Plato and Brouwer, sitting in a binary tree

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This talk reports on my joint project with Dag Normann (U. of Oslo) on the Reverse Mathematics and computability theory of the uncountable.

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See arXiv for some of our papers!

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Solution? An alternative hierarchy, going back to Brouwer, is identified. The 'Big Five' equivalences are a reflection of (part of) this new hierarchy, following Plato's allegory of the cave.

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ν-functional produces witness to $(\exists f : \mathbb{N} \to \mathbb{N})A(f)$, yielding Z_2.

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Connection: $\mathsf{Z}_2 \equiv_{L_2} \mathsf{Z}_2^\omega \equiv_{L_2} \mathsf{Z}_2^\Omega$. Note 3rd vs 4th order!

Gödel hierarchy

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- $\vdots$
- large cardinals
- $\vdots$
- ZFC
- ZC (Zermelo set theory)
- simple type theory

medium

- Z_2 (second-order arithmetic)
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- ACA_0 (arithmetical comprehension)

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- WKL_0 (weak König's lemma)
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- PRA (primitive recursive arithmetic)
- bounded arithmetic

It is striking that a great many foundational theories are linearly ordered by [consistency strength] $<$. Of course it is possible to construct pairs of artificial theories which are incomparable under $<$. However, this is not the case for the "natural" or non-artificial theories which are usually regarded as significant in the foundations of mathematics.
(Simpson, Gödel Centennial Volume; also: Koelner, Burgess, Friedman,. . .)

Gödel hierarchy

= 'comprehension' hierarchy

$\uparrow$

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...

LESS sets exist

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12. There is no injection from $[0,1]$ to $\mathbb{N}$ (Cantor, 1874).

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Theorem (Arzela, 1885)

Let $f_n : ([0,1] \times \mathbb{N}) \to \mathbb{R}$ be a sequence such that

1. *Each f_n is Riemann integrable on $[0,1]$.*
2. *There is $M > 0$ such that $(\forall n \in \mathbb{N}, x \in [0,1])(|f_n(x)| \leq M)$.*
3. *$\lim_{n\to\infty} f_n = f$ exists and is Riemann integrable.*

Then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

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2. *There is $M > 0$ such that $(\forall n \in \mathbb{N}, x \in [0,1])(|f_n(x)| \leq M)$.*
3. *$\lim_{n\to\infty} f_n = f$ exists and is Riemann integrable.*

Then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

This theorem, called Arz, is provable in Z_2^{Ω} but not in Z_2^{ω}.

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The *convergence thms* show that the Lebesgue integral is superior to the Riemann integral. Such thms do exist for Riemann integrals:

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Riemann's *Habilschrift* (1854) entrenched discontinuous functions in the mainstream.

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We use 'totally bounded' and 'separable' in the sense of RM.

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These 'new' proofs led us to...

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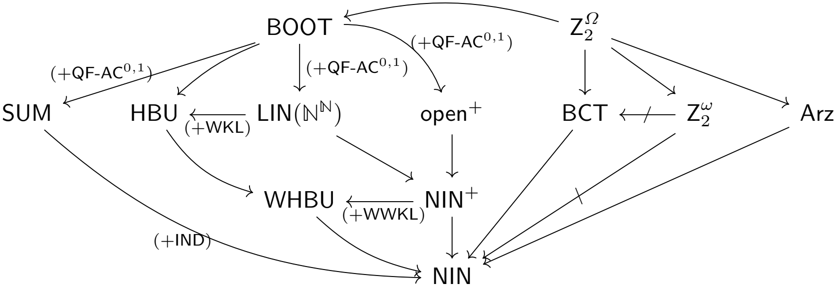
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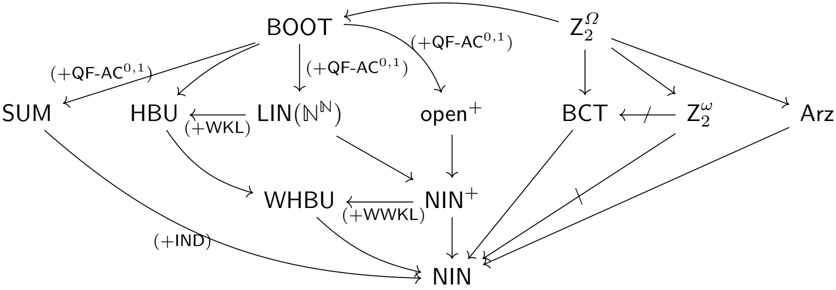

