(Not) Computing linear orders

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April 23, 2020
Breaking uniform computations: linear orders with "wild" intervals
Measures of complexity, 1/2

By *structure* we will mean a countable structure in a finite language. A *copy* of a structure is an isomorphic structure with domain $\omega$. We identify copies with their atomic diagrams. A few natural ways to compare structures:

- Muchnik (weak, nonuniform) reducibility: $A \leqsw B$ iff for every copy $B$ of $B$ there is a copy $A$ of $A$ with $A \leq T B$.
- Medvedev (strong, uniform) reducibility: $A \leqss B$ iff there is some $e \in \omega$ such that $\Phi_B e \sim A$ for every $B \sim B$.
- Medvedev-mod-parameters: $A \leqss/p B$ iff there is some tuple $c \in B$ such that $A \leqss (B, c)$.

Theorem (Kallimullin) Only the obvious implications hold in general (although in all known natural examples, $\leqsw$ and $\leqss/p$ coincide).

Theorem (Turetsky, Westrick) "$\leqss/p$" is not transitive in general.
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Muchnik reducibility is much better understood than Medvedev reducibility.

Theorem (Essentially Sacks)

For countable ordinals $\alpha, \beta$ (thought of as linear orders) we have $\alpha \leq_w \beta$ iff $\alpha < \omega_1^{CK}(\beta)$.

Question (Hamkins, Li)

What does the degree structure $D^\text{ord}_s$ of the countable ordinals modulo $\leq_s$ look like?

This is still largely open;
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Things are a bit clearer if we look at uniform *hyperarithmetic* reducibility, but still largely unknown.
Definition

A wild interval in a linear order $L$ is an endpointed interval $[a, b]_L$ with $[a, b]_L \not\leq_s L$. $L$ is wild if $L$ has a wild interval.

(Obviously impossible for $\leq_w$.)

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(Obviously impossible for $\leq w$.) Since well-orderings compare nicely, $\leq_s$-incomparable ordinals yield wild intervals. Unfortunately, the examples produced are hideously large: images of $\omega_1$ under Mostowski collapse of “sufficiently large” countable substructures of the universe. Can we do better?
A simple wild interval

Theorem (Knight, Soskova)

*There wild linear orders low in the arithmetical hierarchy.*
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(Sacks: impossible for ordinals.) Key observation:

$$\mathcal{A} \leq_{s} \mathcal{B} \implies \Sigma_{n}^{c} Th(\mathcal{A}) \leq_{e} \Sigma_{n}^{c} Th(\mathcal{B}).$$

Now look for "shapeable" linear orders with well-understood \(\Sigma_{n}^{c}\)-theories for some rich enough \(n\): shuffle sums.

Definition

For \(X \subseteq \omega + 1\), \(Shuff(X)\) is the unique-up-to-isomorphism linear order of the form

$$\sum_{q \in \mathbb{Q}} L_{q}$$

where each \(L_{q}\) has ordertype \(n\) for some \(n \in X\) and \(\{q \in \mathbb{Q} : |L_{q}| = n\}\) is dense for \(n \in X\).

Knight-Soskova use \(Shuff(X \cup \{\omega\} \cup \mathbb{Q}) + 1 + \mathcal{A}\) for \(X \not\in \Sigma_{0}^{3}\).
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for \( X \not\in \Sigma_3^0 \).
Beyond shuffle sums

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**Question**

*Is there a scattered wild linear order low in the arithmetical hierarchy?*

**Question**

*Can a linear order be “totally wild” (whatever that means)?*

Former appears to point towards ordinals; latter interesting on its own.
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Theorem (S.)
Yes to both ... unhelpfully.

