Plato and Brouwer, sitting in a binary tree

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This talk reports on my joint project with Dag Normann (U. of Oslo) on the Reverse Mathematics and computability theory of the uncountable.
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See arXiv for some of our papers!
Beyond our comprehension

What? The usual comprehension hierarchy (by itself) is not appropriate for studying third-order arithmetic. Why? Many natural/basic statements of third-order arithmetic need 'crazy much' comprehension for a proof. Same for Kleene's higher-order computation based on S1-S9. Solution? An alternative hierarchy, going back to Brouwer, is identified. The 'Big Five' equivalences are a reflection of (part of) this new hierarchy, following Plato's allegory of the cave.
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History of comprehension (and vice versa)

In Grundlagen der Mathematik, Hilbert and Bernays formalise (a lot of) mathematics in a logical system $H$. System $H$ makes (essential) use of third-order parameters, but is 'more second-order' than previous systems (with Ackermann). $H$ inspired second-order arithmetic $\mathbb{Z}_2$ based on comprehension:

$$(\exists X \subseteq \mathbb{N}) (\forall n \in \mathbb{N}) (n \in X \iff \varphi(n))$$

for any formula $\varphi(n)$ in $L_2$, language of $\mathbb{Z}_2$.

Indeed, the following is (explicitly) introduced in $H$:

$$(\exists n \in \mathbb{N}) (f(n) = 0) \rightarrow f(\mu(f)) = 0$$

(Feferman's $\mu$) yielding arithmetical comprehension. Similarly:

$\nu$-functional produces witness to $(\exists f : \mathbb{N} \to \mathbb{N}) A(f)$, yielding $\mathbb{Z}_2$. 

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$Z^\omega_2$ is based on comprehension as follows:

$$(\exists f : \mathbb{N} \to \mathbb{N})A(f) \leftrightarrow A(\nu_{k+1}g.A(g)) \quad (*)$$

for $A \in \Pi^1_k \cap L_2$ and any $k$. (Feferman, Sieg, Suslin, Kohlenbach)
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Connection: $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. 

Note 3rd vs 4th order!
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Connection: $Z_2 \equiv_{L_2} Z^\omega_2 \equiv_{L_2} Z^\Omega_2$. Note 3rd vs 4th order!
Gödel hierarchy

<table>
<thead>
<tr>
<th>Strong</th>
<th>Medium</th>
<th>Weak</th>
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<tbody>
<tr>
<td></td>
<td>large cardinals</td>
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<td>ZFC</td>
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<td>ZC (Zermelo set theory)</td>
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<td>simple type theory</td>
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<td>Z₂ (second-order arithmetic)</td>
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It is striking that a great many foundational theories are linearly ordered by [consistency strength] <. Of course it is possible to construct pairs of artificial theories which are incomparable under <. However, this is not the case for the “natural” or non-artificial theories which are usually regarded as significant in the foundations of mathematics.

(Simpson, Gödel Centennial Volume; also: Koelner, Burgess, Friedman,...)
Gödel hierarchy

= ‘comprehension’ hierarchy

MORE sets exist

\[
\begin{align*}
\text{strong} & \quad \{ \ldots \} \\
\text{medium} & \quad \{ Z_2 \text{ (second-order arithmetic)} \} \\
\text{weak} & \quad \{ WKL_0 \text{ (weak König’s lemma)} \}
\end{align*}
\]

\ldots

\LARGE{\uparrow}

\LARGE{\downarrow}

LESS sets exist

\ldots

large cardinals

ZFC

\begin{align*}
ZC & \quad (\text{Zermelo set theory}) \\
\text{simple type theory} & \quad \{ \ldots \}
\end{align*}

\begin{align*}
\Pi^1_2-\text{CA}_0 & \quad (\text{comprehension for } \Pi^1_2\text{-formulas}) \\
\Pi^1_1-\text{CA}_0 & \quad (\text{comprehension for } \Pi^1_1\text{-formulas}) \\
\text{ATR}_0 & \quad (\text{arithmetical transfinite recursion}) \\
\text{ACA}_0 & \quad (\text{arithmetical comprehension}) \\
\text{WKL}_0 & \quad (\text{weak König’s lemma}) \\
\text{RCA}_0 & \quad (\text{recursive comprehension}) \\
\text{PRA} & \quad (\text{primitive recursive arithmetic}) \\
\text{bounded arithmetic}
\end{align*}
Gödel hierarchy

strong

Zermelo-Fraenkel set theory with choice
aka ‘the’ foundation of mathematics

ZFC
ZC (Zermelo set theory)
simple type theory

medium

\ \begin{align*}
Z_2 \text{ (second-order arithmetic)} \\
\vdots \\
\end{align*}

\ \begin{align*}
II_2^1 \text{-CA}_0 \text{ (comprehension for } II_2^1 \text{-formulas)} \\
II_1^1 \text{-CA}_0 \text{ (comprehension for } II_1^1 \text{-formulas)} \\
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weak

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WKL_0 \text{ (weak König's lemma)} \\
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## Gödel hierarchy