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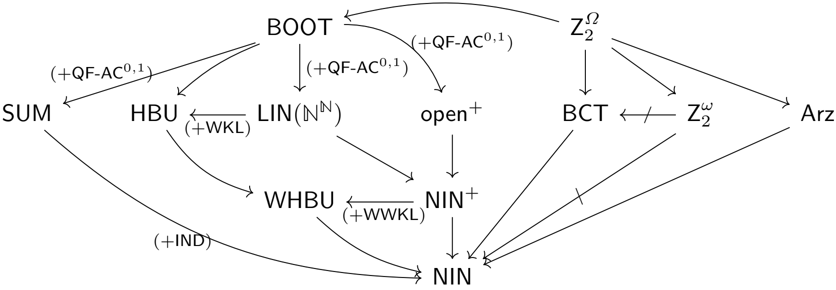

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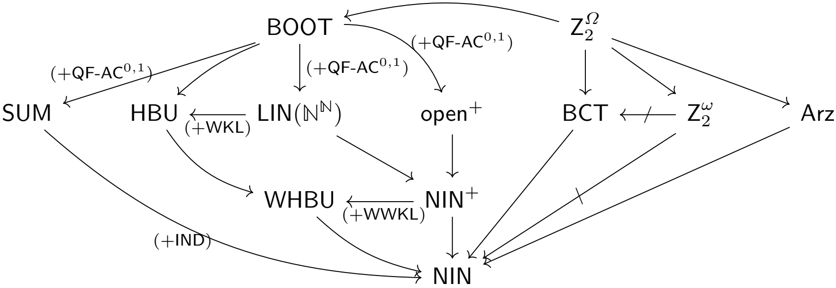

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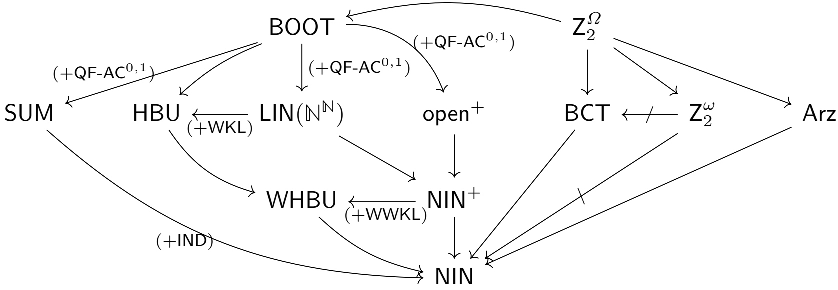

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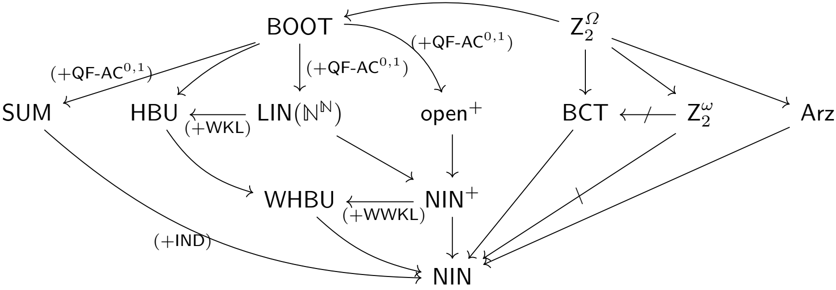
Negative results do not change if we add $\text{QF-AC}^{0,1}$ to Z_2^ω.
$\text{QF-AC}^{0,1}$ is 'weakest' fragment of CC not provable in ZF.