Both arguments via forcing. Nice feature: “coarse” argument relativizes to generally uniform reducibility notions (although we eventually need large cardinals), while Σⁿᶜ-analysis-approach doesn’t seem to.
A scattered wild linear order

We code reals into linear orders, then build a \(\mathbb{Z}\)-sum with separators out of a Cohen array.
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For $X \subseteq \omega$, $n \in \omega$ let

$$A(X, i) = \begin{cases} \zeta + i + 2 + \zeta & \text{if } i \in X, \\ \zeta + i + 3 + \zeta & \text{if } i \notin X \end{cases}$$

and set $A(X) = \zeta + \sum_{i \in \omega} A(X, i) + \zeta$. 
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- For $\mathcal{X} = (X_z)_{z \in \mathbb{Z}}$ a doubly-infinite sequence of reals, let

$$S(\mathcal{X}) = \sum_{z \in \mathbb{Z}} [1 + A(X_z)]$$
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$S(\mathcal{X})$ is scattered. We’ll look at $S(\mathcal{X})$ for $\mathcal{X}$ sufficiently (?) generic:

Definition

$\mathbb{P}_{\text{sca}}$ is the set of arrays $p \in (2^{<\omega})^{\mathbb{Z}}$ with $p_z = \emptyset$ for cofinitely many $z \in \mathbb{Z}$. (This is just Cohen forcing rephrased.)
\[ S(\mathcal{X}) = \ldots + 1 + A(X_{-1}) + 1 + A(X_0) + 1 + A(X_1) + 1 + \ldots \]

Let \( I^\mathcal{X}_z \) be the \( (1 + A(X_z) + 1) \)-part of \( S(\mathcal{X}) \); under mild genericity, this is unique.
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**Theorem**

Suppose \( \mathcal{X} \) is sufficiently \((?)\) \( P_{sca} \)-generic. Then for each \( z \in \mathbb{Z} \) we have \( I_z \nsubseteq_s S(\mathcal{X}) \).
$$S(\mathcal{X}) = ... + 1 + A(X_{-1}) + 1 + A(X_0) + 1 + A(X_1) + 1 + ...$$

Let $I^\mathcal{X}_z$ be the $(1 + A(X_z) + 1)$-part of $S(\mathcal{X})$; under mild genericity, this is unique.

**Theorem**

*Suppose $\mathcal{X}$ is sufficiently (?) $P_{sca}$-generic. Then for each $z \in \mathbb{Z}$ we have $I_z \not\leq_s S(\mathcal{X})$.***

**Idea:**

- Shifts of $\mathbb{Z}$ yield automorphisms of $P_{sca}$ which preserve the isomorphism type of the linear order built.
- Finite support (cofinitely many partial reals empty at each stage) lets us build divergent extensions of conditions.

**Convention**

*Look at $z = 0$ for simplicity, let $\mathcal{G}$ be (name for) $P_{sca}$-generic filter, and assume $(p_z)_{z \in \mathbb{Z}} = p \in P_{sca}$ forces $\Phi^-_e : S(\mathcal{G}) \geq_s I^\mathcal{G}_0$. (?)*
Shift and split

Set $\sigma : z \mapsto z + k$ for some “big enough” $k$ (and conflate $\sigma$ obvious extension to conditions/filters),
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Set $\sigma : z \mapsto z + k$ for some “big enough” $k$ (and conflate $\sigma$
obreakspace{}obvious extension to conditions/filters), and note that if $\mathcal{Y}_0, \mathcal{Y}_1$ differ by a shift then $S(\mathcal{Y}_0) \cong S(\mathcal{Y}_1)$.

Choose extensions $p^{left}, p^{right} \leq p$ such that $\sigma(p^{left}) = p^{right}$ and $p_0^{left}(n) \nmid p_0^{right}(n)$ for some “big enough” $n$. 
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Picking generics \( G^{\text{left}} \ni p^{\text{left}}, G^{\text{right}} \ni p^{\text{right}} \) with \( \sigma(G^{\text{left}}) = G^{\text{right}} \) we have:
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- $S(G^{\text{left}}) \cong S(G^{\text{right}})$ (call this “$S$”),
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- $I^G_0^{\left} \not\cong I^G_0^{\right}$,
- $S(G^{\left}) \cong S(G^{\right})$ (call this “$S$”),
- $\Phi^{S(G^{\left})}_e \cong I^G_0^{\left}$ and $\Phi^{S(G^{\right})}_e \cong I^G_0^{\right}$, but
- $\Phi_e$ is isomorphism-invariant on copies of $S$
Even more wild intervals