### strong
- Zermelo-Fraenkel set theory with choice
- aka ‘the’ foundation of mathematics

- Hilbert-Bernays’s *Grundlagen der Mathematik*

### medium
- $\Pi^1_2$-CA$_0$ (comprehension for $\Pi^1_2$-formulas)
- $\Pi^1_1$-CA$_0$ (comprehension for $\Pi^1_1$-formulas)
- ATR$_0$ (arithmetical transfinite recursion)
- ACA$_0$ (arithmetical comprehension)

### weak
- WKL$_0$ (weak König’s lemma)
- RCA$_0$ (recursive comprehension)
- PRA (primitive recursive arithmetic)
- bounded arithmetic

### Other Foundations
- ZFC (Zermelo set theory)
- ZC (Zermelo set theory)
- simple type theory
Gödel hierarchy

- **strong**
  - Zermelo-Fraenkel set theory with choice
    - aka ‘the’ foundation of mathematics

- **medium**
  - Hilbert-Bernays’s *Grundlagen der Mathematik*
  - Russell-Weyl-Feferman
    - predicative mathematics

- **weak**
  - WKL₀ (weak König’s lemma)
  - RCA₀ (recursive comprehension)

...
Gödel hierarchy

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**weak**
- The ‘Big Five’ of Reverse Mathematics

- : large cardinals
- : ZFC
- : ZC (Zermelo set theory)
- : simple type theory

- Z₂ (second-order arithmetic)
- :
- : \(\Pi^1_2\)-CA\(_0\) (comprehension for \(\Pi^1_2\)-formulas)
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- WKL\(_0\) (weak König’s lemma)
- : RCA\(_0\) (recursive comprehension)
- : PRA (primitive recursive arithmetic)
- : bounded arithmetic
Gödel hierarchy

Strong

Zermelo-Fraenkel set theory with choice
aka ‘the’ foundation of mathematics

Hilbert-Bernays’s Grundlagen
der Mathematik

medium

Russell-Weyl-Feferman
predicative mathematics

The ‘Big Five’ of Reverse Mathematics

Hilbert’s finitary math

weak

weak

WKL₀ (weak König’s lemma)
RCA₀ (recursive comprehension)
PRA (primitive recursive arithmetic)
bounded arithmetic

Medium

Z₂ (second-order arithmetic)

Π₁²-CA₀ (comprehension for Π₁²-formulas)
Π₁¹-CA₀ (comprehension for Π₁¹-formulas)
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ACA₀ (arithmetical comprehension)

Weak

large cardinals

ZFC

ZC (Zermelo set theory)
simple type theory

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Bounded arithmetic
Gödel hierarchy

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**Π\(_1^1\)-CA\(_0\)** (comprehension for Π\(_1^1\)-formulas)
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**Π\(_1^2\)-CA\(_0\)** (comprehension for Π\(_1^2\)-formulas)
- Z\(_2\) (second-order arithmetic)
- \(\vdots\)

**large cardinals**
- \(\vdots\)

Received view: **natural/important** systems form **linear** Gödel hierarchy
Received view: **natural/**important systems form linear Gödel hierarchy and 80/90% of ordinary mathematics is provable in $\text{ACA}_0/II^1_1\text{-CA}_0$. 
Incomprehensible!

Recall that $Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega$. 
Recall that $Z_2 \equiv_{L^2} Z_2^\omega \equiv_{L^2} Z_2^\Omega$. The following \textit{third-order} theorems are provable in $Z_2^\Omega$, but not in $Z_2^\omega$. 
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11. An open set in $[0, 1]$ is a countable union of open intervals.
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5. There is a function $f : \mathbb{R} \to \mathbb{R}$ not in Baire class 2.
6. Baire characterisation theorem for Baire class 1.
7. Heine-Borel/Vitali/Lindelöf for *uncountable* coverings.
8. Basic Lebesgue measure/integral and gauge integral.
9. Unordered sums are countable (E.H. Moore)
10. Convergence theorems for nets indexed by $\mathbb{N}^\mathbb{N}$ (Moore-Smith).
11. An open set in $[0, 1]$ is a countable union of open intervals.
12. There is no injection from $[0, 1]$ to $\mathbb{N}$ (Cantor, 1874).
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Let \( f_n : ([0, 1] \times \mathbb{N}) \to \mathbb{R} \) be a sequence such that

1. Each \( f_n \) is **Riemann integrable** on \([0, 1]\).
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Riemann’s *Habilschrift* (1854) entrenched discontinuous functions in the mainstream.
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We use ‘totally bounded’ and ‘ separable’ in the sense of RM.
Baire category theorem

For this slide, we assume ‘open sets’ are given by (third-order) characteristic functions: ‘$x \in O$’ means $Y(x) = 1$ for some $Y : \mathbb{R} \to \{0, 1\}$;
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These ‘new’ proofs led us to...
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4 ON THE UNCOUNTABILITY OF $\mathbb{R}$

audience of mathematicians as we identify surprising results about a very well-studied topic in the foundations of mathematics, namely the genesis of set theory. Beyond this, we have formulated the below proofs in such a way as to appeal to an as broad as possible audience. In particular, the below is meant to showcase the techniques used to establish the results in [48–51], which are part of our ongoing project on the logical and computational properties of the uncountable. Indeed, $\text{NIN}$ is implied by most of the (third-order) principles we have hitherto studied, e.g. the Lindelöf lemma ($\text{LIN}$), the Heine-Borel ($\text{HBU}$), Vitali ($\text{WHBU}$), and Baire category ($\text{BCT}$) theorems. Our results for $\text{NIN}$ imply that all stronger principles behave in the same way, as depicted in Figure 1.

