sentences.
2. One-way Łoś's theorem holds for Σ_3 sentences.

Theorem (Łoś's theorem for cohesive powers)

Let L be a computable language, $\mathcal{A}$ be an n-decidable structure, and let C be cohesive. Then

1. If Φ is a Δ_{n+3} sentence then
$$\Pi_C \mathcal{A} \models \Phi \quad \text{if and only if} \quad \mathcal{A} \models \Phi$$
2. If Φ is a Σ_{n+3} sentence, then
$$\mathcal{A} \models \Phi \quad \text{implies} \quad \Pi_C \mathcal{A} \models \Phi$$

If $\mathcal{A}$ is decidable structure then $\Pi_C \mathcal{A} \equiv \mathcal{A}$.

An observation

Example

Consider $\mathbb{Q}$ as a linear order (i.e., as a structure in the language $\{<\}$.)

$\mathbb{Q}$ is a countable dense linear order without endpoints.
If $\mathcal{L}$ is a countable dense linear order without endpoints, then $\mathcal{L} \cong \mathbb{Q}$.
"Dense linear order w/o endpoints" is axiomatized by a Π_2 sentence θ.

If C is any cohesive set, then $\Pi_C\mathbb{Q} \models \theta$ by Łoś for cohesive powers.

So $\Pi_C\mathbb{Q}$ is a countable dense linear order without endpoints.

Thus $\Pi_C\mathbb{Q} \cong \mathbb{Q}$.

(Not an accident: $\Pi_C\mathcal{A} \cong \mathcal{A}$ whenever $\mathcal{A}$ is uniformly locally finite ultrahomogeneous, i.e. every isomorphism between two finitely-generated substructures in a sufficiently effective way extends to an automorphism on $\mathcal{A}$. Examples are the computable presentations of the Rado graph and the countable atomless Boolean algebra.)

Reducts and substructures

Let $L \subseteq L^+$ be two languages, and let $\mathcal{A}$ be an L^+-structure. Then the reduct $\mathcal{A} \upharpoonright L$ of $\mathcal{A}$ is the L-structure obtained from $\mathcal{A}$ by forgetting about the symbols of $L^+ \setminus L$.

Proposition

Let $L \subseteq L^+$ be computable languages, $(\mathcal{A}_n \mid n \in \mathbb{N})$ be a uniformly computable sequence of L^+-structures and $C \subseteq \mathbb{N}$ be cohesive. Then

$$\Pi_C(\mathcal{A}_n \upharpoonright L) \cong (\Pi_C \mathcal{A}_n) \upharpoonright L$$

Proposition

Let L be a computable language with a unary relation symbol U. Let $\mathcal{A}$ be a computable L-structure, and suppose that $\{a \in |\mathcal{A}| \mid \mathcal{A} \models U(a)\}$ forms the domain of a computable substructure $\mathcal{B}$ of $\mathcal{A}$. Let C be a cohesive set. Then $\{[\varphi] \in |\Pi_C(\mathcal{A})| : \Pi_C \mathcal{A} \models U([\varphi])\}$ forms the domain of a substructure $\mathcal{D}$ of $\Pi_C \mathcal{A}$ and $\Pi_C \mathcal{B} \cong \mathcal{D}$.

Disjoint unions

Let L be a relational language, and let $\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}$ be L-structures. Then the disjoint union of $\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}$ is the L-structure $\bigsqcup_{i<k} \mathcal{A}_i$ with domain $\bigcup_{i<k}\{i\} \times |\mathcal{A}_i|$ and $R^{\bigsqcup_{i<k} \mathcal{A}_i}((i_0, x_0), \ldots, (i_{m-1}, x_{m-1}))$ if $i_0 = \cdots = i_{m-1} = i$ for some $i < k$ and $R^{\mathcal{A}_i}(x_0, \ldots, x_{m-1})$.

Proposition

Let L be a computable language and let $\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}$ be L-structures, and $C \subseteq \mathbb{N}$ be cohesive. Then

$$\Pi_C \bigsqcup_{i<k} \mathcal{A}_i \cong \bigsqcup_{i<k} \Pi_C \mathcal{A}_i.$$

Saturation

Fact: for a countable language, ultraproducts over countably incomplete ultrafilters (i.e., ultrafilters that are not closed under countable intersections) are always $\aleph_1$-saturated.

A structure $\mathcal{A}$ is recursively saturated if it realizes every computable type over $\mathcal{A}$.