Next, we’ll build a linear order which has as many wild intervals as possible - even relative to parameters - in a precise sense.
The extreme example

Definition
A non-computably-presentable linear order \( L \) is a thicket if for every finite tuple \( \overline{c} \in L \) and every infinite \([a, b]_L\) we have \((L, \overline{c}) \geq_s [a, b]_L\) iff there are \(c_a, c_b \in \overline{c}\) finitely far from \(a, b\) respectively.
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Easy to see that:

- Every thicket has the form

\[
\sum_{i \in \hat{\eta}} B_i
\]

for \( \hat{\eta} \in \{\eta, 1 + \eta, \eta + 1, 1 + \eta + 1\} \) and \( B_i \in \hat{\omega} := \omega \cup \{\omega, \omega^*, \zeta\} \).
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- Thickets are basically interwoven nowhere-dense sets: for each nontrivial interval $J \subseteq \hat{\eta}$, the set \{q \in J : B_q$ appears nowhere densely in $J$\} is dense in $J$. 
Co-shuffle sums

Simplest examples of thickets happen to be *co-shuffle-sums*:

**Definition**

A *co-shuffle sum is a linear order of the form*

\[ \sum_{q \in \mathbb{Q}} n_q \]

*with* \( n_q \in \omega \) *such that for each* \( i \in \omega \) *the set* \( \{ q \in \mathbb{Q} : n_q = i \} \) *is nowhere dense.*

Contra shuffle-sums, there are *many* co-shuffle sums up to isomorphism and their \( \Sigma^c_n \)-theories look complicated.
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Contra shuffle-sums, there are many co-shuffle sums up to isomorphism and their $\Sigma^c_n$-theories look complicated. As before, we’ll build a co-shuffle sum from an appropriately-generic family of reals, then use a symmetry-and-splitting argument to kill off potential Medvedev reductions. We can accommodate parameters now since $\eta$ is sufficiently homogeneous (couldn’t do that above).
The forcing

Vaguely Mathias-flavored
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Conditions: \( \pi = \langle a, B \rangle \) where:

- \( a : \mathbb{Q} \rightarrow \mathbb{N} \) partial finite,
- \( B \) is a finite partial map from closed intervals in \( \mathbb{R} \) with distinct finite algebraic irrational endpoints to \( \mathbb{N} \),
- for each \( \langle [x, y], n \rangle \in B \) and \( q \in dom(a) \), if \( x < q < y \) then \( a(q) \neq n \).

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Vaguely Mathias-flavored
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Order conditions as with Mathias.
Using irrational endpoints for the prohibitions in $B$ isn’t necessary but gives us “sufficiently homogeneous” intervals for convenience.
Orderings from filters

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For rationals $q_0 < q_1$ we write “$[q_0, q_1]^G$” for the sub-order of $L(G)$ consisting of all points with left coordinate $q$ satisfying $q_0 \leq q \leq q_1$. For $\bar{c} \in \mathbb{Q}$ and $G$ sufficiently $\mathbb{P}$-generic, we let $\langle L(G); \bar{c} \rangle$ be the expansion of $L(G)$ by constants naming each element with left coordinate in $\bar{c}$. 
Automorphisms

It will be enough to show that for every tuple $\overline{c}$ of rationals and every pair of rationals $p < q$ not both in $\overline{c}$, we have

$$\langle L(G); \overline{c} \rangle \not\preceq_s [p, q]^G.$$
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Definition

By automorphism we’ll mean an automorphism of the linear order $\mathbb{R}$ which sends rationals to rationals and irrationals to irrationals whose restriction to the rationals is computable.
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Definition

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Automorphisms preserve genericity, and extend naturally to maps on conditions and filters (with which we conflate them). Automorphic filters yield isomorphic orderings.
Some structure

Definition

Fix a condition \( \pi = \langle a, B \rangle \in \mathbb{P} \), irrationals \( x < y \), and a rational \( q \in [x, y] \). The interval \([x, y]\) is \( \langle \pi, q \rangle \)-homogeneous iff

\[ \text{dom}(a) \cap [x, y] = \{q\}, \text{ and} \]

useful here: irrationality of prohibitions implies that no element of \( \text{dom}(a) \) is "critical."
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1. \( \text{dom}(a) \cap [x, y] = \{q\} \), and
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For every condition $\pi = \langle a, B \rangle$ and every rational $q \in \text{dom}(a)$ there is a $\langle \pi, q \rangle$-homogeneous interval.