Moreover, our results have a number of interesting conceptual consequences. Indeed, $\text{NIN}$ seems to be the weakest natural third-order statement not provable in $\text{Z}_{2}^{\omega}$, a system conservative over second-order arithmetic $\text{Z}_{2}$. We admit that such claims are inherently vague. Moreover, a number of early critics, including Borel, of the Axiom of Choice actually implicitly used this axiom in their work (see [20, p. 315]). A similar observation can be made for $\text{NIN}$ as follows: around 1874, Weierstrass seems to have held the belief that there cannot be essential differences between infinite sets (see [20, p. 184]), although basic compactness results, pioneered in part by Weierstrass himself, imply the uncountability of $\mathbb{R}$.

Finally, the following figure provides an overview of some of the results in this paper. Here, $\text{NIN}^+$ expresses that any $[0, 1] \rightarrow \mathbb{N}$-functional maps some sequence of (distinct) reals to the same natural number. Further definitions can be found in Section 2 while implications not involving $\text{NIN}$ or $\text{NIN}^+$ are in [48–51, 57]. We do point out that $\text{Arz}$ is Arzelà's convergence theorem for the Riemann integral, published in 1885 ([1]), i.e. ordinary mathematics if ever there was such.

Note that $\text{Z}_{2}^{\omega} \implies \text{RCA}_0$ and $\text{Z}_{2}^{\omega}$ are both conservative extensions of $\text{Z}_{2}$, and where $\text{IND}$ is as above. The negative results in Figure 1 do not change if we add countable choice as in $\text{QF-AC}_0$ to $\text{Z}_{2}^{\omega}$. Lest we be accused of comparing apples and oranges, we point out that the functionals $\text{S}_k^2$ used to define $\text{Z}_{2}^{\omega}$ are third-order and that $\text{NIN}$ is part of the language of third-order arithmetic. By contrast, Kleene's $\text{K}_3$ used to define $\text{Z}_{2}^{\omega}$ is fourth-order.

4 Weierstrass seems to have changed his mind by 1885, which he expressed in a letter to Mittag-Leffler (see [20, p. 185]).

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Note that $\mathbb{Z}_\omega^{\Omega}$ and $\mathbb{Z}_2^\omega$ are both conservative extensions of $\mathbb{Z}_2^\omega$, and where $\uparrow_1^k$-$\text{CA}_0$ is as above. The negative results in Figure 1 do not change if we add countable choice as in QF-AC to $\mathbb{Z}_2^\omega$. Lest we be accused of comparing apples and oranges, we point out that the functionals $S^k_2$ used to define $\mathbb{Z}_2^\omega$ are third-order and that NIN is part of the language of third-order arithmetic. By contrast, Kleene’s $\uparrow_3$ used to define $\mathbb{Z}_\omega^{\Omega}$, is fourth-order.

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**Diagram: (with definitions and implications)**

- **HBU**: Heine-Borel theorem for uncountable coverings of \([0, 1]\).
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- **LIN(\( \mathbb{N}^\mathbb{N} \))**: Lindelöf lemma for uncountable coverings of \( \mathbb{N}^\mathbb{N} \).
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Negative results do not change if we add QF-AC$^{0,1}$ to $Z_2^\omega$.

QF-AC$^{0,1}$ is ‘weakest’ fragment of CC not provable in ZF.
4 ON THE UNCOUNTABILITY OF \( \mathbb{R} \)

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ON THE UNCOUNTABILITY OF $\mathbb{R}$

Audience of mathematicians as we identify surprising results about a very well-studied topic in the foundations of mathematics, namely the genesis of set theory. Beyond this, we have formulated the below proofs in such a way as to appeal to an as broad as possible audience. In particular, the below is meant to showcase the techniques used to establish the results in [48–51], which are part of our ongoing project on the logical and computational properties of the uncountable. Indeed, $NIN$ is implied by most of the (third-order) principles we have hitherto studied, e.g. the Lindelöf lemma ($LIN(N^N)$) and the Heine-Borel ($HBU$), Vitali ($WHBU$), and Baire category ($BCT$) theorems. Our results for $NIN$ imply that all stronger principles behave in the same way, as depicted in Figure 1.

Moreover, our results have a number of interesting conceptual consequences. Indeed, $NIN$ seems to be the weakest natural third-order statement not provable in $\mathbb{Z}^2_\kappa$, a system conservative over second-order arithmetic $\mathbb{Z}^2$. We admit that such claims are inherently vague. Moreover, a number of early critics, including Borel, of the Axiom of Choice actually implicitly used this axiom in their work (see [20, p. 315]). A similar observation can be made for $NIN$ as follows: around 1874, Weierstrass seems to have held the belief that there cannot be essential differences between infinite sets (see [20, p. 184]), although basic compactness results, pioneered in part by Weierstrass himself, imply the uncountability of $\mathbb{R}$.