$\mathcal{A}$ is Σ_n-recursively saturated if it realizes every computable Σ_n-type over $\mathcal{A}$.

Theorem

Let L be a computable language, and $C \subseteq \mathbb{N}$ be cohesive.

1. Let $(\mathcal{A}_n \mid n \in \mathbb{N})$ be a sequence of uniformly decidable L-structures. Then $\Pi_C \mathcal{A}_n$ is recursively saturated.
2. Let $(\mathcal{A}_n \mid n \in \mathbb{N})$ be a sequence of uniformly n-decidable L-structures. Then $\Pi_C \mathcal{A}_n$ is recursively Σ_n-saturated.
3. For a decidable L-structure $\mathcal{A}$, $\Pi_C \mathcal{A}$ is recursively saturated.
4. For an n-decidable L-structure $\mathcal{A}$, $\Pi_C \mathcal{A}$ is recursively Σ_n-saturated.

Saturation and isomorphism

Theorem

Let L be a computable language, and $C \subseteq \mathbb{N}$ be co-c.e. cohesive.

1. Let $(\mathcal{A}_n \mid n \in \mathbb{N})$ be a sequence of uniformly n-decidable L-structures. Then $\prod_C \mathcal{A}_n$ is recursively Σ_{n+1}-saturated.
2. For an n-decidable L-structure $\mathcal{A}$, $\prod_C \mathcal{A}$ is recursively Σ_{n+1}-saturated.

Theorem

Let L be a computable language, let $\mathcal{A}_0$ and $\mathcal{A}_1$ be computable L-structures that are computably isomorphic, and let C be cohesive. Then $\prod_C \mathcal{A}_0 \cong \prod_C \mathcal{A}_1$.

Corollary

If $\mathcal{A}$ is a computable L-structures which is computably categorical, then for every structure $\mathcal{B} \cong \mathcal{A}$ we have $\prod_C \mathcal{A} \cong \prod_C \mathcal{B}$.

Linear orders

Theorem

Let $\mathcal{L} = (\mathrm{L}, \prec_{\mathcal{L}})$ and $\mathcal{M} = (\mathrm{M}, \prec_{\mathcal{M}})$ be computable linear orders, and let C be a cohesive.

1. Sum $\prod_{\mathrm{C}}(\mathcal{L} + \mathcal{M}) \cong \prod_{\mathrm{C}}\mathcal{L} + \prod_{\mathrm{C}}\mathcal{M}$,
2. Product $\prod_{\mathrm{C}}(\mathcal{L}\mathcal{M}) \cong \left(\prod_{\mathrm{C}}\mathcal{L}\right)\left(\prod_{\mathrm{C}}\mathcal{M}\right)$, and
3. Reverse $\prod_{\mathrm{C}}(\mathcal{L}^*) \cong \left(\prod_{\mathrm{C}}\mathcal{L}\right)^*$.

The product $\mathcal{L}\mathcal{M}$ is a linear order $\mathcal{P} = (\mathrm{P}, \prec_{\mathcal{P}})$, where $\mathrm{P} = \mathrm{M} \times \mathrm{L}$ and

$$(\mathrm{x}, \mathrm{a}) \prec_{\mathcal{P}} (\mathrm{y}, \mathrm{b}), \quad \text{if and only if} \quad (\mathrm{x} \prec_{\mathcal{M}} \mathrm{y}) \text{ or } (\mathrm{x} = \mathrm{y} \text{ and } \mathrm{a} \prec_{\mathcal{L}} \mathrm{b}).$$

- ω — the order type of $(\mathbb{N}; <)$.
- ζ — the order type of $(\mathbb{Z}; <)$.
- η — the order type of $(\mathbb{Q}; <)$.

Linear orders: condensation

Let $\mathcal{L} = (\mathrm{L}, \prec_{\mathcal{L}})$ be a linear order.

Definition

A condensation of $\mathcal{L}$ is any linear order $\mathcal{M} = (\mathrm{M}, \prec_{\mathcal{M}})$ obtained by partitioning L into a collection of non-empty intervals M and, for $\mathrm{I}, \mathrm{J} \in \mathrm{M}$, $\mathrm{I} \prec_{\mathcal{M}} \mathrm{J}$ if and only if $(\forall \mathrm{a} \in \mathrm{I})(\forall \mathrm{b} \in \mathrm{J})(\mathrm{a} \prec_{\mathcal{L}} \mathrm{b})$.