Useful here: irrationality of prohibitions implies that no element of $\text{dom}(a)$ is “critical.”
More shifting and splitting, 1/3

Theorem

If $G$ is sufficiently (?!) generic, then $L(G)$ is a thicket.

For $w \in I \cup N$ and $\pi$ a condition, say $w$ is $\pi$-huge if for each $\langle q, n \rangle \in a$ and each $\langle x, y, k \rangle \in B$ we have $w > q, n, x, y, k$.

Fix sufficiently (?) generic $G$ and $\pi = \langle a, B \rangle \in G$ with $q_0, c \in \text{dom}(a)$ such that $\pi \models \Phi_{ge}: \langle L(G); c \rangle \geq s[q_0, q_1]$ $G$.

Have the following:

▶ an $s \in I$ with $s > \max(\text{dom}(a))$;

▶ a positive natural $N > \max(\text{ran}(a) \cup \text{ran}(B))$;

▶ a $t \in Q$ and $r_0, r_1 \in I$ such that $q_0 < t < r_0 < r_1$ $[u, r_1]$ is $\langle \pi, q_0 \rangle$-homogeneous (which exists by Lemma 0.19); and

▶ an automorphism $\alpha$ which is the identity outside $(u, r_1)$ and has $\alpha(r_0) < q_0$ (which exists by the homogeneity of $Q$).
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More shifting and splitting, 2/3

\(\pi\) forces an unwanted computation, and we’ve chosen some useful auxiliary objects. Define two extensions of \(\pi\) as follows:

\[
\pi + = \langle a \cup \{\langle t, N \rangle\}, \langle \alpha^{-1}(q_0), a(q_0) \rangle \rangle, B \cup \{\langle \alpha(r_0), s, N \rangle\} \rangle,
\]

\[
\pi - = \alpha(\pi +) = \langle a \cup \{\langle \alpha(t), N \rangle\}, \langle \alpha(q_0), a(q_0) \rangle \rangle, B \cup \{\langle \alpha(r_0), s, N \rangle\} \rangle.
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These are indeed conditions via useful choices above (in particular, note that \(\alpha(s) = s\)).

Let \(H = \alpha(G)\). Have \(\pi \in H\) and WLOG \(\pi + \in G\) so \(\pi - \in H\).
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These are indeed conditions via useful choices above (in particular, note that \( \alpha(s) = s \)).
Let \( H = \alpha(G) \). Have \( \pi \in H \) and WLOG \( \pi^+ \in G \) so \( \pi^- \in H \).
More shifting and splitting, 3/3
Comparing $G$ and $H$, we have $[q_0, q_1]^H \not\cong [q_0, q_1]^G$ (the latter has a block of size $N$ while the former doesn’t).
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$$\Phi_e : \langle L(G); \bar{c} \rangle \geq_s [q_0, q_1]^G$$

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Since $\alpha(\bar{c}) = \bar{c}$ have $\langle L(G); \bar{c} \rangle \cong \langle L(H); \bar{c} \rangle$. And this contradicts isomorphism-invariance of $\Phi_e$. 
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Key point: in neither proof did we need to force isomorphism, just equality of invariants (“which bits are coded in?”) after simple modifications (shifts and automorphisms). That’s low-level arithmetic.
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Question
What are the Turing degrees of thickets? (Are there $\Delta^0_2$ thickets? Are there PA-degrees not computing thickets?)
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Also, this generalizes beyond $\leq_S$: any “reasonably-definable” uniform reducibility notion has thickets (possibly after assuming large cardinals).
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Back to ordinals

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A concrete problem to help things along:

**Problem**

*Either find a pair of Medvedev-incomparable ordinals below the first recursively inaccessible, or show that no such exists.*