Finally, the following figure provides an overview of some of the results in this paper. Here, $NIN^+$ expresses that any $[0,1]$-functional maps some sequence of (distinct) reals to the same natural number. Further definitions can be found in Section 2 while implications not involving $NIN$ or $NIN^+$ are in [48–51, 57]. We do point out that Arz is Arzelà’s convergence theorem for the Riemann integral, published in 1885 ([1]), i.e. ordinary mathematics if ever there was such.

Note that $\mathbb{Z}^\kappa_\omega$ and $\mathbb{Z}^\kappa_2$ are both conservative extensions of $\mathbb{Z}^2$, and where $\kappa_{-CA^0}$ is as above. The negative results in Figure 1 do not change if we add countable choice as in $\mathbb{QF-AC}^0_0$ to $\mathbb{Z}^\kappa_2$. Lest we be accused of comparing apples and oranges, we point out that the functionals $S^k_2$ used to define $\mathbb{Z}^\kappa_2$ are third-order and that $NIN$ is part of the language of third-order arithmetic. By contrast, Kleene’s $\kappa_3$ used to define $\mathbb{Z}^\kappa_2$, is fourth-order.

---

**HBU:** Heine-Borel theorem for uncountable coverings of $[0, 1]$.  
**WHBU:** Vitali covering theorem for uncountable coverings of $[0, 1]$.  
**LIN($N^N$):** Lindelöf lemma for uncountable coverings of $N^N$.

**Similar computational results:** $\exists^3$ computes realiser $\Theta$ for HBU, which computes realiser for NIN; no $\nu_n$ computes a realiser for NIN.
HBU: Heine-Borel theorem for uncountable coverings of $[0, 1]$.

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LIN($\mathbb{N}^\mathbb{N}$): Lindelöf lemma for uncountable coverings of $\mathbb{N}^\mathbb{N}$.
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All these third-order theorems are provable in Z_2^\Omega + QF-AC^{0,1}, but not provable in Z_2^\omega + QF-AC^{0,1}, where Z_2 \equiv_{L_2} Z_2^\omega \equiv_{L_2} Z_2^\Omega.
4 ON THE UNCOUNTABILITY OF R
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Why do we need ‘crazy much’ comprehension for basic theorems?

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Why do we need ‘crazy much’ comprehension for basic theorems?

Because apples and oranges: the ‘comprehension functionals’ in $Z_2^\omega$ and $Z_2^\Omega$ are discontinuous, while the other theorems (HBU, NIN, etc) are consistent with Brouwer’s (continuity) theorem.
Brouwer and continuity to the rescue

L.E.J. Brouwer is (in)famous for his \textit{intuitionism}. 
Brouwer and continuity to the rescue

L.E.J. Brouwer is (in)famous for his *intuitionism*. Intuitionistic mathematics is formalised using non-classical continuity axioms that have a (non-classical) weak counterpart.
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\[
\text{Definition (NFP, 1970, Kreisel-Troelstra)}
\]

For any formula \(A\), we have

\[
\left(\forall f \in \mathbb{N}\right) \left(\exists n \in \mathbb{N}\right) A(f^n) \rightarrow \left(\exists \gamma \in K^0\right) \left(\forall f \in \mathbb{N}\right) A(f\gamma(f))
\]

where ‘\(\gamma \in K^0\)’ essentially means that \(\gamma\) is an RM-code/associate.

Note that \(f^n\) is the finite sequence \(<f(0), f(1),..., f(n-1)>\). NFP expresses that there are (many) continuous choice functions.
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**Definition (NFP, 1970, Kreisel-Troelstra)**

For any formula $A$, we have

$$(\forall f \in \mathbb{N}^\mathbb{N})(\exists n \in \mathbb{N})A(\overline{f}n) \rightarrow (\exists \gamma \in K_0)(\forall f \in \mathbb{N}^\mathbb{N})A(\overline{f}\gamma(f)),$$

where ‘$\gamma \in K_0$’ essentially means that $\gamma$ is an RM-code/associate.

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1) Theorems like BOOT, HBU, and the Lindelöf lemma are equivalent to natural fragments of NFP.
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NFP has great properties (in contrast to comprehension):
1) Theorems like BOOT, HBU, and the Lindelöf lemma are equivalent to natural fragments of NFP.
2) The equivalences from 1) map to the Big Five equivalences, under the canonical embedding of HOA in SOA.
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NFP has **great** properties (in contrast to comprehension):

1) Theorems like **BOOT**, **HBU**, and the Lindelöf lemma are equivalent to natural fragments of NFP.

2) The equivalences from 1) map **map** to the Big Five equivalences, under the canonical embedding of HOA in SOA. The second item reminds one of Plato’s allegory of the cave.
Plato and his -ism

Plato is well-known in (foundations of) mathematics for his eponymous philosophy platonism, i.e. the theory that mathematical objects are objective, timeless entities, independent of the physical world and the symbols that represent them. Plato's allegory of the cave provides a powerful visual: We can only know reflections/shadows/... of ideal objects.
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Plato’s allegory of the cave provides a powerful visual:

![Illustration of Plato's allegory of the cave](image)

We can only know reflections/shadows/… of ideal objects.
What are the current foundations of mathematics reflections of?
Plato and his -ism

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Fragments of NFP and equivalences

Big Five and equivalences

ECF
Plato and his -ism

Plato’s allegory of the cave provides a powerful visual:

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What are the current foundations of mathematics reflections of?