Definition

For $\mathrm{x} \in \mathrm{L}$, let $\mathrm{c_F}(\mathrm{x})$ denote the set of $\mathrm{y} \in \mathrm{L}$ for which there are only finitely many elements between x and y:

$$\mathrm{c_F}(\mathrm{x}) = \{\mathrm{y} \in \mathrm{L} : \text{the interval } [\mathsf{min}_{\prec_{\mathcal{L}}}\{\mathrm{x}, \mathrm{y}\}, \mathsf{max}_{\prec_{\mathcal{L}}}\{\mathrm{x}, \mathrm{y}\}]_{\mathcal{L}} \text{ in } \mathcal{L} \text{ is finite}\}.$$

The set $\mathrm{c_F}(\mathrm{x}) \neq \emptyset$, as $\mathrm{x} \in \mathrm{c_F}(\mathrm{x})$. The finite condensation $\mathrm{c_F}(\mathcal{L})$ of $\mathcal{L}$ is the condensation obtained from the partition $\{\mathrm{c_F}(\mathrm{x}) : \mathrm{x} \in \mathrm{L}\}$.

For example, $\mathrm{c_F}(\omega) \cong 1$, $\mathrm{c_F}(\zeta) \cong 1$, $\mathrm{c_F}(\eta) \cong \eta$, and $\mathrm{c_F}(\omega + \zeta\eta) \cong 1 + \eta$. Notice that the order-type of $\mathrm{c_F}(\mathrm{x})$ is always either finite, ω, ω^*, or ζ.

Linear orders

Let $(\mathcal{L}_n \mid n \in \mathbb{N})$ be a uniformly computable sequence of linear orders, let C be cohesive.

Lemma

Let $[\psi]$ and $[\varphi]$ be elements of $\prod_C \mathcal{L}_n$. Then the following are equivalent.

(1) $[\varphi]$ is the $\prec_{\prod_C \mathcal{L}_n}$-immediate successor of $[\psi]$.

(2) $(\forall^\infty n \in C)(\varphi(n)$ is the $\prec_{\mathcal{L}_n}$-immediate successor of $\psi(n))$.

(3) $(\exists^\infty n \in C)(\varphi(n)$ is the $\prec_{\mathcal{L}_n}$-immediate successor of $\psi(n))$.

Moreover $[\psi] \preccurlyeq_{\prod_C \mathcal{L}_n} [\varphi]$ iff $\lim_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}_n}| = \infty$.

Theorem

Let $(\mathcal{L}_n \mid n \in \mathbb{N})$ be a uniformly computable sequence of linear orders, let C be cohesive. If either $(\mathcal{L}_n \mid n \in \mathbb{N})$ is uniformly 1-decidable or C is co-c.e. then $c_F(\prod_C \mathcal{L}_n)$ is dense.

Cohesive powers of computable copies of ω

Let $\mathcal{L}$ be a computable copy of ω, and let C be cohesive.

Lemma

- The image of the canonical embedding of $\mathcal{L}$ into $\prod_C \mathcal{L}$ is an initial segment of $\prod_C \mathcal{L}$ of order-type ω.
- So, $\prod_C(\mathcal{L}) \cong \omega + \mathcal{M}$, for some linear order $\mathcal{M}$. ω-standard part and $\mathcal{M}$-nonstandard.
- If $[\varphi]$ is an element of $\prod_C \mathcal{L}$ then $[\varphi]$ is non-standard if and only if $\lim_{n \in C} \varphi(n) = \infty$.
- If $[\varphi]$ is nonstandard element of $\prod_C \mathcal{L}$ then there are nonstandard elements $[\psi^-]$ and $[\psi^+]$ of $\prod_C \mathcal{L}$, in other blocks of $[\varphi]$, such that $[\psi^-] \preccurlyeq_{\prod_C \mathcal{L}} [\varphi] \preccurlyeq_{\prod_C \mathcal{L}} [\psi^+]$. ($\lim_{n \in C} |(\psi^-(n), \varphi(n))_{\mathcal{L}}| = \infty$).

Cohesive powers of computable copies of ω

Let $\mathcal{L}$ be a computable copy of ω, and let C be cohesive.

Theorem

- If either $\mathcal{L}$ is 1-decidable or C is co-c.e. then $c_F(\Pi_C \mathcal{L}) = 1 + \eta$.
- If $\mathcal{L}$ is computably isomorphic to the standard presentation of ω then $\Pi_C \mathcal{L}$ has order type $\omega + \zeta\eta$.

