(Not) Computing linear orders

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April 23, 2020

Breaking uniform computations: linear orders with “wild” intervals

Measures of complexity, 1/2

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- Medvedev-mod-parameters: $\mathcal{A} \leq_{s/p} \mathcal{B}$ iff there is some tuple $\overline{c} \in \mathcal{B}$ such that $\mathcal{A} \leq_s (\mathcal{B}, \overline{c})$.

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Theorem (Turetsky, Westrick)

"$\leq_{s/p}$" is not transitive in general.

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Muchnik reducibility is much better understood than Medvedev reducibility.

Theorem (Essentially Sacks)

For countable ordinals α, β (thought of as linear orders) we have $\alpha \leq_w \beta$ iff $\alpha < \omega_1^{CK}(\beta)$.

Question (Hamkins, Li)

What does the degree structure $\mathcal{D}_s^{ord}$ of the countable ordinals modulo $\leq_s$ look like?

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Is $\omega_1^{CK} \leq_s \omega_2^{CK}$? Do joins exist in $\mathcal{D}_s^{ord}$?

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Things are a bit clearer if we look at uniform *hyperarithmetic* reducibility, but still largely unknown.

Endpointed intervals

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(Obviously impossible for $\leq_w$.) Since well-orderings compare nicely, $\leq_s$-incomparable ordinals yield wild intervals. Unfortunately, the examples produced are hideously large: images of ω_1 under Mostowski collapse of "sufficiently large" countable substructures of the universe.

Can we do better?

A simple wild interval

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For $X \subseteq \omega + 1$, $Shuff(X)$ is the unique-up-to-isomorphism linear order of the form $\sum_{q \in \mathbb{Q}} L_q$ where each L_q has ordertype n for some $n \in X$ and $\{q \in \mathbb{Q} : |L_q| = n\}$ is dense for $n \in X$.

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Knight-Soskova use

$$Shuff(X \cup \{\omega\}) + 1 + Shuff(\omega + 1)$$

for $X \notin \Sigma^0_3$.

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Former appears to point towards ordinals; latter interesting on its own.

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Theorem (S.)

Yes to both ... unhelpfully.

Both arguments via forcing. Nice feature: "coarse" argument relativizes to generally uniform reducibility notions (although we eventually need large cardinals), while Σ^c_n-analysis-approach doesn't seem to.

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- For $X \subseteq \omega, n \in \omega$ let

$$A(X,i) = \begin{cases} \zeta + i + 2 + \zeta & \text{if } i \in X, \\ \zeta + i + 3 + \zeta & \text{if } i \notin X \end{cases}$$

and set $A(X) = \zeta + \sum_{i \in \omega} A(X,i) + \zeta$.

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$S(\mathcal{X})$ is scattered. We'll look at $S(\mathcal{X})$ for $\mathcal{X}$ sufficiently (?) generic:

Definition

$\mathbb{P}_{sca}$ is the set of arrays $p \in (2^{<\omega})^{\mathbb{Z}}$ with $p_z = \emptyset$ for cofinitely many $z \in \mathbb{Z}$. (This is just Cohen forcing rephrased.)

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Let $I_z^{\mathcal{X}}$ be the $(1 + A(X_z) + 1)$-part of $S(\mathcal{X})$; under mild genericity, this is unique.

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Idea:

- Shifts of $\mathbb{Z}$ yield automorphisms of $\mathbb{P}_{sca}$ which preserve the isomorphism type of the linear order built.
- Finite support ($=$ cofinitely many partial reals empty at each stage) lets us build divergent extensions of conditions.

Convention

Look at $z = 0$ for simplicity, let $\mathcal{G}$ be (name for) $\mathbb{P}_{sca}$-generic filter, and assume $(p_z)_{z \in \mathbb{Z}} = p \in \mathbb{P}_{sca}$ forces $\Phi_e^- : S(\mathcal{G}) \geq_s I_0^{\mathcal{G}}$. (?)

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- Φ_e is isomorphism-invariant on copies of $\mathcal{S}$

Even more wild intervals

Next, we'll build a linear order which has as many wild intervals as possible - even relative to parameters - in a precise sense.

The extreme example

Definition

A non-computably-presentable linear order L is a thicket *if for every finite tuple $\overline{c} \in L$ and every infinite $[a, b]_L$ we have $(L, \overline{c}) \geq_s [a, b]_L$ iff there are $c_a, c_b \in \overline{c}$ finitely far from a, b respectively.*

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Easy to see that:

- Every thicket has the form

$$\sum_{i \in \hat{\eta}} B_i$$

for $\hat{\eta} \in \{\eta, 1+\eta, \eta+1, 1+\eta+1\}$ and $B_i \in \hat{\omega} := \omega \cup \{\omega, \omega^*, \zeta\}$.