Fragments of NFP and equivalences

Big Five and equivalences

ECF is canonical embedding of HOA into SOA (Kleene-Kreisel).
The Big Five as a reflection
The Big Five as a reflection

\[ II_1^{1}-CA_0 \]

\[ ATR_0 \]

\[ ACA_0 \]

\[ WKL_0 \]

\[ RCA_0 \]
The Big Five as a reflection

- $II_1^1$-CA$_0$
- ATR$_0$
- ACA$_0$
- WKL$_0$
- RCA$_0$ proves $\Delta^0_1$-comprehension
The Big Five as a reflection

\[ II^1_1 \text{-CA}_0 \]
\[ \text{ATR}_0 \]
\[ \text{ACA}_0 \]
\[ \text{WKL}_0 \leftrightarrow \text{Dini’s theorem.} \]
\[ \leftrightarrow \text{countable Heine-Borel compactness} \]
\[ \leftrightarrow \text{Riemann int. thms} \]
\[ \text{RCA}_0 \text{ proves } \Delta^0_1 \text{-comprehension} \]
The Big Five as a reflection

- $II^1_1$-CA$_0$
- ATR$_0$
- ACA$_0$
  - $\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
  - $\leftrightarrow$ Monotone conv. thm
  - $\leftrightarrow$ Ascoli-Arzela
  - $\leftrightarrow$ thms about closed sets
    (as countable unions)
- WKL$_0$
  - $\leftrightarrow$ Dini’s theorem.
    - $\leftrightarrow$ countable Heine-Borel compactness
    - $\leftrightarrow$ Riemann int. thms
- RCA$_0$
  - proves $\Delta^0_1$-comprehension
The Big Five as a reflection

\[ I_1^{1\text{-CA}_0} \]

\[ ATR_0 \leftrightarrow \text{perfect set theorem} \]

\[ ACA_0 \leftrightarrow \text{range of } f : \mathbb{N} \to \mathbb{N} \text{ exists} \]

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\[ ACA_0 \leftrightarrow \text{thms about closed sets} \]

\[ ACA_0 \leftrightarrow \text{thms about closed sets (as countable unions)} \]

\[ WKL_0 \leftrightarrow \text{Dini’s theorem.} \]

\[ WKL_0 \leftrightarrow \text{countable Heine-Borel compactness} \]

\[ WKL_0 \leftrightarrow \text{Riemann int. thms} \]

\[ RCA_0 \text{ proves } \Delta_1^0\text{-comprehension} \]
The Big Five as a reflection

- $\mathcal{II}_{1}^{1}$-CA$_{0}$ $\iff$ Cantor-Bendixson thm
- ATR$_{0}$ $\iff$ perfect set theorem
  $\iff$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
- ACA$_{0}$ $\iff$ Monotone conv. thm
  $\iff$ Ascoli-Arzela
  $\iff$ thms about closed sets
  (as countable unions)
- WKL$_{0}$ $\iff$ Dini’s theorem.
  $\iff$ countable Heine-Borel
  compactness
  $\iff$ Riemann int. thms
- RCA$_{0}$ proves $\Delta_{1}^{0}$-comprehension

SECOND-ORDER arithmetic
The Big Five as a reflection

\[ II^1_1 \text{-CA}_0 \iff \text{Cantor-Bendixson thm} \]
\[ ATR_0 \iff \text{perfect set theorem} \]
\[ ACA_0 \iff \text{range of } f : \mathbb{N} \to \mathbb{N} \text{ exists} \]
\[ \iff \text{Monotone conv. thm} \]
\[ \iff \text{Ascoli-Arzelà} \]
\[ \iff \text{thms about closed sets} \]
\[ \text{(as countable unions)} \]
\[ WKL_0 \iff \text{Dini’s theorem.} \]
\[ \iff \text{countable Heine-Borel compactness} \]
\[ \iff \text{Riemann int. thms} \]
\[ \text{RCA}_0 \text{ proves } \Delta^0_1 \text{-comprehension} \]

SECOND-ORDER arithmetic

\[ \text{BOOT}_2 \]
\[ \Sigma \text{-TR} \]
\[ \text{BOOT} \]
\[ \text{WKL}^1 \]
\[ \text{RCA}_0^\omega \]