Examples

Example

Let C be a cohesive set. Let $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Q}$ denote the standard presentations of ω, ζ, and η.

- $\Pi_{\mathrm{C}}\mathbb{N}^* \cong (\Pi_{\mathrm{C}}\mathbb{N})^* \cong (\omega + \zeta\eta)^* \cong \zeta\eta + \omega^*$.
- $\Pi_{\mathrm{C}}\mathbb{Z} \cong \Pi_{\mathrm{C}}(\mathbb{N}^* + \mathbb{N}) \cong \zeta\eta + \omega^* + \omega + \zeta\eta \cong \zeta\eta + \zeta + \zeta\eta \cong \zeta\eta$.
- $\Pi_{\mathrm{C}}\mathbb{Z}\mathbb{Q} \cong (\Pi_{\mathrm{C}}\mathbb{Z})(\Pi_{\mathrm{C}}\mathbb{Q}) \cong \zeta\eta\eta \cong \zeta\eta$.
- $\Pi_{\mathrm{C}}(\mathbb{N} + \mathbb{Z}\mathbb{Q}) \cong (\Pi_{\mathrm{C}}\mathbb{N}) + (\Pi_{\mathrm{C}}\mathbb{Z}\mathbb{Q}) \cong (\omega + \zeta\eta) + (\zeta\eta) \cong \omega + \zeta\eta$.

Are there other cohesive powers of $\mathbb{N}$?

More properly:

Is there a computable copy $\mathcal{L}$ of $\mathbb{N}$ with $\Pi_C \mathcal{L} \ncong \omega + \zeta\eta$?

Such an $\mathcal{L}$ cannot be isomorphic to $\mathbb{N}$ via a computable isomorphism.

Classic computable copy $\mathcal{L} = (\mathbb{N}, \prec)$ of $\mathbb{N}$ with non-computable isomorphism (the successor is not computable).

- Let $f\colon \mathbb{N} \to \mathbb{N}$ be computable injection with $\mathsf{ran}(f) = K = \{e : \Phi_e(e)\downarrow\}$.
- Put the evens in their usual order: $2a \prec 2b$ if $2a < 2b$.
- For each s, put $2s + 1$ between $2f(s)$ and $2f(s) + 2$: $2f(s) \prec 2s + 1 \prec 2f(s) + 2$.

However:

We still get $\Pi_C \mathcal{L} \cong \omega + \zeta\eta$ for every cohesive C.

So, it is not enough just to ensure that the isomorphism $\mathcal{L} \cong \mathbb{N}$ is non-computable!

A different cohesive power of $\mathbb{N}$

Theorem

For every co-c.e. cohesive set C, there is a computable copy $\mathcal{L}$ of $\mathbb{N}$ such that $\Pi_C\mathcal{L} \not\cong \omega + \zeta\eta$.

Idea: Build $\mathcal{L} = (\mathbb{N}, \prec)$ so that $[\mathrm{id}]$ does not have an immediate successor in the cohesive power $\Pi_C\mathcal{L}$.

To do this, ensure that

$$\forall^\infty n \in C\ (\varphi_e(n)\downarrow \Rightarrow \varphi_e(n) \text{ is not the } \prec\text{-immediate successor of } n)$$

Then $[\varphi_e]$ is not the immediate successor of $[\mathrm{id}]$ in $\Pi_C\mathcal{L}$.

Corollary

There is a computable linear order $\mathcal{L}$, a cohesive set C, and a Π_3-sentence Φ such that $\mathcal{L} \models \Phi$, but $\Pi_C\mathcal{L} \not\models \Phi$.

Proposition

There is a unif. comp. seq. of finite linear orders $(\mathcal{L}_n \mid n \in \mathbb{N})$ such that the cohesive product $\Pi_C\mathcal{L}_n$ is a linear order with no maximum element.

Coloured linear orders

Definition

A coloured linear order is a structure $\mathcal{O} = (\mathrm{L}, \mathbb{N}, \prec_{\mathcal{L}}, \mathrm{F})$, where $\mathcal{L} = (\mathrm{L}, \prec_{\mathcal{L}})$ is a linear order and F is (the graph of) a function $\mathrm{F} : \mathrm{L} \to \mathbb{N}$, thought of as a colouring of L.