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  for $\hat{\eta} \in \{\eta, 1+\eta, \eta+1, 1+\eta+1\}$ and $B_i \in \hat{\omega} := \omega \cup \{\omega, \omega^*, \zeta\}$.
- Thickets are basically interwoven nowhere-dense sets: for each nontrivial interval $J \subseteq \hat{\eta}$, the set $\{q \in J : B_q \text{ appears nowhere densely in } J\}$ is dense in J.

Co-shuffle sums

Simplest examples of thickets happen to be *co-shuffle-sums*:

Definition

A co-shuffle sum is a linear order of the form

$$\sum_{q \in \mathbb{Q}} n_q$$

with $n_q \in \omega$ *such that for each* $i \in \omega$ *the set* $\{q \in \mathbb{Q} : n_q = i\}$ *is nowhere dense.*

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Contra shuffle-sums, there are *many* co-shuffle sums up to isomorphism and their Σ^c_n-theories look complicated.
As before, we'll build a co-shuffle sum from an appropriately-generic family of reals, then use a symmetry-and-splitting argument to kill off potential Medvedev reductions. We can accommodate parameters now since η is sufficiently homogeneous (couldn't do that above).

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Conditions: $\pi = \langle a, B \rangle$ where:

- $a : \mathbb{Q} \to \mathbb{N}$ partial finite,
- B is a finite partial map from closed intervals in $\mathbb{R}$ with distinct finite algebraic irrational endpoints to $\mathbb{N}$,
- for each $\langle [x, y], n \rangle \in B$ and $q \in dom(a)$, if $x < q < y$ then $a(q) \neq n$.

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Using irrational endpoints for the prohibitions in B isn't necessary but gives us "sufficiently homogeneous" intervals for convenience.

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For $\overline{c} \in \mathbb{Q}$ and G sufficiently $\mathbb{P}$-generic, we let $\langle L(G); \overline{c}\rangle$ be the expansion of $L(G)$ by constants naming each element with left coordinate in $\overline{c}$.

Automorphisms

It will be enough to show that for every tuple $\overline{c}$ of rationals and every pair of rationals $p < q$ not both in $\overline{c}$, we have

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Automorphisms preserve genericity, and extend naturally to maps on conditions and filters (with which we conflate them). Automorphic filters yield isomorphic orderings.

Some structure

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Fix a condition $\pi = \langle a, B \rangle \in \mathbb{P}$, *irrationals* $x < y$, *and a rational* $q \in [x, y]$. *The interval* $[x, y]$ *is* $\langle \pi, q \rangle$-homogeneous *iff*

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Lemma

For every condition $\pi = \langle a, B \rangle$ *and every rational* $q \in dom(a)$ *there is a* $\langle \pi, q \rangle$*-homogeneous interval.*

Useful here: irrationality of prohibitions implies that no element of $dom(a)$ is "critical."

More shifting and splitting, 1/3

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For $w \in \mathbb{I} \cup \mathbb{N}$ and π a condition, say w is π-*huge* if for each $\langle q, n \rangle \in a$ and each $\langle x, y, k \rangle \in B$ we have $w > q, n, x, y, k$.

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Fix sufficiently (?) generic G and $\pi = \langle a, B \rangle \in G$ with $q_0, \overline{c} \in dom(a)$ such that

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- an automorphism α which is the identity outside (u, r_1) and has $\alpha(r_0) < q_0$ (which exists by the homogeneity of $\mathbb{Q}$).

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These are indeed conditions via useful choices above (in particular, note that $\alpha(s) = s$).
Let $H = \alpha(G)$. Have $\pi \in H$ and WLOG $\pi^+ \in G$ so $\pi^- \in H$.

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Since $\alpha(\overline{c}) = \overline{c}$ have $\langle L(G); \overline{c} \rangle \cong \langle L(H); \overline{c} \rangle$. And this contradicts isomorphism-invariance of Φ_e.

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Question

What are the Turing degrees of thickets? (Are there Δ^0_2 thickets? Are there PA-degrees not computing thickets?)

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Also, this generalizes beyond $\leq_s$: any "reasonably-definable" uniform reducibility notion has thickets (possibly after assuming large cardinals).

Back to ordinals

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A concrete problem to help things along:

Problem

Either find a pair of Medvedev-incomparable ordinals below the first recursively inaccessible, or show that no such exists.