HIGHER-ORDER arithmetic
The Big Five as a reflection

\[ \Pi_1^1-CA_0 \iff \text{Cantor-Bendixson thm} \]
\[ \text{ATR}_0 \iff \text{perfect set theorem} \]
\[ \iff \text{range of } f : \mathbb{N} \to \mathbb{N} \text{ exists} \]
\[ \iff \text{Monotone conv. thm} \]
\[ \iff \text{Ascoli-Arzela} \]
\[ \iff \text{thms about closed sets} \]
\[ \text{(as countable unions)} \]
\[ \text{WKL}_0 \iff \text{Dini’s theorem.} \]
\[ \iff \text{countable Heine-Borel compactness} \]
\[ \iff \text{Riemann int. thms} \]
\[ \text{RCA}_0 \text{ proves } \Delta^0_1-\text{comprehension} \]

\[ \text{SECOND-ORDER arithmetic} \]

\[ \text{BOOT} \]
\[ \Sigma-\text{TR} \]
\[ \text{BOOT}_2 \]

\[ \text{HIGHER-ORDER arithmetic} \]
\[ \text{plus a fragment of countable choice} \]
The Big Five as a reflection

\[ II_1^1 \text{-CA}_0 \iff \text{Cantor-Bendixson thm} \]
\[ \text{ATR}_0 \iff \text{perfect set theorem} \]
\[ \text{ACA}_0 \iff \begin{array}{l}
\text{range of } f : \mathbb{N} \to \mathbb{N} \text{ exists} \\
\text{Monotone conv. thm} \\
\text{Ascoli-Arzela} \\
\text{thms about closed sets (as countable unions)}
\end{array} \]
\[ \text{WKL}_0 \iff \begin{array}{l}
\text{Dini’s theorem.} \\
\text{countable Heine-Borel compactness} \\
\text{Riemann int. thms}
\end{array} \]
\[ \text{RCA}_0 \text{ proves } \Delta^0_1 \text{-comprehension} \]

\[ \text{BOOT}_2 \]
\[ \Sigma \text{-TR} \]
\[ \text{BOOT} \]

\[ \text{WKL}^1 \iff \begin{array}{l}
\text{Dini’s theorem for nets.} \\
\text{uncountable Heine-Borel compactness: HBU} \\
\text{gauge integral thms}
\end{array} \]
\[ \text{RCA}_0^\omega \text{ plus a fragment of countable choice} \]

SECOND-ORDER arithmetic

HIGHER-ORDER arithmetic
The Big Five as a reflection

\[ \Pi^1_1 - \text{CA}_0 \leftrightarrow \text{Cantor-Bendixson thm} \]

\[ \text{ATR}_0 \leftrightarrow \text{perfect set theorem} \]
\[ \leftrightarrow \text{range of } f : \mathbb{N} \rightarrow \mathbb{N} \text{ exists} \]
\[ \leftrightarrow \text{Monotone conv. thm} \]
\[ \leftrightarrow \text{Ascoli-Arzela} \]
\[ \leftrightarrow \text{thms about closed sets} \]
\[ \text{(as countable unions)} \]

\[ \text{WKL}_0 \leftrightarrow \text{Dini’s theorem.} \]
\[ \leftrightarrow \text{countable Heine-Borel compactness} \]
\[ \leftrightarrow \text{Riemann int. thms} \]

\[ \text{RCA}_0 \text{ proves } \Delta^0_1 - \text{comprehension} \]

SECOND-ORDER arithmetic

\[ \text{BOOT}_2 \]

\[ \Sigma - \text{TR} \]
\[ \leftrightarrow \text{range of } Y : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N} \text{ exists} \]

\[ \text{BOOT} \leftrightarrow \text{Mon. conv. thm for nets} \]
\[ \leftrightarrow \text{Ascoli-Arzela for nets} \]
\[ \leftrightarrow \text{thms about closed sets} \]
\[ \text{(as uncountable unions)} \]

\[ \text{WKL}^1_0 \leftrightarrow \text{Dini’s theorem for nets.} \]
\[ \leftrightarrow \text{uncountable Heine-Borel compactness: HBU} \]
\[ \leftrightarrow \text{gauge integral thms} \]

\[ \text{RCA}_0^\omega \text{ plus a fragment of countable choice} \]

HIGHER-ORDER arithmetic
The Big Five as a reflection

- $II_1^{\mathcal{C}A_0} \iff$ Cantor-Bendixson thm
- $\mathcal{A}TR_0 \iff$ perfect set theorem
  - $\iff$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
  - $\iff$ Monotone conv. thm
  - $\iff$ Ascoli-Arzela
  - $\iff$ thms about closed sets
    (as countable unions)
- $\mathcal{W}KL_0 \iff$ Dini’s theorem.
  - $\iff$ countable Heine-Borel compactness
  - $\iff$ Riemann int. thms
- $\mathcal{R}CA_0$ proves $\Delta^0_1$-comprehension

SECOND-ORDER arithmetic

Cantor-Bendixson thm

- $\mathcal{B}OOT_2 \iff$ (uncountable unions)
- $\Sigma-\mathcal{TR} \iff$ range of $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ exists
- $\mathcal{B}OOT \iff$ Mon. conv. thm for nets
  - $\iff$ Ascoli-Arzela for nets
  - $\iff$ thms about closed sets
    (as uncountable unions)
- $\mathcal{W}KL_1 \iff$ Dini’s theorem for nets.
  - $\iff$ uncountable Heine-Borel compactness: HBU
  - $\iff$ gauge integral thms
- $\mathcal{R}CA^\omega_0 \iff$ plus a fragment of countable choice