If $\mathcal{O}$ is a computable coloured linear order and C is a cohesive set, then the cohesive power $\Pi_{\mathrm{C}}\mathcal{O}$ consists of a linear order $\Pi_{\mathrm{C}}\mathcal{L}$, a set $\Pi_{\mathrm{C}}\mathbb{N}$ thought of as a collection of colours, and a (graph of a) function F thought of as a colouring of $\Pi_{\mathrm{C}}\mathcal{L}$.
Call a colour $\| \delta \| \in \Pi_{\mathrm{C}}\mathbb{N}$ a solid colour if δ is eventually constant on C (i.e., if $\| \delta \|$ is in the range of the canonical embedding of $\mathbb{N}$ into $\Pi_{\mathrm{C}}\mathbb{N}$).
Otherwise, call $\| \delta \|$ a striped colour.
If $\mathcal{L}$ is a copy of ω then we call $\mathcal{O}$ a coloured copy of ω.

Colourful linear orders

Definition

Call the cohesive power $\Pi_C \mathcal{O}$ colourful if the following items hold:
For every pair of non-standard elements $[\phi], [\psi] \in \Pi_C \mathcal{L}$ with
$[\psi] \prec_{\Pi_C \mathcal{L}} [\varphi]$

- and every solid colour $\| \delta \| \in \Pi_C \mathbb{N}$, there is a $[\theta] \in \Pi_C \mathcal{L}$ with $[\psi] \prec_{\Pi_C \mathcal{L}} [\theta] \prec_{\Pi_C \mathcal{L}} [\varphi]$ and $\mathrm{F}([\theta]) = \| \delta \|$.
- there is a $[\theta] \in \Pi_C \mathcal{L}$ with $[\psi] \prec_{\Pi_C \mathcal{L}} [\theta] \prec_{\Pi_C \mathcal{L}} [\varphi]$ where $\mathrm{F}([\theta])$ is a striped colour.

Theorem

Let C be a co-c.e. cohesive set. Then there is a computable coloured copy $\mathcal{O}$ of ω such that $\Pi_C \mathcal{O}$ is colourful.

If C is a co-c.e. cohesive set, then the first bullet of Definition implies the second.

Colourful linear orders

We construct a linear order $\mathcal{O} = (\mathrm{L}, \mathbb{N}, \prec_{\mathcal{L}}, \mathrm{F})$, with $\mathcal{L} \cong \omega$.

- C - co-c.e. cohesive set, then $[\phi] \in \Pi_{\mathrm{C}}\mathcal{L}$ has a total computable el.
- $[\phi]$ is non-standard if and only if $\lim_{\mathrm{n} \in \mathrm{C}} \varphi(\mathrm{n}) = \infty$.
- for every pair of total computable functions φ and ψ with $\lim_{\mathrm{n} \in \mathrm{C}} \varphi(\mathrm{n}) = \lim_{\mathrm{n} \in \mathrm{C}} \psi(\mathrm{n}) = \infty$:

$$(\forall^{\infty} \mathrm{n} \in \mathrm{C})(\psi(\mathrm{n}) \downarrow \prec_{\mathcal{L}} \varphi(\mathrm{n}) \downarrow \Rightarrow$$
$$(\forall \mathrm{d} \leq \max_{<}(\varphi(\mathrm{n}), \psi(\mathrm{n}))(\exists \mathrm{k})(\psi(\mathrm{n}) \prec_{\mathcal{L}} \mathrm{k} \prec_{\mathcal{L}} \varphi(\mathrm{n}) \ \& \ (\mathrm{F}(\mathrm{k}) = \mathrm{d}))$$

- Thus between $[\psi]$ and $[\varphi]$ there are elements of $\Pi_{\mathrm{C}}\mathcal{L}$ of every solid colour and also at least one element of a striped colour.

A computable copy of ω with a cohesive power of order-type $\omega + \eta$

Theorem

For every co-c.e. cohesive set C, there is a computable copy $\mathcal{L}$ of $\mathbb{N}$ such that

$$\prod_{\mathrm{C}} \mathcal{L} \;\cong\; \omega + \eta.$$

Proof.