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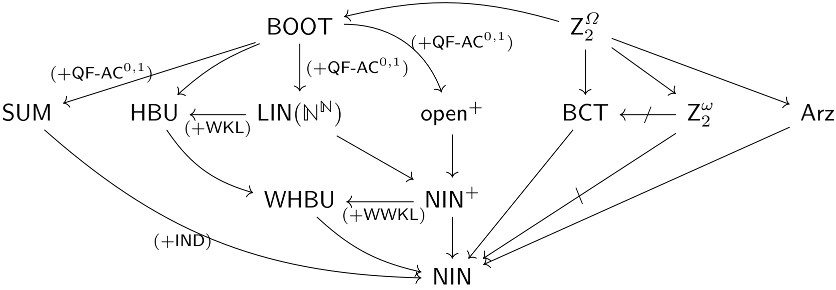

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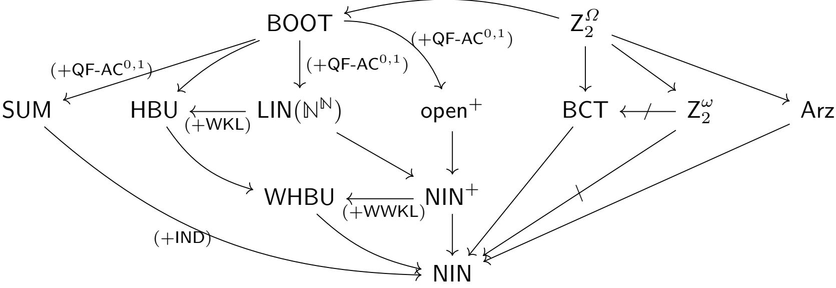
Similar computational results: $\exists^3$ computes realiser Θ for HBU, which computes realiser for NIN; no ν_n computes a realiser for NIN.



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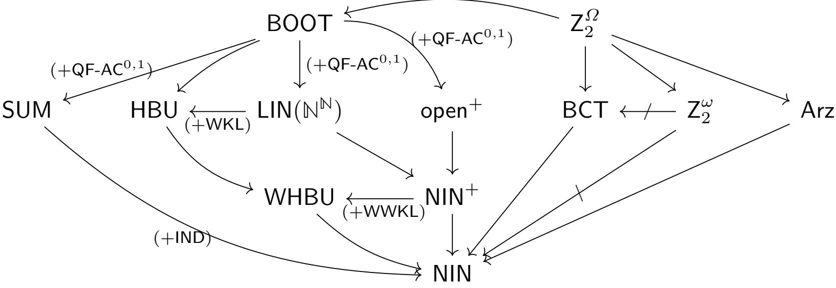

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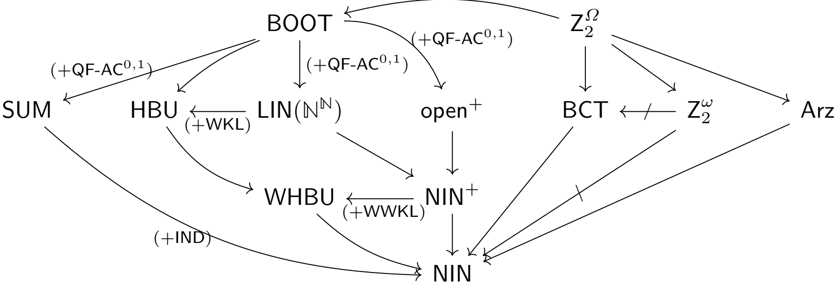

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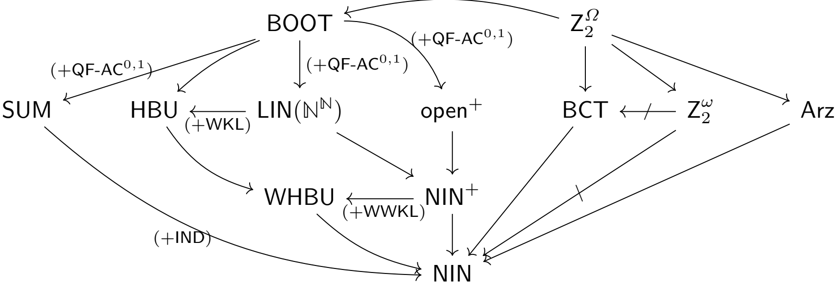

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Because apples and oranges: the ‘comprehension functionals’ in Z_2^ω and Z_2^Ω are discontinuous, while the other theorems (HBU, NIN, etc) are consistent with Brouwer’s (continuity) theorem.

Brouwer and continuity to the rescue

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For any formula A, we have

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Note that $\overline{f}n$ is the finite sequence $\langle f(0), f(1), \ldots, f(n-1)\rangle$.

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The second item reminds one of Plato's allegory of the cave.