HIGHER-ORDER arithmetic

- $\mathcal{E}CF \iff$ replaces uncountable objects by countable representations/RM-codes
- $\mathcal{E}CF \iff$ converts right-hand side to left-hand side, including equivalences!
The Big Five as a reflection

\[\begin{align*}
\Pi^1_1-C\text{A}_0 & \iff \text{Cantor-Bendixson thm} \\
\text{ATR}_0 & \iff \text{perfect set theorem} \\
\text{ACA}_0 & \iff \text{range of } f : \mathbb{N} \to \mathbb{N} \text{ exists} \\
& \quad \iff \text{Monotone conv. thm} \\
& \quad \iff \text{Ascoli-Arzelà} \\
& \quad \iff \text{thms about closed sets} \\
& \quad \quad \text{(as countable unions)} \\
\text{WKL}_0 & \iff \text{Dini’s theorem.} \\
& \quad \iff \text{countable Heine-Borel compactness} \\
& \quad \iff \text{Riemann int. thms} \\
\text{RCA}_0 & \text{proves } \Delta^0_1 \text{-comprehension} \\
\text{SECOND-ORDER arithmetic} & \quad \quad \\
\text{Cantor-Bendixson thm} & \text{BOOT}_2 \iff \text{(uncountable unions)} \\
\Sigma^{\text{-TR}} & \iff \text{perfect set thm (idem)} \\
\text{BOOT} & \iff \text{range of } Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N} \text{ exists} \\
& \quad \iff \text{Mon. conv. thm for nets} \\
& \quad \iff \text{Ascoli-Arzelà for nets} \\
& \quad \iff \text{thms about closed sets} \\
& \quad \quad \text{(as uncountable unions)} \\
\text{WKL}^1 & \iff \text{Dini’s theorem for nets.} \\
& \quad \iff \text{uncountable Heine-Borel compactness: HBU} \\
& \quad \iff \text{gauge integral thms} \\
\text{RCA}_0^\omega & \text{plus a fragment of countable choice} \\
\text{HIGHER-ORDER arithmetic} & \quad \\
\end{align*}\]
The Big Five as a reflection

ECF replaces uncountable objects by countable representations/RM-codes

- $\Pi^1_1$-CA$_0$ $\iff$ Cantor-Bendixson thm
- ATR$_0$ $\iff$ perfect set theorem
  $\iff$ range of $f : \mathbb{N} \to \mathbb{N}$ exists
  $\iff$ Monotone conv. thm
  $\iff$ Ascoli-Arzelà
  $\iff$ thms about closed sets (as countable unions)
- ACA$_0$ $\iff$ Dini’s theorem.
  $\iff$ countable Heine-Borel compactness
  $\iff$ Riemann int. thms
- WKL$_0$ $\iff$ Dini’s theorem for nets.
  $\iff$ uncountable Heine-Borel compactness: HBU
  $\iff$ gauge integral thms
- RCA$_0$ proves $\Delta^0_1$-comprehension

SECOND-ORDER arithmetic

Cantor-Bendixson thm
- BOOT$_2$ $\iff$ (uncountable unions)
- $\Sigma$-TR $\iff$ perfect set thm (idem)
- BOOT $\iff$ Mon. conv. thm for nets
  $\iff$ Ascoli-Arzelà for nets
  $\iff$ thms about closed sets (as uncountable unions)
- WKL$_1$ $\iff$ Dini’s theorem for nets.
  $\iff$ uncountable Heine-Borel compactness: HBU
  $\iff$ gauge integral thms
- RCA$_0^\omega$ plus a fragment of countable choice

HIGHER-ORDER arithmetic

ECF
The Big Five as a reflection

ECF replaces uncountable objects by countable representations/RM-codes
ECF converts right-hand side to left-hand side, including equivalences!

\[ \Pi^1_1\text{-CA}_0 \leftrightarrow \text{Cantor-Bendixson thm} \]
\[ \text{ATR}_0 \leftrightarrow \text{perfect set theorem} \]
\[ \text{ACA}_0 \leftrightarrow \text{range of } f : \mathbb{N} \rightarrow \mathbb{N} \text{ exists} \]
\[ \leftrightarrow \text{Monotone conv. thm} \]
\[ \leftrightarrow \text{Ascoli-Arzelà} \]
\[ \leftrightarrow \text{thms about closed sets} \]
\[ \text{(as countable unions)} \]
\[ \text{WKL}_0 \leftrightarrow \text{Dini’s theorem.} \]
\[ \leftrightarrow \text{countable Heine-Borel compactness} \]
\[ \leftrightarrow \text{Riemann int. thms} \]
\[ \text{RCA}_0 \text{ proves } \Delta^0_1\text{-comprehension} \]

SECOND-ORDER arithmetic

Cantor-Bendixson thm
\[ \text{BOOT}_2 \leftrightarrow (\text{uncountable unions}) \]
\[ \text{Σ-TR} \leftrightarrow \text{perfect set thm (idem)} \]
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\[ \leftrightarrow \text{thms about closed sets} \]
\[ \text{(as uncountable unions)} \]
\[ \text{WKL}^1 \leftrightarrow \text{Dini’s theorem for nets.} \]
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\[ \leftrightarrow \text{gauge integral thms} \]
\[ \text{RCA}_0^\omega \text{ plus a fragment of countable choice} \]

HIGHER-ORDER arithmetic
The Big Five as a reflection

ECF replaces uncountable objects by countable/continuous RM-codes

Kohlenbach’s RM: based on discontinuity; Plato hierarchy: based on continuity.