Let C be co-c.e. and cohesive. Let $\mathcal{O} = (\mathrm{L}, \mathbb{N}, \prec_{\mathcal{L}}, \mathrm{F})$ be the computable coloured copy of ω. Let $\mathcal{L} = (\mathrm{L}, \prec_{\mathcal{L}})$ denote the computable copy of ω. The cohesive power $\prod_{\mathrm{C}} \mathcal{L}$ has an initial segment of order-type ω. There is neither a least nor greatest non-standard element of $\prod_{\mathrm{C}} \mathcal{L}$. By the previous theorem the non-standard elements of $\prod_{\mathrm{C}} \mathcal{L}$ are dense. So $\prod_{\mathrm{C}} \mathcal{L}$ consists of a standard part of order-type ω and a non-standard part that forms a countable dense linear order without endpoints. So, $\prod_{\mathrm{C}} \mathcal{L} \cong \omega + \eta$. □

Non-elementary equivalent

Example

Let C be a co-c.e. cohesive set, and let $\mathcal{L}$ is a computable copy of ω with $\Pi_{\mathrm{C}}\mathcal{L} \cong \omega + \eta$.

1. Let $\mathrm{k} \geq 1$, and $\overline{\mathrm{k}}$ denote a linear order with k elements $0 < 1 < \cdots < \mathrm{k} - 1$. Then $\overline{\mathrm{k}}\mathcal{L} \cong \omega$

   $$\Pi_{\mathrm{C}}(\overline{\mathrm{k}}\mathcal{L}) \;\cong\; \big(\Pi_{\mathrm{C}}\overline{\mathrm{k}}\big)\big(\Pi_{\mathrm{C}}\mathcal{L}\big) \;\cong\; \overline{\mathrm{k}}(\omega + \eta) \;\cong\; \omega + \overline{\mathrm{k}}\eta.$$

   The linear orders $\omega + \overline{\mathrm{k}}\eta$ for $\mathrm{k} \geq 1$ are pairwise non-elementarily equivalent.

2. Consider the computable linear orders $\mathcal{L}$ and $\mathcal{L} + \mathbb{Q}$. They are not elementarily equivalent because the sentence "every element has an immediate successor" is true of $\mathcal{L}$ but not of $\mathcal{L} + \mathbb{Q}$. However, using the last theorem and the fact that $\Pi_{\mathrm{C}}\mathbb{Q} \cong \eta$, we calculate

   $$\Pi_{\mathrm{C}}(\mathcal{L} + \mathbb{Q}) \;\cong\; \Pi_{\mathrm{C}}\mathcal{L} + \Pi_{\mathrm{C}}\mathbb{Q} \;\cong\; (\omega + \eta) + \eta \;\cong\; \omega + \eta \;\cong\; \Pi_{\mathrm{C}}\mathcal{L}.$$

A generalized sum

Definition

Let $\mathcal{L}$ be a linear order, and let $(\mathcal{M}_l \mid l \in |\mathcal{L}|)$ be a sequence of linear orders indexed by $|\mathcal{L}|$. The generalized sum $\Sigma_{l\in|\mathcal{L}|}\mathcal{M}_l$ of $(\mathcal{M}_l \mid l \in |\mathcal{L}|)$ over $\mathcal{L}$ is the linear order $\mathcal{S} = (S, \prec_{\mathcal{S}})$ defined as follows:
$S = \{(l, m) \mid l \in L \ \&\ m \in \mathcal{M}_l\}$, and $(l_0, m_0) \prec_{\mathcal{S}} (l_1, m_1)$ if and only if $(l_0 \prec_{\mathcal{L}} l_1) \vee (l_0 = l_1 \ \&\ m_0 \prec_{\mathcal{M}_{l_0}} m_1)$.

Example

$\mathcal{L}_1 + \mathcal{L}_2 = \Sigma_{i\in\overline{2}}\mathcal{L}_i$ and $\mathcal{L}_1\mathcal{L}_2 = \Sigma_{l\in|\mathcal{L}_2|}\mathcal{L}_1$

Theorem

Let $\mathcal{L}$ be a computable linear order, and let $(\mathcal{M}_l \mid l \in |\mathcal{L}|)$ be a uniformly computable sequence of linear orders indexed by $|\mathcal{L}|$. Let C be a cohesive set. Then

$$\Pi_C \Sigma_{l\in|\mathcal{L}|}\mathcal{M}_l \cong \Sigma_{[\theta]\in\Pi_C\mathcal{L}}\Pi_C\mathcal{M}_{\theta(n)}$$

A shuffle sum

Definition

Let X be a non-empty collection of linear orders with $|X| \leq \aleph_0$. Let $f : \mathbb{Q} \to X$ be a function such that $f^{-1}(\mathcal{M})$ is dense in $\mathbb{Q}$ for each linear order $\mathcal{M} \in X$. Let $\mathcal{S} = \Sigma_{q \in \mathbb{Q}} f(q)$ be the generalized sum of the sequence $(f(q) \mid q \in \mathbb{Q})$ over $\mathbb{Q}$. By density, the order-type of $\mathcal{S}$ does not depend on the particular choice of f. Therefore $\mathcal{S}$ is called the shuffle of X and is denoted $\sigma(X)$.

