The Biggest Five of Reverse Mathematics

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Secondly, we discuss the very different logical and mathematical limits of this extension. (chasm, abyss)

Thirdly, we may discuss foundational implications, though . . .
Reverse Mathematics
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= finding the **minimal** axioms needed to **prove** a theorem of ordinary mathematics.
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= finding the \textbf{minimal} axioms needed to \textit{prove} a theorem of ordinary mathematics.

Harvey Friedman & Steve Simpson (courtesy of MFO).
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The Biggest Five of Reverse Mathematics

The Big Five

Reverse Mathematics = finding the minimal axioms needed to prove a theorem of ordinary mathematics.

One always assume the base theory RCA\(_0\), a weak system formalising 'computable mathematics'.

Big Five phenomenon = most/many theorems of ordinary mathematics are either provable in RCA\(_0\), or equivalent to one of the other 'Big Five' systems:

- (countable) Heine-Borel compactness: WKL\(_0\)
- Sequential compactness: ACA\(_0\)
- Transfinite recursion: ATR\(_0\)
- Some descriptive set theory: Π\(_1^1\)-comprehension.

Mummert: a few equivalences for Π\(_1^2\)-comprehension and topology.
The Big Five

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The base theory $\text{RCA}_0^\omega$

$\text{RCA}_0^\omega$ makes use of the language of finite types: $n \in \mathbb{N}$ or $n^0$, $f \in \mathbb{N}^{\mathbb{N}}$ or $f^1$, $Y : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ or $Y^2$, et cetera.
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$\text{RCA}_0^\omega$ has axioms for primitive recursion and induction (variation of $\text{RCA}_0$), axiom of function extensionality, and QF-AC$^{1,0}$:

$$(\forall f^1)(\exists n^0)(Y(f, n) = 0) \to (\exists G^2)(\forall f^1)(Y(f, G(f)) = 0),$$
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Real numbers and ‘$=_{\mathbb{R}}$’ defined as in $\text{RCA}_0$; $\mathbb{R} \to \mathbb{R}$-functions are $\mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$-functions extensional relative to ‘$=_{\mathbb{R}}$’.
Real analysis has been studied in second-order RM, mostly for \textit{continuous functions}.
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**Coding** continuous functions (on $\mathbb{N}^\mathbb{N}$ and $\mathbb{R}$) does not change the RM of the Big Five (Kleene, Kohlenbach, Normann, Sanders).
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Recently, Dag Normann and I have obtained a plethora of equivalences (over RCA\(^0\)) between:

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Recently, Dag Normann and I have obtained a plethora of equivalences (over RCA\(_{\omega}^0 \)) between:

- **second-order** Big Five systems
- **third-order** theorems about (slightly) discontinuous functions.

These third-order theorems are called second-order-ish for obvious reasons. A similar phenomenon does not exist for first- and second-order theorems (AFAIK).
Weak König’s lemma

The following are equivalent to $WKL_0$ over $RCA_0$:
Weak König’s lemma

The following are equivalent to WKL₀ over RCA₀:

- A continuous function on [0, 1] is bounded.
- A bounded continuous function on [0, 1] has a supremum.
- A continuous function on [0, 1] attains a maximum.
- Cousin’s lemma for continuous functions.
Weak König’s lemma

The following are equivalent to WKL\(_0\) over \(\text{RCA}_0\):
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- Cousin’s lemma for \textbf{continuous} functions.

There is \textbf{no mathematical need} to restrict to continuity here!
Weak König’s lemma

The following third-order thms are equivalent to WKL$_0$ over RCA$^\omega$: 

- A regulated function on $[0, 1]$ is bounded.
- A bounded Baire 1 function on $[0, 1]$ has a supremum.
- A upper semi-continuous function on $[0, 1]$ attains a max.
- A bounded quasi-continuous function on $[0, 1]$ has a sup.
- Cousin’s lemma for quasi-continuous functions.

Regulated means: the left and right limits exist.
Baire 1 means: pointwise limit of continuous functions.
Upper semi-continuity means: . . . (Baire).
Quasi-continuity means: . . . (Baire and Volterra).

WILD: there are $2^c$ non-measurable quasi-continuous functions and $2^c$ non-Borel bounded and measurable quasi-continuous functions.
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WILD: there are $2^c$ non-measurable quasi-continuous functions and $2^c$ non-Borel bounded and measurable quasi-continuous functions.
Arithmetical comprehension

The following are equivalent to ACA$_0$ over RCA$_0$:

- Let $F : C \to \mathbb{R}$ be continuous where $C \subset [0, 1]$ is an RM-closed set. Then $\sup_{x \in C} F(x)$ exists.
- Let $F : C \to \mathbb{R}$ be continuous where $C \subset [0, 1]$ is an RM-closed set. Then $F$ attains a maximum value on $C$.
- Jordan decomposition theorem restricted to codes.
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There is no mathematical need to restrict to continuity here!
Arithmetical comprehension

These third-order thms are equivalent to ACA\(_0\) over RCA\(_0^\omega\):

- Let \( F : C \to \mathbb{R} \) be cadlag where \( C \subset [0, 1] \) is an RM-closed set. Then \( \sup_{x \in C} F(x) \) exists.