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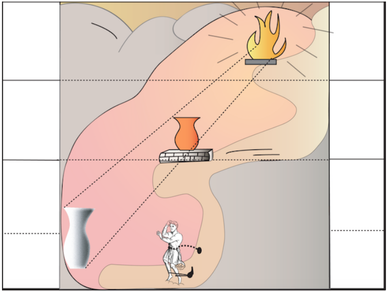

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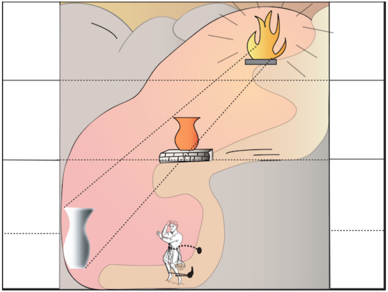

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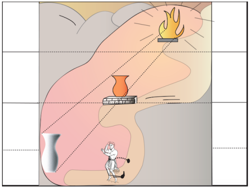

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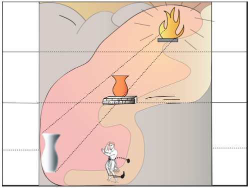

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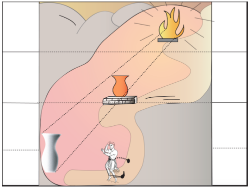

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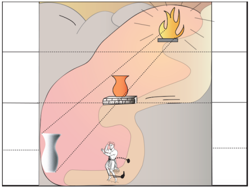

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ECF is canonical embedding of HOA into SOA (Kleene-Kreisel).

The Big Five as a reflection

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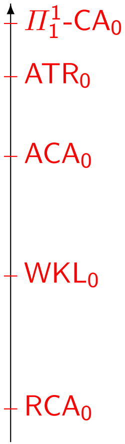

The Big Five as a reflection

Π^1_1-CA_0

ATR_0

ACA_0

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RCA_0 proves Δ^0_1-comprehension

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SECOND-ORDER arithmetic

BOOT$_2$

Σ-TR

BOOT

WKL^1

RCA$_0^\omega$

HIGHER-ORDER arithmetic

The Big Five as a reflection

SECOND-ORDER arithmetic		HIGHER-ORDER arithmetic	
Π_1^1-CA_0	$\leftrightarrow$ Cantor-Bendixson thm	BOOT_2	
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ECF replaces uncountable objects by countable representations/RM-codes

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ECF converts right-hand side to left-hand side, including equivalences!

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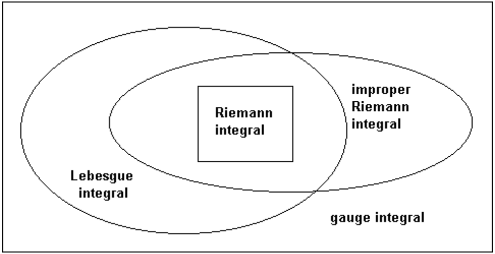

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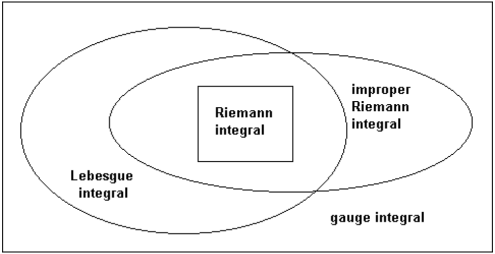

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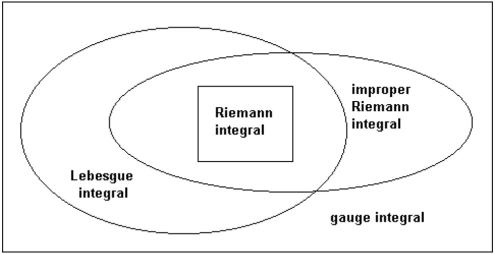

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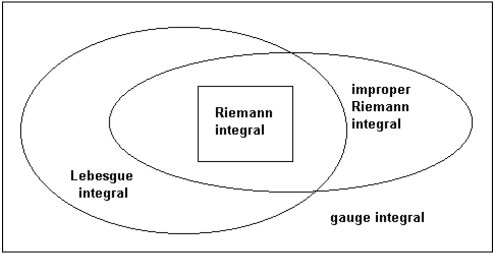

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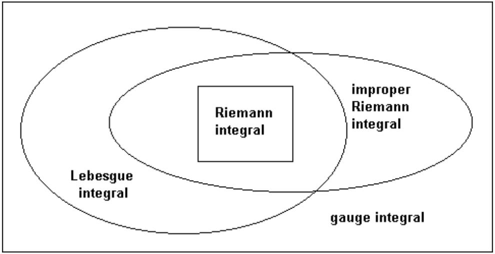

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Many basic thms about gauge integral are equivalent to HBU. ECF maps these to thms about Riemann integral.