ECF

$I\!I^1_1$-CA$_0$ $\iff$ Cantor-Bendixson thm

ATR$_0$ $\iff$ perfect set theorem

$\iff$ range of $f : \mathbb{N} \rightarrow \mathbb{N}$ exists

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$\iff$ higher-order arithmetic

plus a fragment of countable choice

HIGHER-ORDER arithmetic

ECF

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Beyond Riemann and Lebesgue: the gauge integral

The gauge integral was introduced in 1912 by Denjoy (in a different form) and generalises Lebesgue’s integral (1904).
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![Diagram showing the relationship between Riemann integral, Lebesgue integral, and gauge integral](image)

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Many basic thms about gauge integral are equivalent to HBU. ECF maps these to thms about Riemann integral.
The gauge integral: Riemann’s cousin!
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**Definition (Riemann integral)**

\[ f : \mathbb{R} \to \mathbb{R} \text{ is Riemann integrable on } I \equiv [0, 1] \text{ with integral } A \in \mathbb{R}: \]

\[
(\forall \varepsilon > 0)(\exists \delta > 0)(\forall P)(\forall i \leq k)(|x_i - x_{i+1}| < \delta) \Rightarrow |S(P, f) - A| < \varepsilon
\]

\[ P = (0, t_1, x_1 \ldots x_k, t_k, 1) \text{ partition } S(P, f) = \sum_i f(t_i)|x_{i+1} - x_i| \text{ Riemann sum} \]
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Weak König’s lemma for binary trees \(T\) where ‘\(\sigma \in T\)’ is given by

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\( \text{HBU} \Leftrightarrow \text{WKL}^1 \) needs some extra choice, which ECF maps to a triviality.
Today: a higher RM

ECF replaces **uncountable** objects by **countable/continuous** RM-codes

Kohlenbach’s RM: based on **discontinuity**; Plato hierarchy: based on **continuity**.

\[ \mathbb{II}_1^1{-}\text{CA}_0 \iff \text{Cantor-Bendixson thm} \]

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\[ \mathbb{RCA}_\omega \iff \text{plus } \Delta{-}\text{comprehension} \]

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Nets: Moore-Smith-Vietoris

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Definition (Nets, ca. 1915)

A set $D \neq \emptyset$ with a binary relation `$\leq$' is directed if

- The relation `$\leq$' is transitive and reflexive.
- For $d, e \in D$, there is $f \in D$ such that $d \leq f \land e \leq f$.

For such $(D, \leq)$ and topological space $X$, any $x : D \to X$ is a net.
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Nets generalise the concept of sequence to (possibly) uncountable index sets. Nets capture topology where sequences fail to.

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\begin{itemize}
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\end{itemize}

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Sequences are nets for \((D, \preceq) = (\mathbb{N}, \leq)\). We write \( x_d \) for \( x(d) \).
Definitions

Definition (Convergence of nets)
A net $x_d$ converges to the limit $y = \lim_d x_d$ if for any neighborhood $U$ of $y$, there is $d_0 \in D$ such that for all $e \succeq d_0$, $x_e \in U$.

If the topological space $X$ has some order $\leq_X$:

Definition (Increasing nets)
A net $x_d : D \to X$ is increasing if $d \preceq e \to x_d \leq x_e$ for $d, e \in D$.

Most notions of convergence carry over to nets mutatis mutandis. We (only) study nets with $D \subseteq N$ and $\preceq_D \subseteq D \times D$. In this way, our nets are third-order objects with extra structure $(D, \preceq)$. Nets yield same (lower type) convergence theory as filters (Bartle).
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$$(\forall Y^2)(\exists X^1)(\forall n^0)[n \in X \leftrightarrow (\exists f^1)(Y(f, n) = 0)].$$

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Many theorems for nets imply (are equivalent to) BOOT: Ascoli-Arzela, anti-Specker, Bolzano-Weierstrass, Cauchy nets, ...
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Many theorems for nets imply (are equivalent to) BOOT: Ascoli-Arzela, anti-Specker, Bolzano-Weierstrass, Cauchy nets, . . . Adding a modulus of convergence to \( \text{MCT}_{\text{net}} \) yields equivalence to \( \text{BOOT} + \text{QF-AC}^{0,1} \). Non-uniqueness and ECF!
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RCA$_0^\omega$ plus $\Delta$-comprehension or countable choice

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Why...

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Two possible meanings of continuous function, namely:

1. second-order RM code for a continuous function.
2. third-order function that satisfies the $\varepsilon$-$\delta$-definition.

See my latest arXiv paper.
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We thank DFG, TU Darmstadt, John Templeton Foundation, and Alexander Von Humboldt Foundation for their generous support!

Thank you for your attention!

Any (content) questions?