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Cadlag means: the continuous on the right with left limit.
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Arithmetical Transfinite Recursion

These **third-order thms** are equivalent to $\text{ATR}_0$ over $\text{RCA}_0^\omega$:

- Jordan decomposition theorem restricted to **arithmetical** (or: $\Sigma^1_1$) functions.
- A non-enumerable **arithmetical** set in $\mathbb{R}$ has a limit point.
- Cousin’s lemma for **effectively Baire 2** functions.
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Baire 1 means: pointwise limit of continuous functions.
Baire 2 means: pointwise limit of Baire 1 functions.
Effectively Baire 2 means: iterated limit of double sequence of continuous functions (\(\approx\) second-order codes for Baire 2).
Arithmetical Transfinite Recursion

These third-order thms are equivalent to ATR₀ over RCA₀:

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Baire (1905) notes that Baire 2 functions can be represented as iterated limits.
\( \Pi^1_1 \)-comprehension

These third-order thms are equivalent to \( \Pi^1_1 \)-CA\(_0\) over \( \text{RCA}_0^\omega + X \):

- For any \( x \in \mathbb{N}^\mathbb{N} \), any bounded \( \Sigma^1_1 \)-class in \( \mathbb{Q}^+ \) has a supremum.
- A bounded effectively Baire 2 \( f : [0, 1] \to \mathbb{R} \) has a supremum.
- For \( n \geq 2 \), a bounded and effectively Baire \( n \) \( f : [0, 1] \to \mathbb{R} \) has a supremum.
**Π₁¹-comprehension**

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- A bounded **effectively Baire 2** \( f : [0, 1] \rightarrow \mathbb{R} \) has a supremum.
- For \( n \geq 2 \), a bounded and **effectively Baire n** \( f : [0, 1] \rightarrow \mathbb{R} \) has a supremum.

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The Biggest Five phenomenon of higher-order RM

Recently, Dag Normann and I have obtained a plethora of equivalences (over $\text{RCA}_0^\omega$) between:

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There are however **hard limits** to the Biggest Five phenomenon, with interesting consequences.
The Biggest Five of Reverse Mathematics

Abyss? Abyss!

In ordinal analysis, the difference between the systems $\Pi^1_1$-$CA_0$ and $\Pi^1_2$-$CA_0$ has been described as an abyss or chasm by Michael Rathjen and Per Martin-Löf. The difference between $\Pi^1_1$-$CA_0$ and $\Sigma^2_2$ is therefore galactic in nature (about 12 parsecs?). Slight variations of the aforementioned second-order-ish theorems are not provable in RCA$_0^\omega$ and stronger systems. The mathematical difference between the original and the variation is infinitesimal.
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The abyss and $\Pi^1_1$-$\text{CA}_0$

This third-order thm is equivalent to $\Pi^1_1$-$\text{CA}_0$ over $\text{RCA}_0^\omega + X$:

An **effectively** Baire 2 function $F : [0, 1] \to [0, 1]$ has a supremum.
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Nota Bene: AC is not the problem!
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The following is equivalent to ACA$_0$ over RCA$_0$:

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\( f : [0, 1] \rightarrow \mathbb{R} \) is quasi-continuous if for all \( \epsilon > 0, N \in \mathbb{N}, x \in [0, 1] \), there is \((a, b) \subset B(x, \frac{1}{2^N})\) with \((\forall y \in (a, b))(|f(x) - f(y)| < \epsilon)\).
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The abyss and weak König’s lemma

This third-order thm is equivalent to WKL$_0$ over RCA$_0^\omega$:

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Many similar results: the classical hierarchy of function classes looks very different in weak (and some strong) systems.
On Kleene’s arithmetical quantifier $\exists^2$

The above was obtained based on the RM of Kleene’s $\exists^2$:

$$(\exists E)(\forall f \in \mathbb{N}^\mathbb{N})(E(f) = 0 \leftrightarrow (\exists n \in \mathbb{N})(f(n) = 0)).$$  ($\exists^2$)
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The system $\text{RCA}_0 + (\exists^2)$ is $L_2$-conservative over ACA$_0$. 
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- the existence of a discontinuous $\mathbb{R} \to \mathbb{R}$-function (Kohlenbach).
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Not provable in $\text{RCA}_0^\omega + (\exists^2) + Z_2$ and stronger systems:

There is a $\mathbb{R} \to \mathbb{R}$-function that is not Baire 2.
Exploring the abyss: the uncountability of $\mathbb{R}$

Cantor’s first set theory paper (1874): uncountability of $\mathbb{R}$. 

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Cantor's theorem: there is no surjection from $\mathbb{N}$ to $[0,1]$. 

NIN: there is no injection from $[0,1]$ to $\mathbb{N}$. 

NBI there is no bijection from $[0,1]$ to $\mathbb{N}$. 

Many many many (third-order) mainstream theorems imply NIN or NBI. 

However, NIN and NBI cannot be proved in RCA$_{\omega + 1}$ and stronger (higher-order) systems (see Normann-Sanders, JSL, 2022).
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What causes this abyss?

A function class is *second-order-ish* if its definition allows one to approximate $f(x)$ for all $x \in \mathbb{R}$ given only $f(q)$ for all $q \in \mathbb{Q}$. 
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Many equivalences for NIN and basic properties of regulated functions. Same for basic properties of measure and category and semi-continuity (Baire, Volterra, . . . ).
Non-second-order-ish mathematics exhibits a number of interesting phenomena that are ‘miniature’ versions of well-known observations in set theory, including:
Foundational musings

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- the mercurial nature of the cardinality of \( \mathbb{R} \),
- basic properties of the Lebesgue measure and integral,
- the special role of the Axiom of Choice,
- the asymmetry between measure and category.
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Thanks!

Questions?

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