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Definition (Riemann integral)

$f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable on $I \equiv [0,1]$ with integral $A \in \mathbb{R}$:

$$(\forall \varepsilon > 0)(\exists \underbrace{\delta > 0}_{\textit{constant}})(\forall P)(\underbrace{(\forall i \leq k)(|x_i - x_{i+1}| < \delta)}_{P \text{ is 'finer' than } \delta} \to |S(P,f) - A| < \varepsilon)$$

$P = (0, t_1, x_1 \dots x_k, t_k, 1)$ partition $S(P,f) = \sum_i f(t_i)|x_{i+1} - x_i|$ Riemann sum

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A function is f Lebesgue integrable IFF f and $|f|$ are gauge integrable.

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$\mathsf{HBU} \leftrightarrow \mathsf{WKL}^1$ needs some extra choice, which ECF maps to a triviality.

Today: a higher RM

ECF replaces uncountable objects by countable/continuous RM-codes

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A set $D \neq \emptyset$ with a binary relation '$\preceq$' is directed if

a. The relation $\preceq$ is transitive and reflexive.
b. For $d, e \in D$, there is $f \in D$ such that $d \preceq f \wedge e \preceq f$.

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Sequences are nets for $(D, \preceq) = (\mathbb{N}, \leq)$. We write x_d for $x(d)$.

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A net x_d *converges* to the limit $y = \lim_d x_d$ if for any neighbourhood U of y, there is $d_0 \in D$ such that for all $e \succeq d_0$, $x_e \in U$.

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Nets yield same (lower type) convergence theory as filters (Bartle).

Monotone convergence

The central object of study in domain theory is the directed complete partial order (dcpo), formulated using nets.

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Many theorems for nets imply (are equivalent to) BOOT: Ascoli-Arzela, anti-Specker, Bolzano-Weierstrass, Cauchy nets, ...

Adding a modulus of convergence to $\mathsf{MCT}_{\mathsf{net}}$ yields equivalence to $\mathsf{BOOT} + \mathsf{QF\text{-}AC}^{0,1}$. Non-uniqueness and ECF!

A higher RM

ECF replaces uncountable objects by countable/continuous RM-codes

Kohlenbach's RM: based on discontinuity; Plato hierarchy: based on continuity.

Π_1^1-CA_0 $\leftrightarrow$ Cantor-Bendixson thm	BOOT_2 $\leftrightarrow$ Cantor-Bendixson thm (uncountable unions)
ATR_0 $\leftrightarrow$ perfect set theorem	Σ-TR $\leftrightarrow$ perfect set thm (idem)
ACA_0 $\leftrightarrow$ range of $f : \mathbb{N} \to \mathbb{N}$ exists $\leftrightarrow$ Monotone conv. thm $\leftrightarrow$ Ascoli-Arzela $\leftrightarrow$ thms about closed sets (as countable unions)	BOOT $\leftrightarrow$ range of $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ exist $\leftrightarrow$ Mon. conv. thm for nets $\leftrightarrow$ Ascoli-Arzela for nets $\leftrightarrow$ thms about closed sets (as uncountable unions)
WKL_0 $\leftrightarrow$ Dini's theorem. $\leftrightarrow$ countabe Heine-Borel compactness $\leftrightarrow$ Riemann int. thms	WKL^1 $\leftrightarrow$ Dini's theorem for nets. $\leftrightarrow$ uncountabe Heine-Borel compactness: HBU $\leftrightarrow$ gauge integral thms
RCA_0 proves Δ_1^0-comprehension	RCA_0^ω plus Δ-comprehension or countable choice
SECOND-ORDER arithmetic	HIGHER-ORDER arithmetic

ECF

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See my latest arxiv paper.

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Any (content) questions?