Example

We want $\mathcal{M} \cong \omega$: $\Pi_C \mathcal{M} \cong \omega + \sigma(\{\overline{2}, \overline{3}\})$.

- Start with $\mathcal{L}$ with $\Pi_C \mathcal{L} \cong \omega + \eta$ and $\mathcal{O} = (L, \mathbb{N}, \prec_{\mathcal{L}}, F)$.
- Collapse F into a colouring $G : L \to \{0, 1\}$. $(G(n) = sg(F(n)))$.
- The colours $\|0\|$ and $\|1\|$ are dense in the non-standard p. of $\Pi_C \mathcal{L}$.
- Replace the elements of $\mathcal{L}$ with colour 0 — with a copy of $\overline{2}$, and with colour 1 — with a copy of $\overline{3}$.

Then $\Pi_C \mathcal{M} \cong \omega + \sigma(\{\overline{2}, \overline{3}\})$.

Shuffle of finite orders

Proposition

Let $k_0, \ldots, k_N$ be nonzero natural numbers and let $\mathcal{O}$ be a computable coloured copy of ω. There is a computable copy $\mathcal{L}$ of ω (constructed from $\mathcal{O}$) such that for every cohesive set C, if $\Pi_C\mathcal{L}$ is colourful, then $\Pi_C\mathcal{L}$ has order type $\omega + \sigma(\{\overline{k}_0, \ldots, \overline{k}_N\})$.

When shuffling infinite collections of finite linear orders into a cohesive power of a computable copy of ω, we start with a computable colored copy of ω and replace its elements by arbitrarily large finite linear orders. If the finite linear orders can be uniformly computably expanded to models of Γ, a set with Π_2 sentences, that says that every element except the last element has an immediate successor, every element except the first element has an immediate predecessor, there is a unique least element and a unique greatest element, then this replacement process naturally shuffles the linear order $\omega + \zeta\eta + \omega^*$ into the cohesive power.

The main result

Expand the language of linear orders to $\mathcal{D}$ with the immediate successor relation, the least element and the greatest element.

Proposition

Let $(\mathcal{M}_n \mid n \in I)$ be a uniformly computable sequence of $\mathcal{D}$ structures, that are all finite models of Γ, indexed by a computable $I \subseteq \mathbb{N}$. Let C be a cohesive set. Let $\theta : \mathbb{N} \to I$ be a partial computable function with $C \subseteq^* \mathsf{dom}(\theta)$. Suppose that $\lim n \in C|M_{\theta(n)}| = \infty$. Then, as a linear order, $\Pi_C \mathcal{M}_{\theta(n)}$ has order-type $\omega + \zeta\eta + \omega^*$.

Theorem

Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets thought of as a set of finite order types. Let C be a co-c.e. cohesive set. There is a computable copy $\mathcal{L}$ of ω such that $\Pi_C \mathcal{L}$ has order type $\omega + \sigma(X \cup \{\omega + \zeta\eta + \omega^*\})$.

Moreover if X is finite and non-empty, then there is also a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C \mathcal{L}$ has order-type $\omega + \sigma(X)$.

A new result

Theorem (Paul Shafer)

Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets thought of as a set of finite order types. There is a computable copy $\mathcal{L}$ of ω such that for every Δ_2 cohesive set C the cohesive power $\Pi_C \mathcal{L}$ has order type $\omega + \sigma(X \cup \{\omega + \zeta\eta + \omega^*\})$.
Moreover if X is finite and non-empty, then there is also a computable copy $\mathcal{L}$ of ω where the cohesive power $\Pi_C \mathcal{L}$ has order-type $\omega + \sigma(X)$.

Byproduct:
Martin, 1963: There is an infinite Π_1 set with no Π_1 cohesive subset.
Shafer, 2022: There is an infinite Π_1 set with no Δ_2 cohesive subset.

THANK YOU!